On the Chromatic Polynomial of a Graph

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Abstract

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G with n vertices, stability number α and clique number ω. We show that for every $\lambda \geq n$,

$$\frac{\lambda - n + \alpha}{\lambda} \left(\frac{\lambda - n + \alpha - 1}{\lambda - n + \alpha} \right)^{\alpha} \leq \frac{P(G, \lambda - 1)}{P(G, \lambda)} \leq \frac{\lambda - \omega}{\lambda} \left(\frac{\lambda - 1}{\lambda} \right)^{n - \omega}.$$

We characterize the graphs that yield the lower bound or the upper bound.
These results give new bounds on the mean colour number $\mu(G)$ of G:

$$n - (n - \omega) \left(\frac{n-1}{n} \right)^{n-\omega} \leq \mu(G) \leq n - \alpha \left(\frac{\alpha - 1}{\alpha} \right)^{\alpha}.$$

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1 Introduction

In this paper, all our graphs will be undirected, simple, and finite. Let G be an arbitrary graph with n vertices. A vertex colouring of G is an assignment

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of colours to the vertices of G so that adjacent vertices get different colours. The minimum number of colours needed in a vertex colouring of G is called the *chromatic number* of G and is denoted by $\chi(G)$. Let $c_k(G)$ denote the total number of vertex colourings of G that use precisely k colours; clearly, $c_k(G) > 0$ if and only if $k \geq \chi(G)$.

Let $P(G, \lambda)$ denote the total number of vertex colourings of G that use at most λ colours. It is well known that $P(G, \lambda)$ is a polynomial in λ:

$$P(G, \lambda) = \sum_{k=1}^{n} c_k(G)(\lambda)_k,$$

where $(\lambda)_k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$; $P(G, \lambda)$ is called the *chromatic polynomial* of G. For instance, if $G = K_n$ (the complete graph with n vertices) then $P(G, \lambda) = (\lambda)_n$; if $G = O_n$ (the graph with n vertices and no edges) then $P(G, \lambda) = \lambda^n$; if $G = T_n$ (an arbitrary tree with n vertices) then $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$. In this paper, λ will be a positive integer.

Consider the set of all colourings of G with at most n colours. The *mean colour number* of G, denoted by $\mu(G)$ is the average of the number of colours used by a colouring in this set. The mean colour number of G is related to the chromatic polynomial of G (see [2]):

$$\mu(G) = n\left(1 - \frac{P(G, n-1)}{P(G, n)}\right).$$

Since

$$\chi(G) \leq \mu(G) \leq n,$$

it follows that

$$0 \leq \frac{P(G, n-1)}{P(G, n)} \leq \frac{n - \chi(G)}{n}.$$

In 1995, Bartels and Welsh [2] conjectured that

$$\frac{P(G, n-1)}{P(G, n)} \leq \left(\frac{n-1}{n}\right)^n,$$

or equivalently,

$$\frac{P(G, n-1)}{P(G, n)} \leq \frac{P(O_n, n-1)}{P(O_n, n)}.$$

This conjecture was proved by Dong [4] in 1998, who showed the following result.

Theorem 1 ([4]) *Let G be a graph with n vertices. Then, for every $\lambda \geq n$,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \left(\frac{\lambda-1}{\lambda}\right)^n. \tag{1}$$

To prove Theorem 1, Dong gave a better bound on $P(G,\lambda-1)/P(G,\lambda)$ in case G was connected.

Theorem 2 ([4]) *Let G be a connected graph with n vertices. Then, for every $\lambda \geq n$,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-2}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-2}. \tag{2}$$

Moreover, if x is a vertex of G such that $G-x$ is connected, then for every $\lambda \geq n$,

$$\frac{P(G-x,\lambda)}{P(G,\lambda)} \geq \frac{2\lambda-3}{(\lambda-1)(2\lambda-d-2)}, \tag{3}$$

where d denotes the degree of x in G.

In this paper we shall improve Theorem 1 and inequality (2) in Theorem 2; moreover, we shall give a lower bound on $P(G,\lambda-1)/P(G,\lambda)$. The proof of our generalization of inequality (2) uses a technique similar to that of Dong, and in particular relies on inequality (3).

We close this section with some easy properties of the chromatic polynomial of a graph that will be used in this paper. For this purpose, let G be a graph and let x and y be two vertices of G; $G+xy$ will denote the graph obtained from G by adding the edge xy (if it does not exists); $G_{|xy}$ will denote the simple graph obtained from G by contracting the vertices x and y (i.e. by identifying x and y and by replacing every multiple edge by a single one); $G-x$ will denote the graph obtained from G by removing vertex x. Let x be a non-isolated vertex of G and let $x_1,\ldots,x_d$ denote the vertices that are adjacent to x; if x_i and x_j, with $i>j$, are non-adjacent, then G^x_{ij} will denote the graph obtained from $G-x$ by adding the edges x_1x_i, x_2x_i, …, $x_{j-1}x_i$ (if they do not exist) and by contracting x_i and x_j. Finally, a *clique-cutset* in a graph $G=(V,E)$ is a clique whose removal from G disconnects the graph; when G has a clique cutset K_t we shall write

$G = G_1 \cup G_2$, where G_1 and G_2 are the two induced subgraphs of G such that $G_1 \cap G_2 = K_t$.

Property 1 *Let G be a graph and let x and y be two non-adjacent vertices in G. Then*

$$P(G,\lambda) = P(G+xy,\lambda) + P(G_{|xy},\lambda).$$

Property 2 *Let G be a graph with n vertices and let x be a vertex of G that is adjacent to all other vertices. Then*

$$P(G,\lambda) = \lambda P(G-x,\lambda-1).$$

Property 3 ([*4*]) *Let $G=(V,E)$ be a graph, let x be a vertex of G, and let $x_1,\ldots,x_d$ be the vertices that are adjacent to x, with $d \geq 1$. Then*

$$P(G,\lambda) = (\lambda-d)P(G-x,\lambda) + \sum_{i=2}^{d} \sum_{1\leq j<i:\, x_ix_j\notin E} P(G^x_{ij},\lambda).$$

Property 4 *Let $G=(V,E)$ be a non-connected graph and let $C_1,\ldots,C_k$ denote its connected components. Then*

$$P(G,\lambda) = \prod_{i=1}^{k} P(C_i,\lambda).$$

Property 5 *Let $G=(V,E)$ be a graph, let K_t denote a clique cutset in G, and let G_1 and G_2 denote the two induced subgraphs of G such that $G = G_1 \cup G_2$ and $K_t = G_1 \cap G_2$. Then*

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{P(K_t,\lambda)}.$$

2 The main results

For every graph G, let $\alpha(G)$ denote the *stability number* of G (maximum cardinality of a subset of pairwise non-adjacent vertices), and let $\omega(G)$ denote the *clique number* of G (maximum cardinality of a subset of pairwise adjacent vertices).

To every graph G with n vertices, we can associate two graphs $L(G)$ and $U(G)$: $L(G)$ is the complete union of an independent set S of size $\alpha(G)$ and a clique of size $n-\alpha(G)$; $U(G)$ is the disjoint union of a clique K of size $\omega(G)$ and an independent set of size $n-\omega(G)$. In Figure 1 we show a graph G along with the corresponding $L(G)$ and $U(G)$. Note that when the graph G is K_n or O_n then both graphs $G(L)$ and $U(L)$ are equal to G.

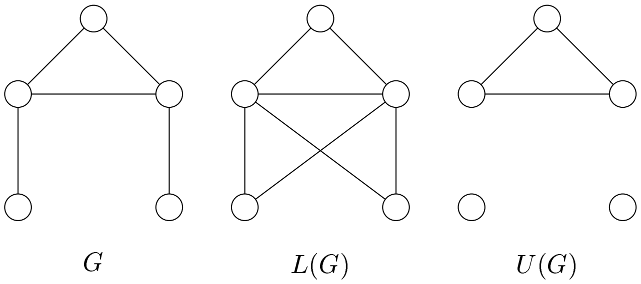


Figure 1

Since

$$P(L(G),\lambda) = P(K_{n-\alpha(G)},\lambda)(\lambda - n + \alpha(G))^{\alpha(G)}$$

and since

$$P(U(G),\lambda) = P(K_{\omega(G)},\lambda)\lambda^{n-\omega(G)},$$

it follows that

$$\frac{P(L(G),\lambda-1)}{P(L(G),\lambda)} = \frac{\lambda-n+\alpha(G)}{\lambda}\left(\frac{\lambda-n+\alpha(G)-1}{\lambda-n+\alpha(G)}\right)^{\alpha(G)}$$

$$\frac{P(U(G),\lambda-1)}{P(U(G),\lambda)} = \frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega(G)}.$$

The following result will show that $P(L(G),\lambda-1)/P(L(G),\lambda)$ is a lower bound on $P(G,\lambda-1)/P(G,\lambda)$ and that $P(U(G),\lambda-1)/P(U(G),\lambda)$ is an upper bound on $P(G,\lambda-1)/P(G,\lambda)$.

Theorem 3 *Let G be a graph with n vertices. Then for every $\lambda \geq n$,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \geq \frac{\lambda-n+\alpha(G)}{\lambda}\left(\frac{\lambda-n+\alpha(G)-1}{\lambda-n+\alpha(G)}\right)^{\alpha(G)} \tag{4}$$

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega(G)}. \tag{5}$$

Theorem 3 implies Theorem 1. To see this, note that for every two integers m, k with $m \geq k \geq 0$, we can write

$$\frac{m-k}{m} = \prod_{i=1}^{k}\frac{m-i}{m-i+1} \leq \prod_{i=1}^{k}\frac{m-1}{m},$$

and so

$$\frac{m-k}{m} \leq \left(\frac{m-1}{m}\right)^{k}, \tag{6}$$

with equality if and only if $k=1$. Hence,

$$\frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega(G)} \leq \left(\frac{\lambda-1}{\lambda}\right)^{n}; \tag{7}$$

inequality (7) follows directly from (6) with $m=\lambda$ and $k=\omega(G)$.

Now, Dong's upper bound on $P(G,\lambda-1)/P(G,\lambda)$ is related to the empty graph O_n, in the sense that

$$\frac{P(O_n,\lambda-1)}{P(O_n,\lambda)} = \left(\frac{\lambda-1}{\lambda}\right)^{n}.$$

Note that O_n can be seen as the "most sparse" graph having n vertices with clique number equal to one; on the other hand, our graph $U(G)$ can be seen as the "most sparse" graph having n vertices with clique number equal to $\omega(G)$.

To the best of our knowledge, the only lower bound on $P(G,\lambda-1)/P(G,\lambda)$ that was known was zero; this lower bound is related to the complete graph K_n, in the sense that

$$\frac{P(K_n,\lambda-1)}{P(K_n,\lambda)} = 0.$$

The graph K_n can be seen as the "most dense" graph having n vertices with stability number equal to one; our graph $L(G)$ can be seen as the "most dense" graph having n vertices with stability number equal to $\alpha(G)$.

It is natural to ask which graphs G reach the lower bound and/or the upper bound. Clearly, when $G = L(G)$ (i.e., G is the complete union of a maximum independent set and a clique) then $P(G, \lambda-1)/P(G, \lambda)$ is equal to the lower bound; when $G = U(G)$ (i.e., G is the disjoint union of a maximum clique and an independent set) then $P(G, \lambda-1)/P(G, \lambda)$ is equal to the upper bound. The following result shows that those are the only possible cases.

Theorem 4 *Let G be a graph with n vertices, let $\alpha = \alpha(G)$, and let $\omega = \omega(G)$. Then for every $\lambda \geq n$,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \frac{\lambda-n+\alpha}{\lambda}\left(\frac{\lambda-n+\alpha-1}{\lambda-n+\alpha}\right)^{\alpha} \quad \textit{if and only if } G = L(G),$$

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega} \quad \textit{if and only if } G = U(G).$$

To prove the validity of (5) in Theorem 3, we shall prove a stronger result for a connected graph.

Theorem 5 *Let G be a connected graph with n vertices. Then, for every $\lambda \geq n$,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega(G)}. \tag{8}$$

Theorem 5 implies inequality (2) in Theorem 2. To see this, note that this is obvious when $\omega(G) \leq 2$ (in this case, the upper bound given by (2) or (8) is the same). When $\omega(G) \geq 3$, we can show that the upper bound given by (8) is smaller than the upper bound given by (2), that is

$$\frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega(G)} < \frac{\lambda-2}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-2}. \tag{9}$$

To see the validity of (9), note that

$$\left(\frac{\lambda-2}{\lambda-1}\right)^{n-2}\left(\frac{\lambda-1}{\lambda-2}\right)^{n-\omega(G)} = \left(\frac{\lambda-2}{\lambda-1}\right)^{\omega(G)-2},$$

[illegible]

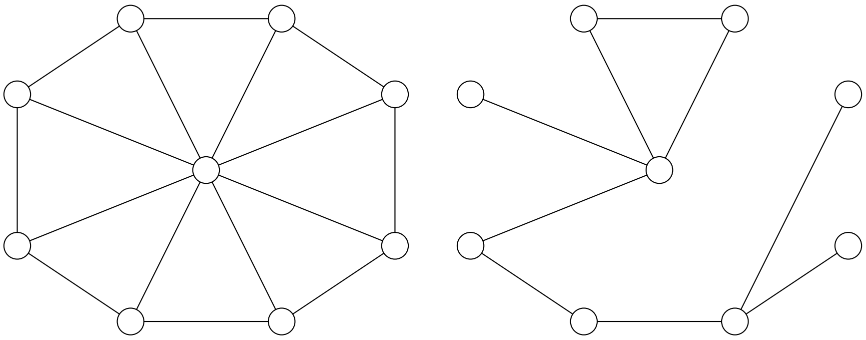
and that

$$\frac{\lambda-\omega(G)}{\lambda-2} = [illegible]$$

and so, by (6),

$$\frac{\lambda-\omega(G)}{\lambda-2} \leq \left(\frac{\lambda-3}{\lambda-2}\right)^{\omega(G)-2} < \left(\frac{\lambda-2}{\lambda-1}\right)^{\omega(G)-2}. \qquad ∎$$

To give a meaning to the upper bound in (8), we need to introduce one more graph related to G: the graph $C(G)$. $C(G) = (V, E)$ is a connected graph with n vertices composed by a clique K and a tree T having precisely one common vertex; K and T have $\omega(G)$ and $n-\omega(G)+1$ vertices, respectively. Since, in general, there are many such trees, the graph $C(G)$ may be not unique. In Figure 2 we show a graph G and a corresponding $C(G)$.



G $\qquad$ $C(G)$

Figure 2

Since

$$P(C(G), \lambda) = \frac{P(K_{\omega(G)}, \lambda) P(T_{n-\omega+1}, \lambda)}{\lambda},$$

it follows that

$$\frac{P(C(G), \lambda-1)}{P(C(G), \lambda)} = \frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega(G)}.$$

Hence, the upper bound in (8) is nothing but $P(C(G), \lambda-1)/P(C(G), \lambda)$. Note that the upper bound in (2) is nothing but $P(T_n, \lambda-1)/P(T_n, \lambda)$.

While the graph T_n can be seen as the "most sparse" connected graph with n vertices ($n \geq 2$) having clique number equal to two, our graph $C(G)$ can be seen as the "most sparse" connected graph with n vertices having clique number equal to $\omega(G)$.

3 Proof of the results

As stated in the previous section, to prove Theorem 3, we first need to prove Theorem 5.

Proof of Theorem 5. Let G be an arbitrary connected graph with n vertices; write $\omega = \omega(G)$. The theorem is obvious when $n \leq 3$ or when $G = K_n$. Hence, we may assume that $n \geq 4$, that $G \neq K_n$, and that (8) holds for all connected graphs with at most $n-1$ vertices.

Let x be an arbitrary vertex of G such that $\omega(G-x) = \omega$ (such a vertex exists since $G \neq K_n$) and let $x_1, \ldots, x_d$ denote all the vertices of G that are adjacent to x. Clearly $d \geq 1$ since G is connected.

First, assume that $G - x$ is not connected. Then, by Property 5, we can write

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{\lambda},$$

where G_1 and G_2 are such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = x$. Since G is connected, both G_1 and G_2 are connected. By assumption, (8) holds for the graphs G_1 and G_2, and so for $i = 1, 2$

$$\frac{P(G_i,\lambda-1)}{P(G_i,\lambda)} \leq \frac{\lambda-\omega(G_i)}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n_i-\omega(G_i)},$$

where n_i denote the number of vertices of G_i. Then

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{(\lambda-\omega(G_1))(\lambda-\omega(G_2))}{\lambda^2}\left(\frac{\lambda-2}{\lambda-1}\right)^{n_1+n_2-\omega(G_1)-\omega(G_2)}\frac{\lambda}{\lambda-1}.$$

Now, ω is equal to $\omega(G_1)$ or to $\omega(G_2)$; without loss of generality, we can assume that $\omega = \omega(G_1)$. Since $n = n_1 + n_2 - 1$, we have

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega}\frac{\lambda-\omega(G_2)}{\lambda-1}\left(\frac{\lambda-1}{\lambda-2}\right)^{\omega(G_2)-1}.$$

But

$$\frac{\lambda - \omega(G_2)}{\lambda - 1} = \frac{(\lambda - 1) - (\omega(G_2) - 1)}{\lambda - 1} \leq \left(\frac{\lambda - 2}{\lambda - 1}\right)^{\omega(G_2)-1},$$

(the last inequality follows from (6)), and so we are done.

Hence, we can assume that $G - x$ is connected. If we set

$$f(G,\lambda) = (\lambda - \omega)(\lambda - 2)^{n-\omega} P(G,\lambda) - \lambda(\lambda - 1)^{n-\omega} P(G,\lambda - 1),$$

to prove the validity of the theorem, we only need show that $f(G,\lambda) \geq 0$. Note that, by Property 3,

$$P(G,\lambda) = (\lambda - d)P(G - x,\lambda) + \sum_{i=2}^{d} \sum_{1 \leq j < i:\, x_i x_j \notin E} P(G^x_{ij},\lambda). \tag{10}$$

Write $\omega_{ij} = \omega(G^x_{ij})$. Since $G - x$ and each G^x_{ij} are connected graphs with less vertices than G, by assumption, inequality (8) holds for these graphs. Hence,

$$\frac{P(G - x,\lambda - 1)}{P(G - x,\lambda)} \leq \frac{\lambda - \omega}{\lambda} \left(\frac{\lambda - 2}{\lambda - 1}\right)^{n-1-\omega} \tag{11}$$

and

$$\frac{P(G^x_{ij},\lambda - 1)}{P(G^x_{ij},\lambda)} \leq \frac{\lambda - \omega_{ij}}{\lambda} \left(\frac{\lambda - 2}{\lambda - 1}\right)^{n-2-\omega_{ij}}. \tag{12}$$

Clearly, either $\omega_{ij} = \omega$ or $\omega_{ij} = \omega + 1$ (because $\omega(G - x) = \omega$). In both cases it is easy to show that

$$\frac{P(G^x_{ij},\lambda - 1)}{P(G^x_{ij},\lambda)} \leq \frac{\lambda - \omega}{\lambda} \left(\frac{\lambda - 2}{\lambda - 1}\right)^{n-2-\omega}. \tag{13}$$

Now, let

$$\begin{aligned} f_1(G,\lambda) = \;& (\lambda - d - 1)(\lambda - 1)[(\lambda - \omega)(\lambda - 2)^{n-1-\omega} P(G - x,\lambda) + \\ & -\lambda(\lambda - 1)^{n-1-\omega} P(G - x,\lambda - 1)], \end{aligned}$$

$$\begin{aligned} f_2(G,\lambda) = \;& (\lambda-1)^2\sum_{i=2}^{d} \sum_{1\le j<i:\, x_ix_j\notin E} [(\lambda-\omega)(\lambda-2)^{n-2-\omega}P(G^x_{ij},\lambda)+ \\ & -\lambda(\lambda-1)^{n-2-\omega}P(G^x_{ij},\lambda-1)], \end{aligned}$$

$$\begin{aligned} f_3(G,\lambda) = \;& (d-1)(\lambda-\omega)(\lambda-2)^{n-1-\omega}P(G-x,\lambda)+ \\ & -(2\lambda-3)(\lambda-\omega)(\lambda-2)^{n-2-\omega}\sum_{i=2}^{d} \sum_{1\le j<i:\, x_ix_j\notin E} P(G^x_{ij},\lambda). \end{aligned}$$

Since

$$(\lambda-d)(\lambda-2) = (\lambda-d-1)(\lambda-1)+(d-1),$$

and since

$$(\lambda-2)^2 = (\lambda-1)^2-(2\lambda-3).$$

using for $P(G,\lambda)$ and $P(G,\lambda-1)$ the corresponding expressions given by (10), it is easy to see that

$$f(G,\lambda) = f_1(G,\lambda)+f_2(G,\lambda)+f_3(G,\lambda).$$

Now, (11) implies that $f_1(G,\lambda)\ge 0$ and (13) implies that $f_2(G,\lambda)\ge 0$. Hence, we only need show that $f_3(G,\lambda)\ge 0$. But,

$$\begin{aligned} f_3(G,\lambda) = \;& (\lambda-\omega)(\lambda-2)^{n-2-\omega}[(d-1)(\lambda-2)P(G-x,\lambda)+ \\ & -(2\lambda-3)P(G,\lambda)+(2\lambda-3)(\lambda-d)P(G-x,\lambda)], \end{aligned}$$

and so $f_3(G,\lambda)\ge 0$ if and only if

$$[(d-1)(\lambda-2)+(2\lambda-3)(\lambda-d)]P(G-x,\lambda)-(2\lambda-3)P(G,\lambda)\ge 0.$$

Now, since $(d-1)(\lambda-2)+(2\lambda-3)(\lambda-d) = (\lambda-1)(2\lambda-d-2)$, inequality (3) implies that $f_3(G,\lambda)\ge 0$. Thus the theorem follows. ■

Proof of Theorem 3. Let G be an arbitrary graph with n vertices and let $\lambda\ge n$. To prove Theorem 3, we shall first prove the validity of the upper bound (inequality (5)) and then the validity of the lower bound (inequality (4)).

Write $\omega = \omega(G)$. If G is connected then Theorem 5 implies that inequality (8) holds. But, since

$$\frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega} \leq \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega},$$

we are done.

Hence, we can assume that G is not connected. Let H be the graph obtained from G by adding a vertex x joined to every vertex of G. Property 2 implies that $P(H,k) = kP(G,k-1)$ for every k, and so

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \frac{\lambda+1}{\lambda}\frac{P(H,\lambda)}{P(H,\lambda+1)}.$$

Now, since H is a connected graph with $n+1$ vertices and since $\omega(H) = \omega+1$, Theorem 5 implies that

$$\frac{P(H,\lambda)}{P(H,\lambda+1)} \leq \frac{\lambda-\omega}{\lambda+1}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega},$$

and so (5) holds.

To prove the validity of the lower bound, write $\alpha = \alpha(G)$. Clearly, (4) is true when $n \leq 3$, or when $G = L(G)$. In particular, (4) holds when $G = K_n$ or when $G = O_n$. Hence, we can assume that $n \geq 4$, that G is different from $L(G)$, and that (4) holds for all graphs with at most $n-1$ vertices and for all graphs with n vertices but more edges than G. Let

$$f(G,\lambda) = (\lambda-n+\alpha)(\lambda-n+\alpha-1)^{\alpha}P(G,\lambda) - \lambda(\lambda-n+\alpha)^{\alpha}P(G,\lambda-1).$$

To prove the validity of (4), we only need show that $f(G,\lambda) \leq 0$. In fact, we shall show that $f(G,\lambda) < 0$.

For this purpose, let x be an arbitrary vertex of G such that $\alpha(G-x) = \alpha$ (such a vertex exists since $G \neq O_n$). Since $G \neq L(G)$, there exists a vertex y not adjacent to x such that $\alpha(G+xy) = \alpha$. Write $G' = G+xy$, $G'' = G_{|xy}$, and $\alpha'' = \alpha(G'')$. Since by Property 1, $P(G,k) = P(G',k) + P(G'',k)$ for every k, it follows that

$$\begin{aligned} f(G,\lambda) &= f'(G,\lambda) + f''(G,\lambda) \\ &= f'(G,\lambda) + (\lambda-n+\alpha-1)g(G,\lambda) + \\ &\quad -\lambda(\lambda-n+\alpha)^{\alpha-1}P(G'',\lambda-1), \end{aligned}$$

where

$$f'(G,\lambda) = (\lambda-n+\alpha)(\lambda-n+\alpha-1)^{\alpha}P(G',\lambda) - \lambda(\lambda-n+\alpha)^{\alpha}P(G',\lambda-1),$$

$$f''(G,\lambda) = (\lambda-n+\alpha)(\lambda-n+\alpha-1)^{\alpha}P(G'',\lambda) - \lambda(\lambda-n+\alpha)^{\alpha}P(G'',\lambda-1),$$

$$g(G,\lambda) = (\lambda-n+\alpha)(\lambda-n+\alpha-1)^{\alpha-1}P(G'',\lambda) - \lambda(\lambda-n+\alpha)^{\alpha-1}P(G'',\lambda-1).$$

By assumption, (4) holds for both graphs G' and G'', and so

$$\frac{P(G',\lambda-1)}{P(G',\lambda)} \geq \frac{\lambda-n+\alpha}{\lambda}\left(\frac{\lambda-n+\alpha-1}{\lambda-n+\alpha}\right)^{\alpha} \tag{14}$$

and

$$\frac{P(G'',\lambda-1)}{P(G'',\lambda)} \geq \frac{\lambda-n+\alpha''+1}{\lambda}\left(\frac{\lambda-n+\alpha''}{\lambda-n+\alpha''+1}\right)^{\alpha''}. \tag{15}$$

Now (14) implies that $f'(G,\lambda) \leq 0$. Clearly, either $\alpha'' = \alpha$ or $\alpha'' = \alpha - 1$.
If $\alpha'' = \alpha$, then (15) reads

$$\frac{P(G'',\lambda-1)}{P(G'',\lambda)} \geq \frac{\lambda-n+\alpha+1}{\lambda}\left(\frac{\lambda-n+\alpha}{\lambda-n+\alpha+1}\right)^{\alpha}, \tag{16}$$

and so

$$\frac{P(G'',\lambda-1)}{P(G'',\lambda)} > \frac{\lambda-n+\alpha}{\lambda}\left(\frac{\lambda-n+\alpha-1}{\lambda-n+\alpha}\right)^{\alpha}. \tag{17}$$

But then, (17) implies that $f''(G,\lambda) < 0$, and so $f(G,\lambda) < 0$.
If $\alpha'' = \alpha - 1$, then (15) reads

$$\frac{P(G'',\lambda-1)}{P(G'',\lambda)} \geq \frac{\lambda-n+\alpha}{\lambda}\left(\frac{\lambda-n+\alpha-1}{\lambda-n+\alpha}\right)^{\alpha-1}. \tag{18}$$

But then, (18) implies that $g(G,\lambda) \leq 0$; moreover $\lambda(\lambda-n+\alpha)^{\alpha-1}P(G'',\lambda-1) > 0$ (because G'' has $n-1$ vertices and $\lambda - 1 \geq n-1$), and again $f(G,\lambda) < 0$. The theorem follows. ■

Proof of Theorem 4. Let G be an arbitrary graph with n vertices, and let $\lambda \geq n$. ¿From the proof of Theorem 3 it follows that (4) holds as equation if and only if $G = L(G)$.

Now consider inequality (5). Clearly, if G is connected, then Theorem 5 implies that (5) holds as an inequality.

Hence, we can assume that G is not connected. Let $C_1, \ldots, C_k$ denote the connected components of G and let n_i denote the number of vertices of C_i. By Property 4,

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \prod_{i=1}^{k} \frac{P(C_i,\lambda-1)}{P(C_i,\lambda)}.$$

Write $\omega_i = \omega(C_i)$ and assume that $\omega_1 \geq \ldots \geq \omega_k$. Hence $\omega = \omega_1$.

First, assume that $\omega_2 = 1$. Then $\omega_i = 1$ for every $i > 1$, and so $n_i = 1$ for every $i > 1$. Hence,

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \frac{P(C_1,\lambda-1)}{P(C_1,\lambda)} \left(\frac{\lambda-1}{\lambda}\right)^{k-1}.$$

Since C_1 is a connected graph with $n-k+1$ vertices, Theorem 5 implies that

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\omega}{\lambda} \left(\frac{\lambda-2}{\lambda-1}\right)^{n-k+1-\omega} \left(\frac{\lambda-1}{\lambda}\right)^{k-1},$$

and so

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\omega}{\lambda} \left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega} \left(\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\right)^{k-1}. \tag{19}$$

If the theorem were false, then there would exist a non-connected graph $G \neq U(G)$ such that

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \frac{\lambda-\omega}{\lambda} \left(\frac{\lambda-1}{\lambda}\right)^{n-\omega}.$$

But then (19) would imply that

$$\left(\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\right)^{n-\omega} \leq \left(\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\right)^{k-1},$$

and so $n-\omega \leq k-1$. Hence, $n_1 = n-k+1 \leq \omega$, that is $n_1 = \omega$, contradicting the assumption that $G \neq U(G)$.

Next, assume that $\omega_2 \geq 2$. Since each graph C_i is connected, Theorem 5 implies that

$$\begin{aligned}\frac{P(G,\lambda-1)}{P(G,\lambda)} &\leq \prod_{i=1}^{k} \frac{\lambda-\omega_i}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n_i-\omega_i} \\ &= \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega}\left(\frac{\lambda-1}{\lambda-2}\right)^{\omega_2+\ldots+\omega_k} \prod_{i=2}^{k}\frac{\lambda-\omega_i}{\lambda}.\end{aligned}$$

Since $\omega_2 \geq 2$, inequality (6) implies that

$$\frac{\lambda-\omega_2}{\lambda} < \left(\frac{\lambda-1}{\lambda}\right)^{\omega_2}$$

and that, for every $i > 2$,

$$\frac{\lambda-\omega_i}{\lambda} \leq \left(\frac{\lambda-1}{\lambda}\right)^{\omega_i}.$$

Hence,

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} < \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\omega}\left(\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\right)^{\omega_2+\cdots+\omega_k}.$$

But, since $n \geq \omega+\omega_2+\cdots+\omega_k$, it follows that $\omega_2+\cdots+\omega_k \leq n-\omega$, and so

$$\left(\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\right)^{\omega_2+\cdots+\omega_k} \leq \left(\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\right)^{n-\omega}.$$

It follows that

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} < \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega}.$$

Thus, the theorem follows. ■

4 Remarks

Our proof of Theorem 5 uses the decomposition of $P(G,\lambda)$ given in Property 3 and inequality (3) in Theorem 2. Theorem 5 is then used to prove the validity of the upper bound in Theorem 3.

One way to prove differently the upper bound in Theorem 3 could be the following. Assume that $G \neq K_n$ and let x be an arbitrary vertex of G such that $\omega(G-x) = \omega(G)$. Since Theorem 3 holds for n small, by induction, we can assume that inequality (5) holds for $G - x$, i.e.

$$\frac{P(G-x,\lambda-1)}{P(G-x,\lambda)} \leq \frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-1-\omega(G)}.$$

Hence, Theorem 3 would follow immediately if the following conjecture were true:

Conjecture 1 *Let G be a graph with n vertices such that $G \neq K_n$. Then there exists a vertex x such that, the following two properties hold:*

(i) $\omega(G-x) = \omega(G)$;

(ii) for every $\lambda \geq n$, $\quad \dfrac{P(G,\lambda-1)}{P(G,\lambda)} \leq \dfrac{\lambda-1}{\lambda} \quad \dfrac{P(G-x,\lambda-1)}{P(G-x,\lambda)}.$

Note that if we remove property (i) in Conjecture 1, then x could be adjacent to all other vertices of G. In this case, Property 2 would imply that

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} = \frac{\lambda-1}{\lambda} \quad \frac{P(G-x,\lambda-2)}{P(G-x,\lambda-1)}.$$

Hence, if we write $H = G - x$, and we set $k = \lambda - 1$, then Conjecture 1 would be equivalent to the following conjecture:

$$[P(H,k)]^2 \geq P(H,k-1)P(H,k+1) \quad \text{for every } k \geq n-1. \tag{20}$$

Note that, if in (20) we replace “for every $k \geq n-1$” with “for every k”, then we obtain the famous “log-concavity” conjecture (see [2] and [3]). This conjecture was disproved by Seymour in [5]. However, in the counterexample given in [5], $k \ll n-1$.

We close this paper with a corollary of Theorem 3. For this purpose, note that, by Theorem 3,

$$\frac{\alpha(G)}{n}\left(\frac{\alpha(G)-1}{\alpha(G)}\right)^{\alpha(G)} \leq \frac{P(G,n-1)}{P(G,n)} \leq \frac{n-\omega(G)}{n}\left(\frac{n-1}{n}\right)^{n-\omega(G)}.$$

Hence, Theorem 3 yields new bounds on the mean colour number of G.

Corollary 1 *For every graph G with n vertices,*

$$n-(n-\omega(G))\left(\frac{n-1}{n}\right)^{n-\omega(G)} \leq \mu(G) \leq n-\alpha(G)\left(\frac{\alpha(G)-1}{\alpha(G)}\right)^{\alpha(G)}.$$

It is natural to ask wether our lower bound on the mean colour number of a graph is stronger than the lower bound given by its chromatic number. Clearly, when $\omega(G)=\chi(G)$ the answer is yes. In fact, for almost all graphs, our lower bound is much bigger than $\chi(G)$.

Theorem 6 *For almost all graphs G with n vertices,*

$$n-(n-\omega(G))\left(\frac{n-1}{n}\right)^{n-\omega(G)} > \chi(G).$$

Proof For almost all graphs G with n vertices (see [1], p.148),

$$\omega(G) \sim 2\log_2 n \quad \text{and} \quad \chi(G) \sim \frac{n}{2\log_2 n}.$$

If we let

$$f(n)=\left(\frac{n-1}{n}\right)^{n-2\log_2 n},$$

then

$$n-(n-\omega(G))\left(\frac{n-1}{n}\right)^{n-\omega(G)} \sim n-(n-2\log_2 n)f(n).$$

Since the limit, as n goes to infinity, of $f(n)$ is $e^{-1}=.367\ldots$, it follows that, for n large enough, $f(n)<0.5$. Hence, for almost all graphs G with n vertices, with n large enough,

$$n-(n-2\log_2 n)f(n) > n-\frac{n-2\log_2 n}{2}=\frac{n}{2}+\log_2 n > \frac{n}{2\log_2 n} \sim \chi(G).$$

■

For dense graphs G, our lower bound on $\mu(G)$ can be smaller than $\chi(G)$. To see this, consider the graph H constructed by Erdős (see, [1], p. 35). This graph has $n/2$ vertices, it has no triangles, and $\alpha(H)<n/18$ for n

large enough. The complement $\bar{H}$ of H has $\alpha(\bar{H}) = 2$, and so $\chi(\bar{H}) \geq n/4$. Let G be the graph obtained by the complete union of $\bar{H}$ and $K_{n/2}$. Note that G is a graph with n vertices, such that

$$\frac{n}{2} < \omega(G) < \frac{5}{9}n, \quad \text{and} \quad \chi(G) \geq \frac{3}{4}n.$$

By adding edges, we can make $\omega(G) = \lfloor 5n/9 \rfloor$ (without decreasing $\chi(G)$). Now, if we take n large enough and divisible by 9, we have

$$n - (n - \omega(G)) \left(\frac{n-1}{n}\right)^{n-\omega(G)} = n \left(1 - \frac{4}{9}\left(\frac{n-1}{n}\right)^{\frac{4}{9}n}\right).$$

But

$$n \left(1 - \frac{4}{9}\left(\frac{n-1}{n}\right)^{\frac{4}{9}n}\right) \leq n \left(1 - \frac{39}{90}e^{-\frac{4}{9}}\right) \leq .73n < \chi(G).$$

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