# On the Fractional Chromatic Index of a Graph and its Complement

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24 September 2004

*ABSTRACT*

The *chromatic index* $\chi_e(G)$ of an undirected graph $G$ is the minimum number of matchings needed to partition its edge set. Let $\Delta(G)$ denote the maximum vertex degree of $G$, and let $\bar{G}$ denote the complement of $G$. Jensen and Toft conjectured that for a graph $G$ with an even number of vertices, either $\chi_e(G) = \Delta(G)$ or $\chi_e(\bar{G}) = \Delta(\bar{G})$. We prove a fractional version of this conjecture.

## 1. The Introduction

The *chromatic index* $\chi_e(G)$ of a graph $G = (V(G), E(G))$ is the minimum number of matchings needed to partition its edge set (for the definition of matching and other standard terms in graph theory, see Bondy and Murty[1] ). Since all the edges incident to a vertex must be in different matchings, we know that $\chi_e(G)$ is at least the maximum degree of $G$, which we denote by $\Delta(G)$. In fact Vizing[5] proved that $\chi_e(G)$ is at most $\Delta(G) + 1$, whilst Holyer[3] proved it is NP-complete to determine if $\chi_e(G)$ is $\Delta(G)$. If $H$ is a subgraph of $G$ with $2t + 1$ vertices, then we need at least $\lceil |E(H)| / t \rceil$ matchings to cover $E(H)$. Thus if $|E(H)| > t\,\Delta(G)$, then $H$ and hence $G$ have chromatic index $\Delta(G) + 1$. A subgraph $H$ of $G$ is called *overfull* if it has an odd number of vertices and

$$|E(H)| > \Delta(G)\ \frac{|V(H)| - 1}{2.} \tag{1}$$

If $G$ is a regular graph with an odd number of vertices then $G$ is overfull, and its complement, $\bar{G}$, being regular, is also overfull. Thus for such graphs $\chi_e(G) = \Delta(G) + 1$ and $\chi_e(\bar{G}) = \Delta(\bar{G}) + 1$. In [4], Jensen and Toft conjectured that this could not happen for graphs with an even number of vertices. They conjectured that if $G$ has an even number of vertices then either $\chi_e(G) = \Delta(G)$ or $\chi_e(\bar{G}) = \Delta(\bar{G})$. As evidence in support of this conjecture, we show that it is true for fractional edge colourings.

A *fractional edge colouring* of $G$ is a non-negative weighting $w(.)$ of the set $\mathbf{M}(G)$ of matchings in $G$ so that for every edge $e \in E(G)$,

$$\sum_{M: e \in M} w(M) = 1.$$

The weight $\alpha$ of this colouring is defined by

$$\alpha = \sum_{M \in \mathbf{M}(G)} w(M),$$

and in this case we say that $G$ has a fractional $\alpha$-edge colouring.

The *fractional chromatic index*, $\chi_e^f(G)$, of $G$ is the minimum $\alpha$ for which $G$ has a fractional $\alpha$-edge colouring. By dividing the weight function by $\alpha$, we see that $G$ has a fractional $\alpha$-edge colouring if and only if the vector $(1/\alpha, \ldots, 1/\alpha) \in R^{|E(G)|}$ is a convex combination of incidence vectors of matchings of $G$. It follows from Edmonds' characterization of the matching polytope[2] that $\chi_e^f(G)$ can be computed in polynomial time, and that in fact

$$\chi_e^f(G) = \max\left\{\Delta(G), \max_{H \subseteq G, H \text{ overfull}} \frac{2\,|\,E(H)\,|}{|V(H)| - 1}\right\}.$$

We will use the corollary that if $\chi_e^f(G) > \Delta(G)$ then $G$ must contain an overfull subgraph. We may now state our result.

**The Theorem.** Let $G$ be a graph such that $\chi_e^f(G) > \Delta(G)$ and $\chi_e^f(\bar{G}) > \Delta(\bar{G})$, then either $G$ or $\bar{G}$ is overfull.

Since overfull subgraphs have an odd number of vertices, this yields:

**The Corollary** If $G$ has an even number of vertices then either $\chi_e^f(G) = \Delta(G)$ or $\chi_e^f(\bar{G}) = \Delta(\bar{G})$.

## 2. The Proof.

Arguing by contradiction, we suppose there is a graph $G$ such that neither $G$ nor its complement is overfull, yet $\chi_e^f(G) > \Delta(G)$ and $\chi_e^f(\bar{G}) > \Delta(\bar{G})$. We may choose an overfull subgraph $H$ of $G$ and an overfull subgraph $F$ of $\bar{G}$. To simplify the exposition, in what follows we let $G$, $F$ and $H$ also stand for their vertex sets where no confusion arises. Since the sum of the degrees of all vertices of a graph is twice the number of its edges, letting $d_H(v)$ be the degree of a vertex $v$ in $H$, we see that (1) is equivalent to

$$\sum_{v \in H} (\Delta(G) - d_H(v)) \le \Delta(G) - 1. \tag{2}$$

and implies that

$$|H| \;\ge\; \Delta(G) + 1 \;\text{ and }\; |F| \;\ge\; \Delta(\bar{G}) + 1. \tag{3}$$

We call $def_G(v) = \Delta(G) - d_G(v)$ the *deficiency*, of a vertex $v$ in $G$, and let $E_G(A, B)$ be the set of edges in $G$ with one endpoint in $A \subseteq V$ and one endpoint in $B \subseteq V$. With these definitions and using (2) and (3) we have

$$\sum_{v \in H} def_G(v) + |E_G(H, G - H)| \le \Delta(G) - 1 \le |H| - 2. \tag{4}$$

Similarly, in the complement $\bar{G}$, we have

$$\sum_{v \in F} def_{\bar{G}}(v) + |E_{\bar{G}}(F, \bar{G} - F)| \le \Delta(\bar{G}) - 1 \le |F| - 2. \qquad (5)$$

The subgraphs $H$ and $F$ naturally partition the vertices of $G$ into four parts with cardinalities $a, b, c, d$, as defined by

$$a = |H \cap F|, \quad b = |H - F|, \quad c = |F - H|, \text{ and } d = |G - F - H|.$$

This partition is illustrated in Table 1. Let $n = |V(G)|$.

| | F | G - F |
|---|---|---|
| H | a | b |
| G - H | c | d |

**Table 1: Partition of $G$ into four subsets**

We note that if $v \in F - H$ and $w \in H - F$ then $vw$ is either an edge of $E_G(H, G - H)$ or of $E_{\bar{G}}(F, G - F)$. The same statement holds when $v \in F \cap H$ and $w \in G - F - H$. This implies the inequality

$$ad + bc \le |E_G(H, G - H)| + |E_{\bar{G}}(F, G - F)|. \qquad (6)$$

For every vertex $v$ of $G$ we have $d_G(v) + d_{\bar{G}}(v) = n - 1$ and so $\Delta(G) + \Delta(\bar{G}) \ge n - 1$. Hence we can define the nonnegative integer $k$ by

$$k = \Delta(G) + \Delta(\bar{G}) - n + 1. \qquad (7)$$

We also have that for all $v$,

$$k = def_G(v) + def_{\bar{G}}(v) \qquad (8)$$

and so

$$\sum_{v \in H} def_G(v) + \sum_{v \in F} def_{\bar{G}}(v) \ge \sum_{v \in H \cap F} (def_G(v) + def_{\bar{G}}(v)) = ak. \qquad (9)$$

Combining the inequalities (4)-(9) we obtain the key inequality:

$$bc + a(d + k) \le \Delta(G) + \Delta(\bar{G}) - 2 = n + k - 3. \qquad (10)$$

Manipulating this inequality will give the desired result. By (3)

$$2a + b + c \ge \Delta(G) + \Delta(\bar{G}) + 2.$$

Combining with (10) we have

$$bc + a(d + k) \le 2a + b + c - 4 \qquad (11)$$

Now if $b, c \ge 1$ then $bc \ge b + c - 1$. If in addition $d + k \ge 2$ then

$$bc + a(d + k) \ge b + c - 1 + 2a > 2a + b + c - 4$$

a contradiction.

The remaining cases to consider are when either $b = 0$ or $c = 0$ or $0 \le d + k \le 1$. We will need the following two observations.

**Observation 1:** $|H|, |F| \le |G| - 2$.

**Proof:** We know by hypothesis that $|H| \ne |G|$. Suppose $|H| = |G| - 1$. Let $w$ be the vertex of $G - H$. From (4) we have

$$d_G(w) = |E(H, G - H)| \le \Delta(G) - 1 - \sum_{v \in H} def_G(v).$$

In fact $d_G(w) \le \Delta(G) - 2$. This follows immediately if $def_G(v) \ge 1$ for some $v \in H$. Otherwise every vertex of $H$ has degree $\Delta(G)$. Since the total degree of $G$ must be even and $|H|$ is odd, $d_G(w)$ must have the same parity as $\Delta(G)$ and again $d_G(w) \le \Delta(G) - 2$. On the other hand, $d_{\bar{G}}(w) \le \Delta(\bar{G})$, which combined with (7) and (8) gives

$$k \ge \Delta(G) + d_{\bar{G}}(w) - n + 1 = \Delta(G) - d_G(w) \ge 2.$$

This also implies that $d_G(w) \ge \Delta(G) - k$. Since $F \cap H = F - w$,

$$\sum_{v \in F \cap H} (def_{\bar{G}}(v) + def_G(v)) + |E(H, G - H)| \ge k(|F| - 1) + \Delta(G) - k$$

$$= k|F| + \Delta(G) - 2k \ge \Delta(\bar{G}) + 1 + \Delta(G) - 2k + (k - 1)|F|,$$

where we used (3) to get the last inequality. Since $F$ is overfull, $|F| \ge 3$ and we have

$$\sum_{v \in F \cap H} (def_{\bar{G}}(v) + def_G(v)) + |E(H, G - H)| \ge \Delta(G) + \Delta(\bar{G}) + k - 2 \ge \Delta(G) + \Delta(\bar{G}).$$

By combining (4) and (5), we see that this cannot happen, giving the desired contradiction. By replacing $G$ by its complement, we obtain $F \le |G| - 2$. □

**Observation 2:** $\min(\Delta(G), \Delta(\bar{G})) \ge n/2 - 1$.

**Proof:** We assume that $\Delta(\bar{G}) \le n/2 - 3/2$ and derive a contradiction. Since $\Delta(G) + \Delta(\bar{G}) \ge n - 1$ it follows that $\Delta(G) \ge n/2 + 1/2$ and hence $|H| \ge n/2 + 3/2$, while $|G - H| \le n/2 - 3/2$. Furthermore every vertex of $G$ must have degree at least $n/2 + 1/2$ and so

$$|E(H, G - H)| \ge |G - H|(n/2 + 1/2 - (|G - H| - 1)) = |G - H|(n/2 + 3/2 - |G - H|).$$

When $|G - H| = 2$ this value is $n - 1$ and when $|G - H| = n/2 - 3/2$ it is $3n/2 - 9/2$.
By Observation 1, we have $|H| \le n - 2$, i.e. $|G - H| \ge 2$, and so

$$|E(H, G - H)| \ge \min(n - 1, 3n/2 - 9/2) \ge n - 3 \ge |H| - 1$$

which violates (4). □

We now return to our manipulation of (10) for the cases when $b = 0$ or $c = 0$ or $0 \le d + k \le 1$. First assume that $b = 0$ and so $H \subseteq F$. By Observation 2 and (3) we have

$$a = |F \cap H| = |H| \ge \Delta(G) + 1 \ge n/2,$$

and so $a \ge b + c + d$. If $d + k \ge 3$ then $a(d + k) \ge 3a \ge 2a + b + c + d$ contradicting (11). If $d + k = 2$ then $a(d + k) = 2a \ge a + b + c + d = n$ contradicting (10). Similarly we obtain contradictions if $c = 0$ and $d + k \ge 2$.

It remains to settle the cases where $d + k = 0$ or 1. If $d = 0$ then neither $b$ nor $c$ can be zero (or else either $G$ or $\bar{G}$ is overfull), and so $bc \ge b + c - 1$. Hence if $d = 0$ and $k = 1$ then

$$bc + a(d + k) \geq a + b + c + d - 1 = n - 1$$

again contradicting (10).

If $k = 0$ then by (8) $G$ is $\Delta(G)$-regular and $\bar{G}$ is $\Delta(\bar{G})$-regular. Since regular graphs with an odd number of vertices are overfull, this means that $n$ is even. We have by (7) that $\Delta(G) + \Delta(\bar{G}) = n - 1$, and so can assume that $\Delta(G) \geq n/2$. Hence by (3) we have that $|H| \geq n/2 + 1$ and so $|G - H| \leq n/2 - 1$. On the other hand, Observation 1 gives $|G - H| \geq 2$, thus

$$|E_G(H, G - H)| \geq |G - H|(n/2 - (|G - H| - 1)) \geq n - 2, \tag{12}$$

where the last inequality is obtained by checking the two extremal values of $|G - H|$. Combining (4) and (12) gives the required contradiction and completes the proof of the theorem. □

**References**

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