REPEATED DISTANCES IN SPACE

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ABSTRACT

For $i = 1, .., n$ let (x_i, r_i) be a circle in the plane with centre x_i and radius r_i. A Repeated Distance graph is a directed graph whose vertices are the centres and where (x_i, x_j) is a directed edge whenever x_j lies on the circle with centre x_i. Special cases are the Nearest Neighbour graph, when r_i is the minimum distance between x_i and any other centre, and the Furthest Neighbour graph which is similar except that maximum replaces minimum. Repeated Distance graphs generalize to any dimension with spheres or hyperspheres replacing circles. Bounds are given on the number of edges in Repeated Distance graphs in d dimensions, with particularly tight bounds for the Furthest Neighbour Graph in three dimensions. The proofs use extremal graph theory.

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1. Introduction

Let $X = \{x_1, \ldots, x_n\}$ be a set of n points in R^d, $d \ge 2$ and let $R = \{r_1, \ldots, r_n\}$ be a set of n positive real numbers. The Repeated Distance Graph $\vec{G}_d(X, R)$ is a directed graph on the point set X with edges (x_i, x_j) whenever $d(x_i, x_j) = r_i$, where d denotes Euclidean distance. In this paper we will be investigating the number $f_d(n)$, which is the maximum number of edges that a repeated distance graph on n points in d dimensions can have. We denote by $G_d(X, R)$ the undirected graph obtained from $\vec{G}_d(X, R)$ by removing the directions on the edges and by deleting any multiple edges. The graph $\overleftrightarrow{G}_d(X, R)$ is an undirected graph on X where (x_i, x_j) is an edge of $\overleftrightarrow{G}_d(X, R)$ whenever both (x_i, x_j) and (x_j, x_i) are edges of $\vec{G}_d(X, R)$. Many special cases of these graphs have been studied, especially when $d = 2$. We will give several examples of these graphs and some selected references. For a complete bibliography the reader is referred to the excellent collection of Moser and Pach[10]. As a notational convention, functions representing the number of edges in directed graphs are denoted with lower case f, and functions for the number of edges in undirected graphs are denoted F.

Example 1. Unit Distance Graph.

This graph is obtained by setting $r_i = 1$, $i = 1, \ldots, n$. Since the Repeated Distance Graph is symmetric $\overleftrightarrow{G}_d(X, R) = G_d(X, R)$. Denote by F_d^{ud} the maximum number of edges in a unit distance graph $G_d(X, R)$ in d dimensions. Chung, Szemerédi and Trotter[4] have shown that

$$F_2^{ud}(n) < cn^{5/4},$$

and Beck[2] has shown that

$$F_3^{ud}(n) < n^{5/3-\varepsilon},$$

which improve on many earlier results. The best lower bounds are

$$F_2^{ud}(n) > n^{1+\frac{\varepsilon}{\log\log n}} \quad \text{and}$$

$$F_3^{ud}(n) > n^{4/3}\log\log n$$

proved by Erdős[5]. In higher dimensions, Lenz (unpublished) gave examples to show that

$$F_d^{ud}(n) > \left(\frac{1}{2} - \frac{1}{2\lfloor d/2 \rfloor}\right) n^2 + c_d n$$

and Erdős[5] showed that

$$F_d^{ud}(n) < \left(\frac{1}{2} - \frac{1}{2\lfloor d/2 \rfloor}\right) n^2 + O(n^{2-\varepsilon_d}).$$

Lenz's examples are constructed by taking $\lfloor d/2 \rfloor$ pairwise perpendicular 2-planes in R^d through the origin, and in each of them by picking $\lfloor n/\lfloor d/2 \rfloor \rfloor$ points on the unit circle around the origin.

Example 2. Minimum Distance Graph

Let $r = \min_{i \neq j} d(x_i, x_j)$. Setting $r_i = r$ we obtain the minimum distance graph, which is also symmetric. Let $F_d^{\min}(n)$ denote the maximum number of edges in a minimum distance graph $G_d(X, R)$ in d dimensions. Harborth[7] has shown that

$$F_2^{\min}(n) = \lfloor 3n - \sqrt{12n - 3} \rfloor.$$

Example 3. Nearest Neighbour Graph.

This non symmetric digraph is formed by setting $r_i = \min_{j \neq i} d(x_i, x_j), i = 1, \ldots, n$. This graph, which contains the minimum distance graph as a subgraph, has many applications in pattern recognition[13]. Let $f_d^{nn}(n)$ denote the maximum number of edges in the nearest neighbour graph $\vec{G}_d(X, R)$. Since this graph is planar for $d = 2$,

$$f_2^{nn}(n) \leq 6n - 12.$$

Example 4. Diameter Graph.

Let $r = \max_{i \neq j} d(x_i, x_j)$. By setting $r_i = r, i = 1, \ldots, n$ we obtain the diameter graph, which is a symmetric digraph. Let $F_d^{diam}(n)$ be the maximum number of edges in the diameter graph $G_d(X, R)$ in d dimensions. Sutherland[12] proved that

$$F_2^{diam}(n) = n,$$

solving a problem of Hopf and Pannwitz. Grünbaum[6] Heppes[8] and Straszewicz[11] independently proved Vázsonyi's conjecture that

$$F_3^{diam}(n) = 2n - 2.$$

In higher dimensions, $d \geq 4$, the previously mentioned results of Erdős and Lenz (Example 1) apply also to the diameter graph.

Example 5. Furthest Neighbour Graph.

By setting $r_i = \max_{i \neq j} d(x_i, x_j)$ we obtain the non symmetric furthest neighbour graph. This graph contains the diameter graph as a subgraph, and also finds application in pattern recognition. Let $f_d^{fn}(n)$ denote the maximum number of edges in a furthest neighbour graph in d dimensions. Avis[1] has shown that

$$f_2^{fn}(2n) = 6n - 3.$$

In this paper we obtain new results for repeated distance graphs, especially in dimension 3 and higher. These results are summarized below.

Theorem 1. There are constants $c_0, c_1, \varepsilon_0, \varepsilon_1$ such that

$$f_2(n) < \sqrt{2}\, n^{3/2} + n/2 \tag{1}$$

$$\frac{n^2}{4} + \frac{3n}{2} \leq f_3(n) < \frac{n^2}{4} + c_0 n^{2-\varepsilon_0} \tag{2}$$

$$n^2(1 - \frac{1}{\lfloor d/2 \rfloor}) < f_d(n) < n^2(1 - \frac{1}{\lceil d/2 \rceil}) + c_1 n^{2-\varepsilon_1} \qquad d \geq 4 \tag{3}$$

$$\frac{n^2}{4} + \frac{3n}{2} < f_3^{fn}(n) < \frac{n^2}{4} + \frac{3n}{2} + 255. \tag{4}$$

2. Repeated Distances

In this section we prove the first three parts of Theorem 1. First we introduce some notation. We denote by $K_{t_1, t_2, \cdots t_r}$ the complete r-partite graph with t_i vertices in colour class i. If all the colour classes have the same number t of vertices, the notation is abbreviated to $K_r(t)$. If all of the edges of a complete r-partite graph are directed in such a way that between any two colour classes the edges are all oriented in the same direction, we call the digraph **homogeneous** and denote it $\vec{K}_{t_1,t_2,\cdots,t_r}$ (respectively, $\vec{K}_r(t)$). Strictly speaking, this notation denotes an equivalence class of 2^r graphs, but for our purposes the actual direction of the edges between each pair of colour classes is normally irrelevant. It will be specified explicitly only where necessary.

The proofs in this section are based on the following three lemmas. The first is an elementary geometric result, and the others are standard results in extremal graph theory. Let $\vec{G}_d(X, R)$ be a repeated distance graph in $d \geq 2$ dimensions. Let T be a subset of t vertices of $\vec{G}_d(X, R)$. A non-empty disjoint subset U of vertices are called **common predecessors** if for every $u \varepsilon U$ and every $t \varepsilon T$, (u, t) is an edge in $\vec{G}_d(X, R)$.

Lemma 1. Let U be a set of common predecessors of T, then

(a) T lies in an orthogonal subspace of R^d to U, hence $dim(T) + dim(U) \leq d$;

(b) If T contains at least 3 points, then $dim(T) \geq 2$.

Let $ex(n, s, t)$ denote the maximum number of edges that a graph with n vertices and no $K_{s,t}$ can have. Let $ex(m, n, s, t)$ denote the maximum number of edges that a bipartite graph $G_{m,n}$ can have if it does not contain a $K_{s,t}$ with s vertices in the first colour class and t vertices in the second colour class. Then an argument of Kovari, Sós and Turan[9] gives the following upper bounds (see[3], p.310).

Lemma 2.

(a) $ex(n, s, t) < \frac{1}{2}(s-1)^{1/t}\,(n-t+1)n^{1-1/t} + \frac{1}{2}(t-1)\,n$ for $2 \leq s$, $2 \leq t$.

(b) $ex(m, n, s, t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$, for $2 \leq s \leq m$, $2 \leq t \leq n$.

The following lemma is an improved version of the Erdős-Stone Theorem proved by Erdős and Simonovits (see[3]).

Lemma 3. For every integer t there is a constant c such that, if n is sufficiently large, every graph of order n with more than

$$\frac{1}{2}(1-\frac{1}{r})n^2 + cn^{2-1/t}$$

edges contains a $K_{r+1}(t)$.

We will now prove the first three parts of Theorem 1. The first part is a simple consequence of Lemma 1.

Proof of (1). We first show that $\vec{G}_2(X, R)$ cannot contain a $\vec{K}_{2,3}$ with all edges directed into the 3 element colour class. Suppose it did. Let T be the 3 element colour class and let U be the other colour class. By Lemma 1(b), $dim(T) \geq 2$ hence $dim(U) = 0$, a contradiction. Let $d^+(x)$ denote the in degree of a vertex. By a well known argument we conclude that

$$\sum_{x \varepsilon X} \binom{d^+(x)}{2} \leq 2\binom{n}{2},$$

and therefore

$$n\binom{f_2(n)/n}{2} \leq 2\binom{n}{2},$$

which implies

$$f_2(n) < \sqrt{2}n^{3/2} + n/2. \qquad \textbf{Q. E. D.}$$

We remark that the right order of magnitude for the bound (1) can be obtained by the following simple argument. Let $k = \lceil n^{1/2} \rceil$. Observe that if there are less than k vertices with degree at least 2k, then the total number of edges in the graph can be at most $kn + 2k(n-k) < 3n^{3/2}$. However, there cannot be k points with degree $2k$ or more. For otherwise, since two circles can intersect in at most two points, the total number of points must be at least

$$2k^2 - 2\binom{k}{2} = k^2 + k > n,$$

which is a contradiction. The bound in (1) is probably far from the truth, in fact we conjecture that

$$f_2(n) < n^{1+\frac{\varepsilon}{loglog\ n}}.$$

Beck has recently proved that

$$f_2(n) = o(n^{3/2})$$

(private communication).

The next two lemmas will be required to prove equations (2) and (3).

Lemma 4. For $d \geq 3$, let $r = \lceil d/2 \rceil + 1$. Then $\vec{G}_d(X, R)$ contains no homogeneous $\vec{K}_r(3)$.

Proof: Suppose the lemma is false. By repeated application of Lemma 1(a), each of the r colour classes of $\vec{K}_r(3)$ lie orthogonal to each other. In addition, at least $r - 1$ colour classes must have incoming edges. Again by Lemma 1(b), these colour classes must have dimension at least 2. The remaining colour class must have dimension at least one, so the total dimension is at least

$$2(r-1) + 1 = 2\lceil d/2 \rceil + 1 > d,$$

a contradiction. **Q.E.D.**

Lemma 5. For positive integers $r \geq 2, t_1, \ldots, t_r$ there is an integer $n = f^r(t_1, \ldots, t_r)$ such that every orientation of $K_r(n)$ contains a homogeneous $\vec{K}_{t_1,\ldots,t_r}$ as a subgraph.

Proof: For positive integers s, t let

$$h(s,t) = \min\{s2^{2t}, t2^{2s}\}.$$

We first show that every orientation of $K_{h(s,t),h(s,t)}$ contains a $\vec{K}_{s,t}$. Indeed, suppose $s \geq t$ so that $h(s,t) = s2^{2t}$. Let S and T denote the two colour classes of an orientation of $K_{h(s,t),h(s,t)}$. For a fixed set of $2t$ vertices in T, there are 2^{2t} ways in which edges can be directed from a vertex in S. Therefore, at least $h(s,t)/2^{2t} = s$ vertices in S behave uniformly with respect to this set. At least t of the $2t$ vertices chosen from T must have all edges from this set of s vertices going in the same direction, giving the required $\vec{K}_{s,t}$.

Recursively define integers

$$a_0 = t_1, \quad a_s = h(a_{s-1}, t_{s+1}), \quad s = 1, \ldots, r-1.$$

We will show that

$$f^r(t_1, \ldots, t_r) \leq f^{r-1}(a_1, \ldots, a_{r-1}),$$

where

$$f^1(t) = t \qquad \text{for all integers } t.$$

The proof is by induction on r. For $r = 2$,

$$f^2(t_1, t_2) \le h(t_1, t_2) = h(a_0, t_2) = f^1(a_1).$$

Let $n = f^{r-1}(a_1, \ldots, a_{r-1})$ for some $r \ge 3$ and any integers $t_1, \ldots, t_r$. We will show that every orientation of $K_r(n)$ contains a homogeneous $\vec{K}_{t_1,\cdots,t_r}$. Let $T_1, \ldots, T_r$ denote the colour classes of a $K_r(n)$. By induction, every orientation of the edges between $T_2, \ldots, T_r$ contains a homogeneous $\vec{K}_{a_1,\ldots,a_{r-1}}$ with a_i vertices in T_{i+1}, $i = 1, \ldots, r-1$. Choose any a_{r-1} vertices in T_1. Since

$$a_{r-1} = h(a_{r-2}, t_r)$$

we can find a homogeneous $\vec{K}_{a_{r-2},t_r}$ with a_{r-2} vertices in T_1 and t_r vertices in T_r. We continue in this way for $i = 1, \ldots, r-2$. Since

$$a_{r-i-1} = h(a_{r-i-2}, t_{r-i}),$$

we can find a homogeneous $\vec{K}_{a_{r-i-2},t_{r-i}}$ amongst the a_{r-i-1} vertices in T_1 found in the preceding iteration. At the final step, we find a homogeneous $\vec{K}_{a_0,t_2} = \vec{K}_{t_1,t_2}$ between T_1 and T_2. By construction, the $t_1, \ldots, t_r$ vertices selected by this procedure give the required homogeneous subgraph. **Q.E.D.**

The bound obtained by this lemma seems extremely loose. In the following proofs, we will use the above lemma with $t_i = 3$, $i = 1, \ldots, r$. We introduce the abbreviated notation

$$\alpha(r) = f^r(3, \ldots, 3).$$

Proof of (2): The lower bound is obtained by placing $\lceil n/2 \rceil$ points on the unit circle $x^2 + y^2 = 1$ and the remaining points on the positive z axis below $z = 1$. For the upper bound, we first show that $\overleftrightarrow{G}_3(X, R)$ cannot contain a $K_{3,3}$. Suppose it did, and let T and U be the colour classes. Then U is a common predecessor set for T, and vice versa. By Lemma 1 we obtain that T and U are orthogonal and that

$$dim(T) \ge 2 \quad \text{and} \quad dim(U) \ge 2,$$

a contradiction. Therefore by Lemma 2(a),

$$|\overleftrightarrow{G}_3(X, R)| < n^{5/3} + n.$$

By Lemma 4, $\vec{G}_3(X, R)$ has no $\vec{K}_3(3)$ and so $G_3(X, R)$ has no $K_3(\alpha(3))$ by Lemma 5. This implies by Lemma 3 that

$$|G_3(X, R)| < \frac{n^2}{4} + c_0 n^{2-1/\alpha(3)}$$

for some constant c_0. The upper bound in equation (2) follows from the fact that

$$f_3(n) = |\overleftrightarrow{G}_3(X, R)| + |G_3(X, R)|.$$

Q.E.D.

Proof of (3): The lower bound will be proved in section 4 as it applies also to the furthest neighbour problem. For the upper bound set $r = \lceil d/2 \rceil + 1$ as in Lemma 4. Therefore $\vec{G}_d(X, R)$ does not contain a $\vec{K}_r(3)$. From Lemma 5 we obtain an integer $\alpha(r)$ such that $G_d(X, R)$ does not contain a $K_r(\alpha(r))$. Therefore, by Lemma 3

$$|G_d(X, R)| < \frac{1}{2}(1 - \frac{1}{r-1})n^2 + \frac{1}{2}\, c_1 n^{2-1/\alpha(r)}$$

for some constant c_1. The upper bound in (3) follows from the fact that

$$f_d(n) \le 2|G_d(X, R)|.$$

Q.E.D.

3. Furthest Neighbours

In this section we obtain a tight bound on the number of edges in the furthest neighbour graph in R^3. Let X be a set of n points in R^3, and let $\vec{G}_3(X, R)$ be the furthest neighbour graph. We call X a **suspension** if it can be transformed into a point set X^* by a suitable rotation, translation and scaling where

$$X^* \subseteq \{(x, y, 0) : x^2 + y^2 = 1\} \cup \{(0, 0, z) : z \in R\}.$$

Informally, the points of a suspension lie either on the polar axis or the equator of a sphere. We first prove that if n is sufficiently large then the furthest neighbour graphs with the maximum number of edges contain a very large suspension.

Lemma 6 Let X be an n point set that maximizes the number of edges in $\vec{G}_3(X, R)$. Then there is an integer n_0 such that if $n \geq n_0$ then X contains a suspension on $n - 6$ points.

Proof: For $n \geq n_0$ let X be an n point set that maximizes the number of edges, denoted $f_3^{fn}(n)$, in the furthest neighbour graph $\vec{G}_3(X, R)$. Let

$$X = H \cup S,$$

where H is the set of convex hull extreme points of X. Further let $h = |H|$ and $s = |S| = n - h$. We begin by showing that h and s are approximately equal. First some notation. For i a vertex and T a set of vertices, we denote the number of edges directed from i to T by $d_{i,T}$ and the number of edges from T to i by $d_{T,i}$. The number of edges with both endpoints in T is denoted d_T. For sets of vertices T and U, let $d_{T,U}$ denote the number of edges directed from T to U.

Let $\vec{G}_3(H, R)$ be the induced subgraph of $\vec{G}_3(X, R)$ on the set H. Observe that $\vec{G}_3(H, R)$ cannot contain an induced $\vec{K}(3, 3)$. For if it did, let U and T be the two colour classes and assume that all edges are directed from U to T. By Lemma 1(b), $dim(T) \geq 2$ and hence $dim(U) = 1$ by Lemma 1(a). But this means that U consists of three collinear points, one of which cannot be an extreme point. Therefore by the argument used in the proof of (1)

$$\sum_{x \in H} \binom{d^+(x)}{3} \leq 2\binom{h}{3},$$

where $d^+(x)$ denotes the number of incoming edges to vertex x from other vertices in H. Recall that d_H denotes the number of edges in $\vec{G}_3(H, R)$. We have

$$h\binom{d_H/h}{3} \leq 2\binom{h}{3},$$

$$d_H < 3h^{5/3}.$$

The example of Lenz, mentioned in Example 1, shows that

$$f_3^{fn}(n) > n^2/4.$$

We show that if $n \geq 3^9$ then $h \leq n/2 + n^{8/9}$. Suppose not. First observe that if y is a furthest neighbour of

$x \in X$, then $y \in H$. So all edges in $\vec{G}_3(X, R)$ terminate in H. Therefore

$$\begin{aligned} f_3^{fn}(n) &\le h\ s + d_H \\ &\le (n/2 - n^{8/9})\ (n/2 + n^{8/9}) + 3n^{5/3} \\ &= n^2/4 - n^{16/9} + 3n^{5/3} \\ &\le n^2/4 \qquad \text{for} \quad n \ge 3^9, \end{aligned}$$

a contradiction. An identical argument shows that $h \ge n/2 - n^{8/9}$. Hence h and s are approximately equal.

Next we show that there exist points x, y in S that have at least $\frac{n}{3} + 1$ common neighbours in H. Suppose that no such pair of vertices exists. Recall that $d_{S,H}$ denotes the number of edges directed from S to H. From the previous discussion we see that

$$f_3^{fn}(n) = d_H + d_{S,H} > n^2/4,$$

whence

$$d_{S,H} \ge \frac{n^2}{4} - 3n^{5/3}.$$

On the other hand, we have that

$$\sum_{i \in H} \binom{d_{S,i}}{2} \le \frac{n}{3} \binom{s}{2}$$

and hence

$$h \binom{d_{S,H}/h}{2} \le \frac{n}{3} \binom{s}{2} \le \frac{ns^2}{6}.$$

Therefore

$$s^2 \ge \frac{6h}{n} \binom{d_{S,H}/h}{2}.$$

However it is easy to check that the right hand side is at least $\frac{3n^2}{8} + o(n^2)$ and so s is at least $(\frac{3}{2})^{1/2} \frac{n}{2} + o(n)$, a contradiction for suitably large n. This establishes the claim that there are two points

x, y in S with at least $n/3$ common neighbours in H. Denote the common neighbours $N(x, y)$. We will now show that every other point u in S is adjacent to $N(x, y)$ in $\vec{G}_3(X, R)$. The points in $N(x, y)$ are cocircular. Therefore if some point u of S is not adjacent to all of them, then it can be adjacent to at most two of them. In this case

$$d_{u,H} \le h - n/3 + 2 \le n/6 + n^{8/9} + 2.$$

Consider replacing u by a point on the segment xy not already in our point set. Note that since u is an interior point, it cannot be the furthest neighbour of any other point. Since its new out degree is at least $n/3$ and the degrees of other points do not decrease, we obtain a set of points with more edges in its furthest neighbour graph, a contradiction. We may therefore assume for that all points in S are collinear.

We next argue that most of the vertices of H are cocircular. Here the argument is complicated by the fact that moving a vertex in H may cause it to become the new furthest neighbour of some vertex x. If x previously had several furthest neighbours, its out degree may now be reduced.

We partition the points in H into those points H_1 that are cocircular with centres in S, and the remaining points H_2. There may be up to two points in H_2 which are also collinear with the points in S. For convenience, we remove these points from H_2 and place them in S. Therefore if $H_2 = \emptyset$, X is a suspension. We will show that if H_2 is sufficiently large, all of its vertices may be moved to H_1 with a net increase in the number of edges in $\vec{G}_3(X, R)$. Let $h_1 = |H_1|$ and $h_2 = |H_2|$. We will show that if n is sufficiently large, then a furthest neighbour graph with maximum number of edges has $h_2 \le 14$.

Let x be any point in H_2. As S contains at most two extreme points,

$$d_{x,S} \le 2,$$

since x is not in H_1 we have

$$d_{S,x} \le 1,$$

and since x is not collinear with S we have

$$d_{x,H_1} \le 2.$$

Consider the bipartite graph G^* with colour classes H_1 and H_2, and all edges from $\vec{G}_3(X,R)$ that were directed from H_1 to H_2. Recall that d_{H_1,H_2} denotes the number of edges in G^*. Since G^* cannot have a $K_{3,3}$, we have from Lemma 2(b) that

$$d_{H_1,H_2} \le ex(h_2, h_1, 3, 3) < 2^{1/3}\ h_2\ h_1^{2/3} + 2h_{1.}$$

Now consider the subgraph of $\vec{G}_3(X,R)$ induced by the points in H_2. Recall that d_{H_2} denotes the number of edges in this subgraph. Then since H_2 is a set of extreme points,

$$d_{H_2} \le 3h_2^{5/3}.$$

Now consider moving all of the vertices in H_2 to the circle through H_1. The total number of edges lost in the furthest neighbour graph by such a move is at most

$$\begin{aligned} d_{H_1,H_2} + d_{H_2} + d_{H,S} + d_{S,H_2} + d_{H_1} &\le 2^{1/3} h_2\ h_1^{2/3} + 2h_1 + 3h_2^{5/3} + 2h + s + 2h_1 \\ &\le 5h_2\ n^{2/3} + 7n \end{aligned}$$

Observe that the set of h_2 vertices moved onto the circle will create at least

$$s\ h_2 \ge (\frac{n}{2} - n^{8/9})\ h_2$$

new edges. If $h_2 \ge 15$, for all sufficiently large n

$$(\frac{n}{2} - n^{8/9})\ h_2 > 5h_2\ n^{2/3} + 7n.$$

Hence for all sufficiently large n, $h_2 \le 14$ and so $\vec{G}_3(X,R)$ must contain a suspension of size $n - 14$. **Q.E.D.**

Proof of (4): Let X be a suspension on n points with h points on a circle and $n - h$ points on the central axis. Each point on the axis has h furthest neighbours on the circle and possibly two additional furthest neighbours on the axis itself. Each point on the circle can have at most two furthest neighbours on the circle and two on the central axis. However, a simple geometric calculation shows that, if the points on the circle have two furthest neighbours on the axis, then these two points are further apart than they are from the circle. This means that they have no furthest neighbours on the circle. It can easily be seen that this does not give a configuration with the maximum number of edges. It can also be shown that in the maximum configuration there are in fact no edges between points on the axis. Hence the total number of furthest neighbour pairs in a suspension on n points is at most

$$h(n-h) + 3h \le \frac{n^2}{4} + \frac{3n}{2} + \frac{9}{4}\,.$$

By Lemma 6, for $n \ge n_0$ the furthest neighbour graph with maximum number of edges contains a suspension of size at least $n-14$. Each of the points not in the suspension can have edges directed towards at most four points in the suspension. Each can be the furthest neighbour of at most one point on the axis, h points on the circle, and the 13 other points not in the suspension. Therefore each point can contribute at most $h+18$ edges. Therefore this graph can have at most

$$h(n-14-h) + 3h + 14(h+18) \le \frac{n^2}{4} + \frac{3n}{2} + 255.$$

This proves the upper bound in (4). For the lower bound, let $n = 4k+3$ and set $h = 2k+3$. Space h points evenly on the unit circle in the $x-y$-plane with centre the origin. Let r denote the farthest distance between points on the circle. It can be verified that $1 < r < 2$. This distance occurs $2h$ times. Now place a point on $(0,0,1)$, a point on $(0,0,\sqrt{r^2-1})$ and $2k-2$ points evenly spaced between them. Each of these points have h furthest neighbours on the circle and each point on the circle has one furthest neighbour on the axis. There are

$$3(2k+3) + 2k(2k+3) = \frac{n^2}{4} + \frac{3n}{2} + \frac{9}{4}$$

edges in this graph. A similar construction for other values of n proves the lower bound in (4).

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