Solitaire Cones

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Abstract

The classical game of Peg Solitaire has uncertain origins, but was certainly popular by the time of LOUIS XIV, and was described by LEIBNIZ in 1710. The modern mathematical study of the game dates to the 1960s, when the solitaire cone was first described by BOARDMAN and CONWAY. Valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. In this paper we study the extremal structure of solitaire cones for a variety of boards, and relate their structure to the well studied metric cone. In particular we give:

1. an equivalence between the multicommodity flow problem with associated dual metric cone and a generalized peg game with associated solitaire cone;
2. a related NP-completeness result;
3. a method of generating large classes of facets;
4. a complete characterization of 0-1 facets;
5. exponential upper and lower bounds (in the dimension) on the number of facets;
6. results on the number of facets, incidence, adjacency and diameter for small rectangular, toric and triangular boards;
7. a complete characterization of the adjacency of extreme rays, diameter, number of 2-faces and edge connectivity for rectangular toric boards;
8. results on the boolean relaxation of solitaire cones.

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Contents

1 Introduction and Basic Properties **3**
- 1.1 Introduction . . . 3
- 1.2 Basic properties . . . 4

2 Relationship with Multicommodity Flows and the Metric Cone **9**
- 2.1 Metric cone and multicommodity flows . . . 9
- 2.2 Multicomomodity flows versus peg game . . . 11

3 Facets of the Solitaire Cone **14**
- 3.1 Rectangular boards . . . 14
- 3.2 Toric boards . . . 18
- 3.3 Bounds on number of facets . . . 21

4 Skeletons and Diameters of Solitaire Cones **23**
- 4.1 Small dimensional solitaire cones . . . 23
- 4.2 Skeletons and diameters . . . 24

5 Related Polyhedra **29**
- 5.1 The boolean solitaire cone . . . 29
- 5.2 The trellis solitaire cone . . . 30
- 5.3 The complete solitaire cone . . . 31
- 5.4 The boolean complete solitaire cone . . . 31

6 Conclusions **32**

7 Appendix **34**
- 7.1 Proof of Lemma 4.1 and Theorem 4.3 . . . 34
  - 7.1.1 Proof of Lemma 4.1 . . . 34
  - 7.1.2 Proof of Theorem 4.3 . . . 35
- 7.2 Proof of Theorem 5.1 . . . 38
  - 7.2.1 Case $m \leq 2$ . . . 38
  - 7.2.2 Case $m = 3$ . . . 38
  - 7.2.3 Case $m \geq 4$ . . . 39

1 Introduction and Basic Properties

1.1 Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes, see Fig. 1.1, although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row and column two consecutive pegs are adjacent to an empty hole, we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of Beasley [7]. The book mentions an engraving of Berey, dated 1697, of a lady with a Solitaire board. The book also contains a quotation of Leibniz [18] which was written for the Berlin Academy in 1710. Although some serious study of the game is reported in the 19th century, the modern mathematical study of the game dates to the 1960s at Cambridge University. The group was lead by Conway who has written a chapter in [8] on various mathematical aspects of the subject.

One of the problems studied by the Cambridge group is the following basic *feasibility* problem of peg solitaire:

> For a given board B, starting configuration S and finishing configuration F, determine if there is a legal sequence of moves from S to F.

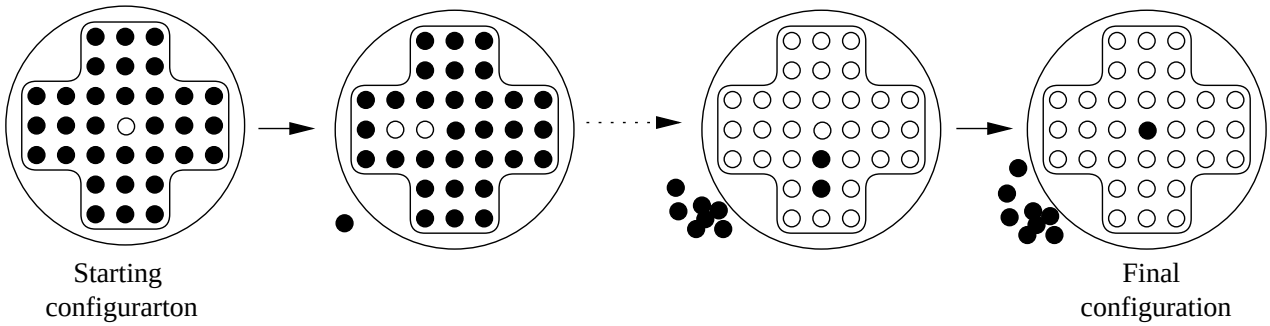


Figure 1.1: A feasible English solitaire peg game with possible first and last moves

The complexity of the feasibility problem for the n by n game was shown by Uehara and Iwata [23] to be NP-complete, so easily checked necessary and sufficient conditions

for feasibility are unlikely to exist. One of the earliest tools used to show the infeasibility of certain starting and finishing configurations is a polyhedral cone, which we will call the *solitaire cone*, S_B, corresponding to some given board B. This paper contains results on the extremal structure of this cone, which we describe in the next subsection.

1.2 Basic properties

For ease of notation, we will mostly be concerned with rectangular boards which we represent by 0-1 matrices. A zero represents an empty hole and a one represents a peg. For example, let S =[1 0 1 1] and F =[0 0 1 0] be starting and finishing positions for the 1 by 4 board. This game is *feasible*, involving two moves and the intermediate position [1 1 0 0] - as showned in Fig. 1.2.

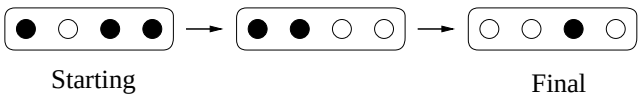
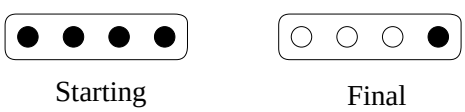


Figure 1.2: A feasible game on the 1 by 4 board

For any move on an m by n board B we can define an m by n *move matrix* which has 3 non-zero entries: two entries of -1 in the positions from which pegs are removed and one entry of 1 for the hole receiving the new peg. The two moves involved in Fig. 1.2 are represented by m_1 =[0 1 -1 -1] and m_2 =[-1 -1 1 0]. Clearly $F = S + m_1 + m_2$. By abuse of language, we use the term *move* for both the move itself and the move matrix. In general it is easily seen that if S, F define a feasible game of k moves there exist move matrices $m_1, \ldots, m_k$ such that

$$F - S = \sum_{i=1}^{k} m_i. \tag{1.1}$$

Lemma 1.1 *Equation 1.1 is necessary but not sufficient for the feasibility of a peg game. See, for example, the following game where S =[1 1 1 1] and F =[0 0 0 1]. We have F − S = [-1 -1 1 0] + [0 1 -1 -1] + [0 -1 -1 1], but S, F do not define a feasible game, in fact there are no legal moves!*



An infeasible game on the 1 by 4 board

Let us relax the conditions of the original peg game to allow any integer (positive or negative) number of pegs to occupy any hole. We call this game the *integer game*, and call the original game the *0-1 game*. Note that in a 0-1 game we require that in every position of the game a hole is either empty or contains a single peg.

Lemma 1.2 *Equation 1.1 is necessary and sufficient for the feasibility of the integer game. In other words, all feasible integer games form the integer cone IS_B defined as the set of all integer non-negatives combinations of moves.*

An integer feasible but 0-1 infeasible game

Unfortunately deciding if $F - S$ can be expressed as the sum of move matrices seems to be a hard computational problem. In Section 2 we show that a variant of the game is NP-complete. A further relaxation of the game leads to a more tractable condition. In the *fractional game* we allow any hole to contain a fractional (positive or negative) number of pegs. A *fractional move matrix* is obtained by multiplying a move matrix by any positive scalar. For example, let $S =$[1 1 1], $F =$[1 0 1]. Then $F - S =$[0 -1 0] $= \frac{1}{2}$ [-1 -1 1] + $\frac{1}{2}$ [1 -1 -1] is feasible as a fractional game by two fractional moves, but is not feasible as a 0-1 or integer game.

A fractional feasible but integer (and therefore 0-1) infeasible game

Let B be a board and n_B the total number of possible moves on the board. The *solitaire cone* S_B is the set of all non-negative combinations of the n_B corresponding move matrices, that is, $F - S \in S_B$ if:

$$F - S = \sum_{i=1}^{n_B} y_i m_i, \;\; y_i \geq 0, \;\; i = 1, \ldots, n_B. \tag{1.2}$$

In the above definition it is assumed that the h_B holes in the board B are ordered in some way and that z, y_i, m_i are all h_B-vectors. When B is a rectangular m by n board it is convenient to display z, y_i, m_i as m by n matrices, although of course all products should be interpreted as dot products of the corresponding mn-vectors.

Lemma 1.3 *For $n_B > 3$, the solitaire cone is full dimensional, that is, the dimension of S_B equals h_B the number of holes of the board B.*

PROOF. Its dual S_B^* contains the $[1, 2]^{h_B}$ cube. □

The following result obtained in 1961 is credited to BOARDMAN (who apparently has not published anything on the subject) by BEASLEY [7], page 87.

Theorem 1.4 *Equation 1.2, that is, $F - S \in S_B$, is necessary and sufficient for the feasibility of the fractional game. In other words, the solitaire cone S_B is the cone of all feasible fractional games.*

Lemma 1.1 and 1.2 and Theorem 1.4 are illustrated in Fig. 1.3

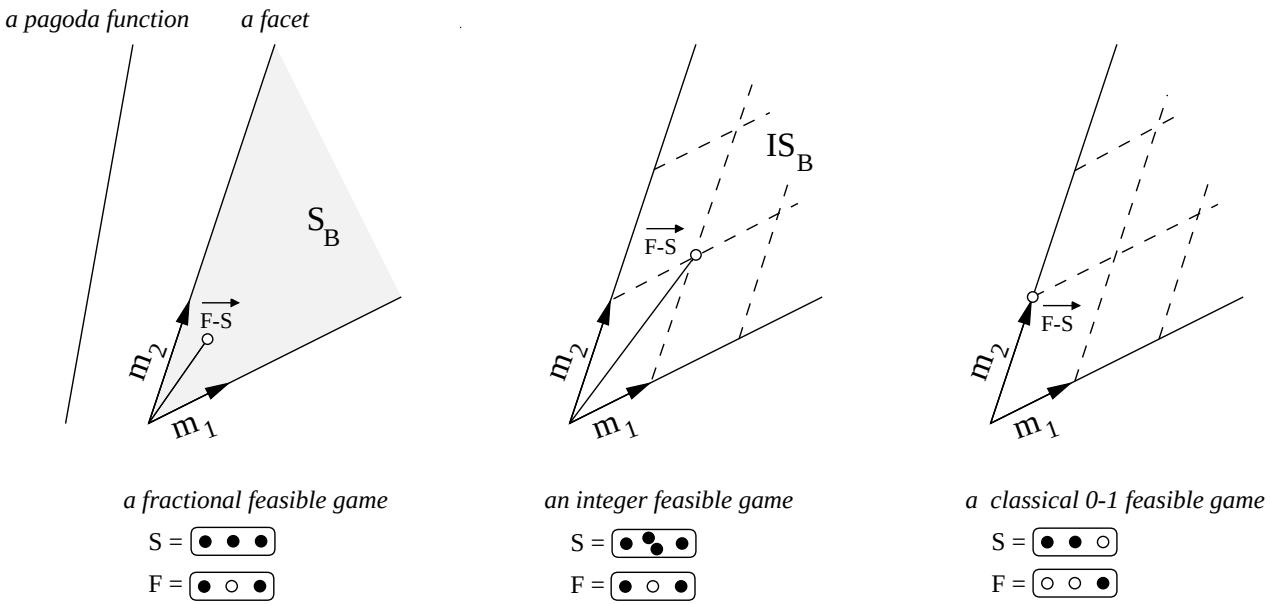
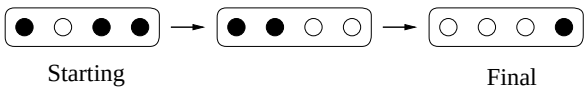


Figure 1.3: Respectively fractional, integer and 0-1 feasible solitaire peg games

The condition $F - S \in S_B$ is therefore a necessary condition for the feasibility of the original peg game and, more usefully, as a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for S_B which is violated by $F - S$. According to [7], page 71, these inequalities "were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway...". They are also known as *resource counts*, and are discussed in some detail in CONWAY [8]. The strongest such inequalities are induced by the facets of S_B. For example, in Section 3 we will give a facet (see Fig. 3.8 (i)) which induces an inequality that is violated for $F - S$ with S,F given in Fig. 1.4. This implies that this game is not feasible even as a fractional game and, therefore, not feasible as an integer game or classical 0-1 game either.

Remark 1.5 *In later sections we will consider the toric closure of a board B which is more symmetric. These boards allow moves that traverse the boundary of the board. For example, the following is game is feasible on a toric closure of the 1 by 4 board.*



A feasible classical game on the toric closure of the 1 by 4 board

Finding all facets of the cone S_B is an example of a *convex hull* or *vertex enumeration* problem, for which various computer programs are available. The results in this paper were obtained using the double description method *cdd* implemented by Fukuda [13], and the reverse search method *lrs* implemented by Avis [3]. We made use of these codes to completely generate all facets for some small boards as reported in later sections (such as the 930 048 facets for the toric closure of the 4 by 4 board). For realistically sized boards the corresponding solitaire cones are too large for these programs. For example the original peg solitaire game gives rise to a cone with 77 extreme rays in 33 dimensions. The reverse search vertex enumeration code *lrs* applied to this cone generated over 300,000 facets from about 6 million bases of the cone that were generated before the program was terminated. An unbiased estimate obtained from *lrs* suggests that the solitaire cone has about 9.2 million facets and 12 billion bases. If correct, this estimate puts the size of the cone just beyond problems currently solvable by a parallel version of *lrs*.

If we dualize the solitaire cone, we obtain a cone whose facets are defined by triangle inequalities. These inequalities, combined in a very different way, also define the well studied metric cone which we define in the next section. One of the motivations of our work was to explore the relationship between the solitaire and metric cones. We give some new results on facets of S_B for various boards in Section 3. We present an algorithm for generating a large class of facets, and which generates all 0-1 facets. These facets are considerably more

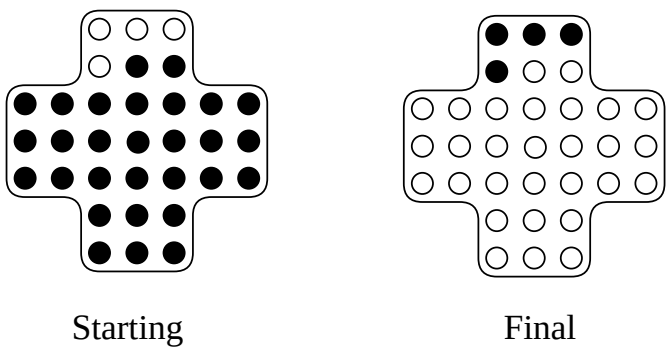


Figure 1.4: An infeasible classical solitaire peg game

complex than the 0-1 facets of the dual of the metric cone, which are generated by cuts in the complete graph.

In Section 4 we give some results and conjectures on the combinatorial and geometric properties of the solitaire cone. We investigate the diameter, edge connectivity, adjacency and incidence relations of the solitaire cone and its dual. In Section 5 we explore the relaxation of the solitaire cone formed by considering only its 0-1 valued facets, and some other related polyhedra. Finally in Section 6 we conclude by comparing and contrasting the solitaire and metric cones. The Appendix 7 contains the proofs of Lemma 4.1 and Theorems 4.3 and 5.1.

2 Relationship with Multicommodity Flows and the Metric Cone

The solitaire cone is generated by a set of extreme rays, each of which is all zero except for three non-zero components which are 1,-1,-1. In this section we relate the solitaire cone to another cone with the same property, the *flow cone* which is dual to the *metric cone*. This cone arises in the study of multicommodity flows (which we abbreviate to multiflows), and we will show similarities between this problem and the peg game. For more details on the relationship between the metric cone and multicommodity flows, see the survey paper of Avis and Deza [4]. We give a brief sketch here.

2.1 Metric cone and multicommodity flows

Let K_n denote the complete graph on n vertices. For each edge ij we define a non-negative integer *capacity* c_{ij} and *demand* d_{ij}. Let c and d denote the respective capacity and demand vectors of length $\binom{n}{2}$. For simplicity in indexing, with vectors of length $\binom{n}{2}$ we identify index ij with index ji. Let $1 \leq r \neq s \neq t \leq n$ be three distinct indices. A *flow* on the triple (r, s, t) is a vector

$$f_{ij}^{rst} = \begin{cases} 1 & i = r, j = s \\ -1 & i = r, j = t \\ -1 & i = s, j = t \\ 0 & \text{otherwise.} \end{cases}$$

The *integral multiflow problem* is to try to find an integral combination of flows, z, such that $z \geq d - c$. If this is possible we say that c, d are *integer feasible*. Such a vector z defines a set of d_{ij} paths between each pair of vertices i and j in K_n. No edge ij in K_n can appear in more than c_{ij} paths. If $z = d - c$, we say the multiflow is *saturated*. This corresponds to the case where each edge ij is contained in exactly c_{ij} paths defined by z. In the example shown in Fig. 2.5, we have $n = 4$, $d =$[3 0 0 0 0 3] and $c =$[0 5 3 4 1 0]. Then $d - c \leq 3f^{123} + 2f^{341} + f^{342} = z$. The problem is integer feasible and the multiflow is unsaturated as there is a residual capacity of 1 on arc $1, 3$.

Even, Itai and Shamir (see Garey and Johnson [14] p. 217) proved the following result showing that integer feasibility is an intractable problem in general.

Theorem 2.1 *It is an NP-complete to decide if c, d are integer feasible for the multiflow problem, even if c is a 0-1 vector and d has only two non-zero components.*

The *fractional multiflow problem* is to try to express $d - c$ as a non-negative combination of flows. If this is possible we say that c, d is *fractionally feasible* For a fractionally feasible problem it is always possible to find a saturated multiflow, since any excess capacity on edge rs can be used by combining flows f^{rst} and f^{srt} (with possibly fractional multipliers). For

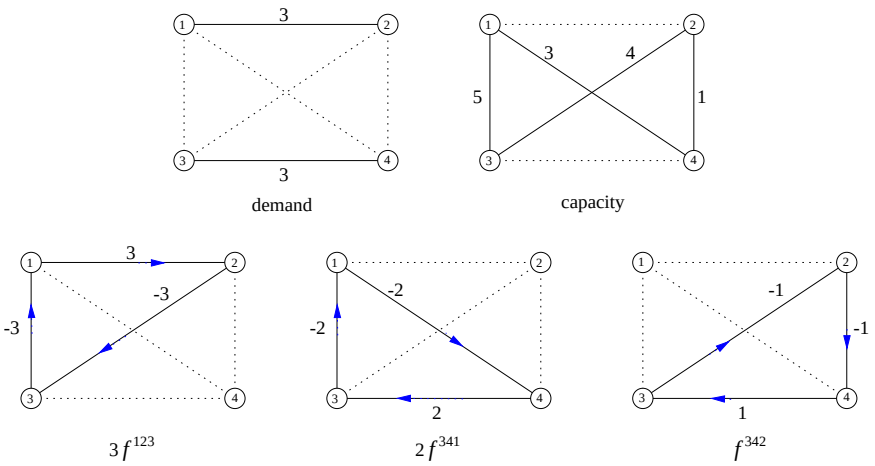


Figure 2.5: An integer feasible multicommodity flow problem

example, set $n = 5$, $d =$[1 1 0 0 1 0 0 0 0 0] and $c =$[0 0 1 1 0 1 1 1 1 0]. This problem has the fractional solution

$$d - c = \frac{1}{2}(f^{124} + f^{125} + f^{134} + f^{135} + f^{234} + f^{235})$$

but no integral solution. Fractional multiflows lead to the study of the *flow cone* which is the set of all non-negative combinations of the $3\binom{n}{3}$ possible flow vectors:

$$F_n = \{z : z = \sum_{r,s,t} y_{rst} f^{rst} \text{ with } y_{rst} \geq 0 \text{ and } 1 \leq r < s < t \leq n\}. \tag{2.3}$$

F_3 is illustrated in Fig. 2.6. The dual of the flow cone is the much studied *metric cone* (see for example, Avis [1], Deza, Deza and Fukuda [11] and Lomonosov [19]):

$$M_n = \{z : z_{rs} \leq z_{rt} + z_{st} \text{ with } 1 \leq r < s < t \leq n\}.$$

The *Japanese theorem* of Iri and Onaga and Kakusho gives a necessary and sufficient condition for the feasibility of the fractional multicommodity flow problem.

Theorem 2.2 *[17, 20] A multiflow problem c, d is fractionally feasible if and only if d–$c \in F_n$.*

Facets of the flow cone are useful to show the infeasibility of multiflow problems. In the previous example, if we change the demand vector to $d =$[1 1 0 0 1 0 0 0 0 1], the problem becomes infeasible. This is demonstrated by the facet $a =$[2 2 1 1 2 1 1 1 1 2]. For each flow f^{ijk}, $af^{ijk} \leq 0$, but $a(d - c) = 2 > 0$ so c, d is infeasible.

The reader should compare this description of multiflows with the development of the solitaire cone in Section 1, and in particular equation 1.2 with 2.3, and Theorem 1.4 with

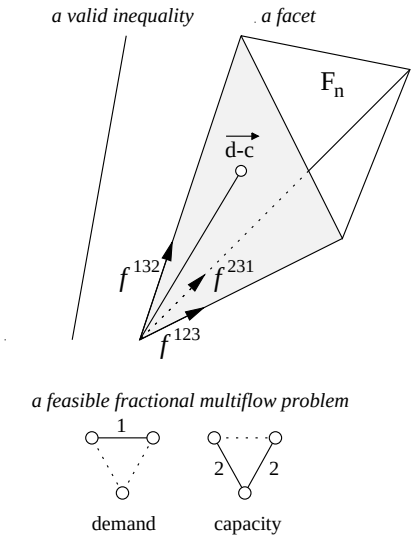


Figure 2.6: All fractionally feasible multicommodity flows problem form the flow cone F_n

Theorem 2.2. There are two obvious differences in the two problems. The first is that in the flow cone we consider all triples from a set of n elements, whereas in the solitaire cone the triples are constrained to be consecutive horizontal or vertical entries in a rectangular array. The second difference is that in the flow cone for a given triple r, s, t we include all three orderings of the non-negative entries 1,-1,-1. In the solitaire cone we only allow the orderings 1,-1,-1 and -1,-1,1. The third ordering, 1,-1,1 , could be considered as an additional move in the peg game where two pegs surrounding an empty hole are replaced by a peg in the empty hole. With this additional move, we can in fact show a strong correspondence between the peg game and multicommodity flows.

2.2 Multicomomodity flows versus peg game

Given a graph $G = (V, E)$, the *line graph* of G is a graph whose vertices correspond to edges in G. Two vertices are adjacent in the line graph if the corresponding edges share an endpoint in G. Let L_n denote the line graph of K_n. For given vectors c and d, we will construct a *generalized peg game* on $B = L_n$. Each vertex of L_n represents a hole of B. The moves on B derive from triangles in K_n. Let i, j, k be three vertices in G. Then the vertices ij, ik, jk in L_n form a *valid triangle* and a *valid move* on the board B. We allow all three possible moves on the triangle: for the integer game this means that the number of pegs on any two of the holes is decremented by one, and the number of pegs on the remaining the hole is incremented by one. We assign initially c_{ij} pegs to vertex ij of L_n. This is the starting configuration S. It is required to reach reach a finishing position with at least d_{ij} pegs at each vertex ij, using valid (respectively valid fractional) moves. We illustrate in Fig. 2.7 the correspondance for

the problem presented in Fig. 2.5.

Theorem 2.3 *A multiflow problem c, d on K_n is integer (respectively fractional) feasible if and only if the corresponding generalized peg game on L_n is integer (respectively fractional) feasible. Therefore, the dual metric cone M_n^* equals the complete solitaire cone CS_{L_n} for a game played on the line graph of the complete graph on n nodes.*

PROOF. First we show that an integer (respectively fractional) multiflow for c, d corresponds to an integral (respectively fractional) solution of the peg game. Suppose z is an integral combination of flows. Each of the flows is in one to one correspondence with a move on a valid triangle of L_n. Since $d \geq c + z$, in the peg game we end up with at least d_{ij} pegs at each vertex ij. Note if the multiflow is saturated, we end up with exactly d_{ij} pegs at each vertex ij. In the same way a solution to the peg game gives the required vector z and its decomposition into flows. The same argument applies to the fractional game. The converse direction is proved in an analogous way. □

Together with Theorem 2.1 we get immediately the following corollary.

Corollary 2.4 *The integer generalized peg game is NP-complete even if in the starting position all holes contain at most one peg and the finishing position has positive requirements for exactly two holes.*

The integer generalized peg game can be solved in polynomial time if only one hole has positive finishing requirement. This is because in this case the corresponding multiflow problem is a single commodity multiflow for which many polynomial time algorithms exist. It is interesting that the 0-1 game is NP-complete on the n by n board, even if the final position contains exactly one peg, see UEHARA AND IWATA [23]

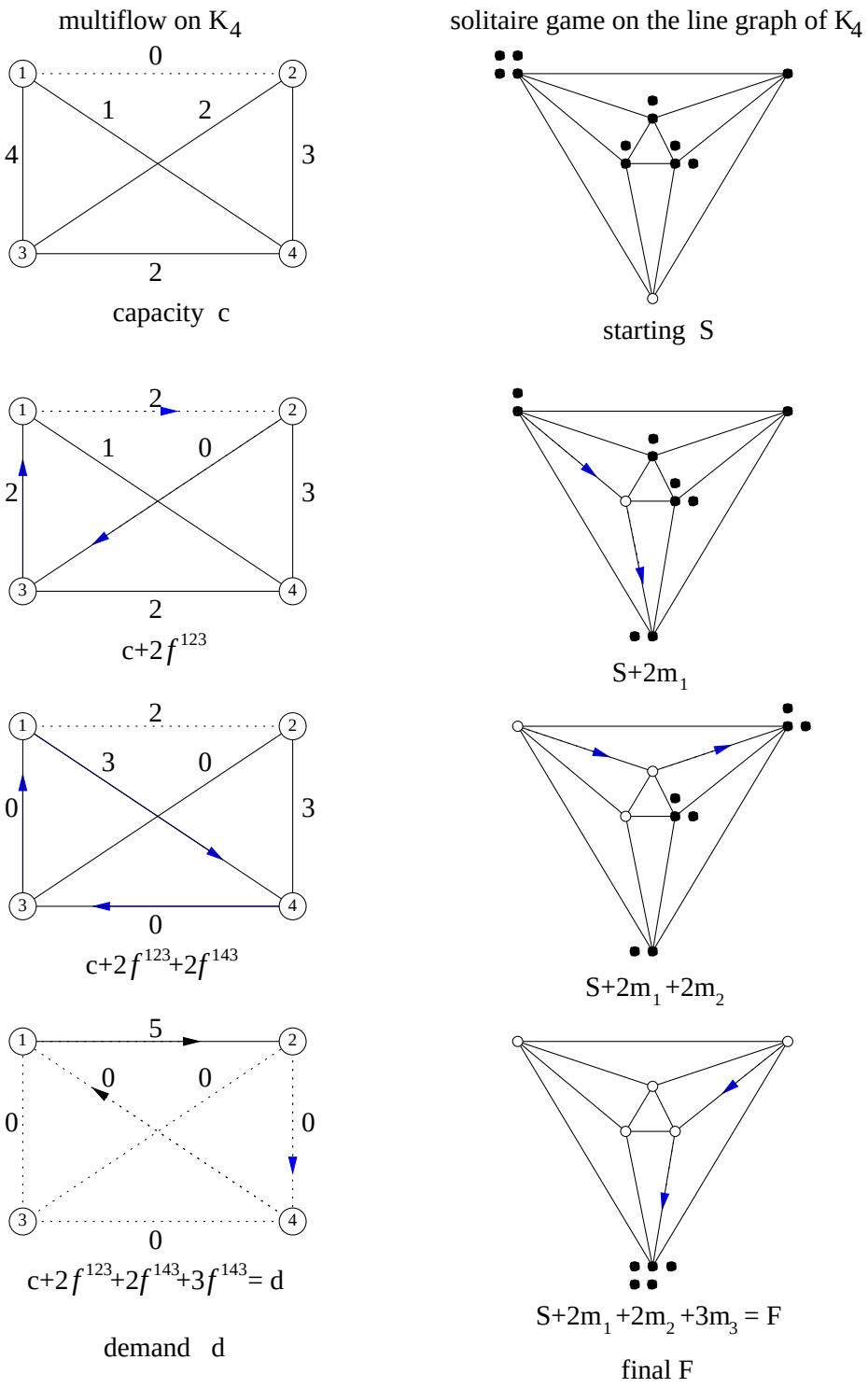


Figure 2.7: Correspondence of multicommodity flows and solitaire peg game

3 Facets of the Solitaire Cone

For simplicity we begin with rectangular boards. Most of the results can be applied to boards which are subsets of the square lattice in the plane, such as the original Peg solitaire board. We also give some results for their toric closures, which are simpler since they avoid many special situations caused by the boundary.

3.1 Rectangular boards

Let B be a rectangular m by n board, with $m \geq 1$ and $n \geq 3$. Using the notation described following equation 1.2, we will represent the coefficients of the facet inducing inequality

$$az \leq 0 \tag{3.4}$$

by the m by n array $a = [a_{i,j}]$. Inequality 3.4 holds for every $z \in S_B$. It is a convenient abuse of terminology to refer to a as a *facet* of S_B. A *corner* of a is a coefficient $a_{i,j}$ with $i \in \{1, m\}$ and $j \in \{1, n\}$.

We will frequently need to refer to three consecutive row or column elements of an m by n array. For this we use the notation $\mathcal{T} = (t_1, t_2, t_3)$ to refer to a *consecutive triple* of row or column indices. For example both $t_1 = i, j$, $t_2 = i, j+1$, $t_3 = i, j+2$ and $t_1 = i+2, j$, $t_2 = i+1, j$, $t_3 = i, j$ are consecutive triples. Using this notation we see that a move matrix for B is an m by n matrix that is all zero except for elements of some consecutive triple which take the values 1,-1,-1. Each consecutive triple defines a *triangle inequality*

$$a_{t_1} \leq a_{t_2} + a_{t_3} \tag{3.5}$$

A triangle inequality is *tight* if equality holds

$$a_{t_1} = a_{t_2} + a_{t_3} \tag{3.6}$$

The following theorem summarizes known results on properties of valid inequalities (pagoda functions) for S_B, see Beasley [7]. We include a proof for completeness.

Theorem 3.1 *For each valid inequality $a = [a_{i,j}]$ for S_B*

1. *The triangle inequality 3.5 must hold for every consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$.*
2. *Negative coefficients of a can only occur in corners.*
3. *If $\mathcal{T} = (t_1, t_2, t_3)$ is a consecutive triple with $a_{t_2} = 0$ then $a_{t_1} = a_{t_3}$.*
4. *If two consecutive row (respectively, column) entries of a are zero the entire row (respectively, column) is zero.*

PROOF.

1. Every move matrix is trivially in S_B so inequality 3.4 with z equal to the move matrix corresponding to $\mathcal{T}$ gives the triangle inequality 3.5.

2. Let a_{t_2} be a coefficient of a that is not a corner. It must lie in two consecutive triples (t_1, t_2, t_3) and (t_3, t_2, t_1). The two triangle inequalities for these triples imply that $a_{t_2} \geq 0$.

3. Again this follows from the two triangle inequalities induced by the triples (t_1, t_2, t_3) and (t_3, t_2, t_1).

4. Follows by repeatedly applying part 3 of the theorem.

□

As mentioned in the introduction, it is not feasible to generate all facets for reasonably sized boards, and in general no characterization of facets is known. A large class of facets can, however, be generated by the following procedure.

GENFACET(B) /*procedure to generate a facet matrix a of S_B */

1. Set a subset of coefficients of a to zero. If two consecutive row or column elements are zero the entire row or column is set to zero. If a corner is zero, a row and/or column of length at least 4 containing it must be set to zero.

2. Choose any undefined coefficient that is not a corner and set it to one.

3. Choose a consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ for which precisely two of the corresponding coefficients of a are defined. Define the remaining coefficient by equation 3.6 providing this does not violate any triangle inequality for a.

4. Repeat step 3 until no further coefficient of a can be defined.

Theorem 3.2 *If GENFACET(B) terminates with all elements of a defined, then a is a facet of S_B.*

PROOF. Let a be a matrix generated by GENFACET and let b be any other m by n matrix that is valid for S_B such that for every consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$,

$$a_{t_1} = a_{t_2} + a_{t_3} \Leftrightarrow b_{t_1} = b_{t_2} + b_{t_3}. \tag{3.7}$$

We will show that b is a positive scalar multiple of a, proving that a is facet inducing. By construction, a satisfies all triangle inequalities and so is valid for S_B, so we may apply results from Theorem 3.1.

First we show that for each $a_{i,j} = 0$ that is set in step 1 of GENFACET we must also have $b_{i,j} = 0$. Initially assume that i, j is not a corner index, so there is some consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ with $t_2 = i, j$. Suppose $a_{t_2} = 0$. It follows from Theorem 3.1 (3) that $a_{t_1} = a_{t_3}$. Applying relation 3.7 twice we obtain the equations

$$b_{t_1} = b_{t_2} + b_{t_3} \qquad\qquad b_{t_3} = b_{t_1} + b_{t_2}$$

and conclude that $b_{t_2} = 0$. An identical argument shows that $b_{i,j} = 0$ implies that $a_{i,j} = 0$. If i, j is a corner with $a_{i,j} = 0$, by construction of a there must be a consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ with $t_1 = i, j$, $a_{t_2} = a_{t_3} = 0$, and such that a_{t_3} is not a corner coefficient. As we showed, this implies $b_{t_2} = b_{t_3} = 0$ and so applying Theorem 3.1 (4) to b we have $b_{t_1} = 0$.

The argument in the previous paragraph implies that all of the entries set to zero in step 1 of GENFACET must be also zero in b. Suppose $a_{i,j}$ is the coefficient set to one in step 2. Since it is not a corner element, Theorem 3.1 (2) implies that $b_{i,j} \geq 0$, and in fact it must be strictly positive since $a_{i,j}$ is non-zero. we may now scale b so that in fact $b_{i,j} = 1$.

We complete the proof with a simple inductive argument to show that at the end of each execution of step 3 of GENFACET, if $a_{i,j}$ has been assigned then $b_{i,j} = a_{i,j}$. Indeed suppose this statement is true when step 3 is about to be executed, and suppose some $a_{i,j}$ is defined by the equation

$$a_{t_1} = a_{t_2} + a_{t_3}.$$

One of the coefficients in this equation is $a_{i,j}$ and the other two coefficients have already been assigned. By induction, these two coefficients are equal to the corresponding coefficients of b. Now by condition 3.7 the equation

$$b_{t_1} = b_{t_2} + b_{t_3}$$

holds for b and so $b_{i,j} = a_{i,j}$. This completes the induction. At termination of GENFACET, all coefficients of a were assigned, so we have shown that b is equal to a up to a positive scalar. □

Remark 3.3 *If the condition in step 1 of GENFACET on corner coefficients is dropped, non-facets may be produced. For example, the matrices*

$$a^{(1)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad\qquad b^{(1)} = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

have the same tight triangle inequalities. Since one is not a scalar multiple of the other, they are not facets. However $a^{(1)}$ would be generated by GENFACET if the condition on a zero corner coefficient being in a row and/or column of zeroes was not included. Such a row

and/or column must have length 4, as shown by the examples below which are not facets.

$$a^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad\qquad b^{(2)} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

It is not known whether this condition is in fact necessary for rectangular boards.

To avoid the special effects created by the boundary of the rectangular board, we were motivated to study their toric closures. Some results on these are given in the Section 3.2 where we simply call them *toric boards*.

Remark 3.4

1. *Not all facets of rectangular boards are generated by GENFACET. For example the following facet (found by computer) of the 3 by 4 rectangular board cannot be generated by GENFACET.*

$$a^{(3)} = \begin{pmatrix} -1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

2. *GENFACET can easily be adapted to non-rectangular boards that are connected subsets of the square grid, such as the original peg solitaire game. The notion of corner generalizes in the obvious way to all holes that have exactly one horizontal and vertical neighbour. For example, the original English game has 8 corners and by GENFACET we can generate the following two facets:*

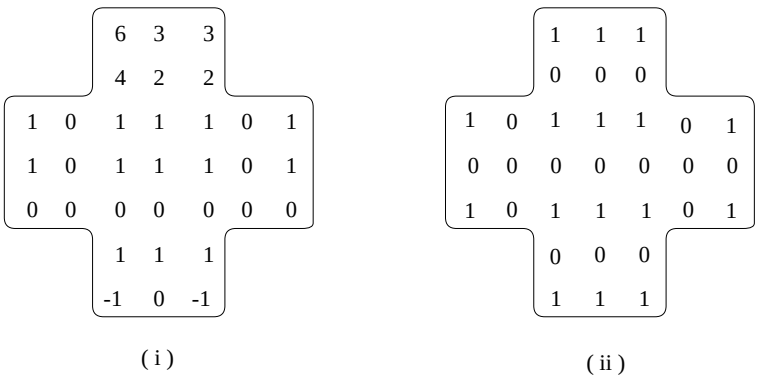


Figure 3.8: Two facets of the English solitaire cone

3. Conway *[8] gives 14 valid inequalities (pagoda functions) for the English solitaire game. Of these, 11 described in his Figures 21(d), 22(a)-(d) are facets of S_B and can be generated by GENFACET. The two pagoda functions in Figures 22(h)(v) are not facets. The*

pagoda function in Figure 21(c) is a facet of S_B, and is not generated by GENFACET.

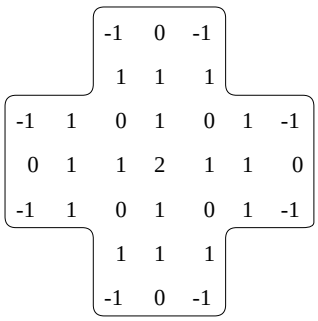


Fig. 21 (c)

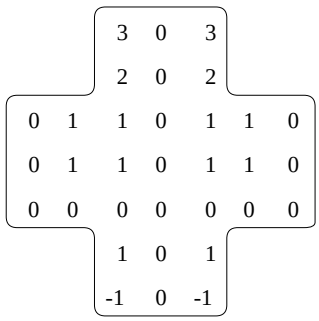


Fig. 21 (d)

3.2 Toric boards

¿From now on, the toric closure of a board is simply called a *toric board*. In other words, the toric m by n board for $m \geq 1$ and $n \geq 3$ is an m by n rectangular board with additional jumps which traverse the boundary. Formally, we extend the definition of *consecutive triple* given in the last subsection by allowing row indices to be taken modulo m and column indices to be taken modulo n. For example, for a 4 by 4 toric board both $t_1 = 2, 3$, $t_2 = 2, 4$, $t_3 = 2, 1$ and $t_1 = 1, 3$, $t_2 = 4, 3$, $t_3 = 3, 3$ are consecutive triples. Similarly we extend the definition of a consecutive string of entries to include strings that traverse the boundary.

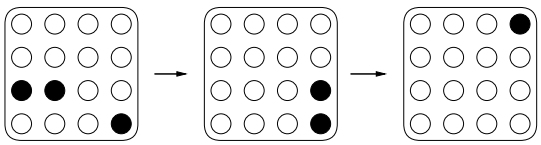


Figure 3.9: Moves on the 4 by 4 toric board

The results of Section 3.1 can easily be adapted to toric boards. Theorem 3.1 applies, except that we get the stronger condition that all coefficients are non-negative since there are no corner coefficients. In GENFACET the condition on corner coefficients is dropped in step 1, and in step 2 any non defined coefficient can be chosen. Theorem 3.2 is easily shown to apply to these boards. Observe that the matrices $a^{(1)}$ and $a^{(2)}$ of the last subsection would be generated by GENFACET for the 3 by 3 toric board, and are facets. There are 6 facets isomorphic to $a^{(1)}$ and 9 facets isomorphic to $a^{(2)}$ and these are all the facets for the solitaire cone of the 3 by 3 toric board. Note that both $b^{(1)}$ and $b^{(2)}$ violate the triangle inequality generated by the triple $t_1 = 2, 1$, $t_2 = 3, 1$, $t_3 = 1, 1$ and so are invalid.

We can use Theorems 3.1 and 3.2 to obtain a complete characterization of 0-1 facets of S_B when B is a toric board. Let a be a m by n 0-1 matrix. We define the a graph G_a on a as follows: vertices of G_a correspond to non-zero coefficients, and two vertices are adjacent if the corresponding coefficients are in some consecutive triple where the remaining coefficient is zero. Note that in fact there must be at least two such triples since if (t_1, t_2, t_3) is such a triple then so is (t_3, t_2, t_1).

Theorem 3.5 [0-1 valued facets characterization]
Let B be the m by n toric board. A m by n 0-1 matrix b is a facet of S_B if and only if

1. *No non-zero row or column contains two consecutive zeroes, and*
2. *G_b is connected.*

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Two pagoda functions of $S_{4\times 4}$, only the first one being a facet

PROOF. For the sufficiency we show that a matrix $a = b$ can be constructed by GENFACET and then apply Theorem 3.2 to show it is a facet. We begin by setting all coefficients of a to zero which are zero in b. Since b is valid for S_B, Theorem 3.1 (4) implies the zeroes of a are a valid choice in step 1 of GENFACET. We will show all other coefficients of a are set to one in GENFACET, so that $a = b$.

In step 2, some non-zero coefficient $a_{i,j}$ of a is set to one, as required. Consider the first execution of step 3. Let v be the vertex in G_b corresponding to $a_{i,j}$. Since G_b is connected, v is adjacent to some vertex w. By construction of G_b, the coefficients corresponding to v and w lie in some consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ where the remaining coefficient is zero. We may choose such a triple so that the zero coefficient is a_{t_2} or a_{t_3}. Then in the equation

$$a_{t_1} = a_{t_2} + a_{t_3}$$

one of the variables has value one and one of the right hand side variables has value zero, so the remaining variable must also have value one. Therefore in step 3, the undefined coefficient, corresponding to vertex w, gets set to one.

In general, every time we execute step 3 we can select an undefined coefficient whose corresponding vertex in G_b is adjacent to some already defined coefficient. Since G_b is connected, this is always possible. In this way all of the coefficients that were not set to zero receive the value one. The conditions on the zeroes are sufficient to ensure that all triangle inequalities are satisfied, so GENFACET has constructed a which is identical to b. It follows from Theorem 3.2 that b is a facet, concluding the proof of sufficiency.

For necessity, suppose (1) is violated, then b violates a triangle inequality and does not generate a valid inequality for S_B. Now suppose (2) is violated. G_b consists of two or more components. We form a new matrix a that is identical to b except that in one of the connected components of G_b we replace all coefficients that have value one with the value $\frac{3}{2}$. Clearly a is not a scalar multiple of b. We show a is valid for S_B and has the same tight triangle inequalities. This is a contradiction since it shows b is not a facet of S_B. To show validity of a, the only triangle inequalities that could fail for a are of the form

$$a_{t_1} > a_{t_2} + a_{t_3}$$

where $a_{t_1} = \frac{3}{2}$. Now the right hand side variables cannot both be zero or the triangle inequality would fail for b, so $\{a_{t_2}, a_{t_3}\} = \{0, 1\}$. But this is impossible because it would imply an edge in G_b between the vertices corresponding to non-zero coefficients, contradicting the fact they lie in separate components. To show a and b have the same tight triangle inequalities, consider a consecutive triple for which the equation

$$a_{t_1} = a_{t_2} + a_{t_3}$$

holds. Since the coefficient values are chosen from $\{0, 1, \frac{3}{2}\}$ there is no way both 1 and $\frac{3}{2}$ can appear in the equation. Therefore non-zero coefficients lie in the same connected component, and the same equation holds for b. Similarly, a tight triangle inequality holding for b must have any non-zero coefficients in the same connected component, so the corresponding equation holds for a. □

Theorem 3.5 is useful for proving large classes of 0-1 matrices are facets. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be two vectors. We say the m by n matrix a is the *product* of x and y if for all $1 \leq i \leq m$ and $1 \leq j \leq n$ $a_{i,j} = x_i y_j$. A simple application of Theorem 3.5 gives the following.

Corollary 3.6

1. *A 0-1 n-vector is a facet of the 1 by n toric board if and only if it has no pair of consecutive zeroes, no string of five or more ones, and at most one string of four ones.*

2. *The product of two 0-1 facets of the 1 by m and 1 by n toric boards gives a 0-1 facet of the m by n toric board.*

We end this section by remarking that a 0-1 vector a is valid over S_B if and only if the position $F = a$ is not reachable from any other position in the 0-1 peg game. To see this note that if $F = a$ is reachable, then the last jump must result in the configuration 0 0 1 in some row or column, violating the triangle inequality, so a is not valid. Conversely if a is not valid, it must contain the string 0 0 1 in some row or column. Replacing this by the string 1 1 0 gives a position leading to $F = a$. A similar statement is not true for the integer game: for example the valid $a = 0\ 1\ 1$ for the 1 by 3 game can be reached from 1 2 0. It would be interesting to see if the 0-1 facets of the 0-1 peg game have some natural interpretation in terms of the game itself.

3.3 Bounds on number of facets

As mentioned in the introduction, experimental evidence and the fact that the game is NP-complete indicates that solitaire cones are likely to have a large number of facets. In this section we give some additional theoretical evidence for this observation. We begin by a simple exponential upper bound applicable to a large number of different boards. For simplicity we restrict ourselves to boards which are a connected subset of the square lattice.

Lemma 3.7 [upper bound]
Let n_B be the number of moves on a board B with h_B holes. S_B has at most $\binom{n_B}{h_B-1} < 2^{n_B}$ facets. In particular the m by n toric board generates at most $\binom{4mn}{mn-1} < (\frac{4^4}{3^3})^{mn}$ facets.

PROOF. This follows from the fact that S_B is a cone in h_B dimensions defined by n_B extreme rays. Each facet is defined by a set of $h_B - 1$ of these rays. □

Theorem 3.8 [lower bound]

1. *There are at least $6^{(m+n)/9}$ 0-1 products of facets generated by the m by n toric board, for m and n divisible by 9.*

2. *There are at least $2^{mn/4}$ 0-1 facets generated by the m by n toric board, for m divisible by 2 and n divisible by 4.*

PROOF.

1. First consider the case $m = 1$ and set $n = 9k$ for any positive integer k. Consider any vector of length $9k$ starting with a zero, with a total of $3k$ zeroes, no two of which are onsecutive, and $3k$ blocks of ones. There are k blocks each of length one, two and three which can be included in any order. Such a vector satisfies the conditions of Corollary 3.6 (1) and so generates a facet of the 1 by n toric board. There are

$$\binom{3k}{k}\binom{2k}{k} > 6^k = 6^{\frac{n}{9}}$$

such facets. The case where $m \geq 2$ follows by combining this result with Corollary 3.6 (2).

2. Consider an m by n matrix defined by

$$a_{ij} = \begin{cases} 1 & i \equiv j \equiv 1 \pmod 2 \\ 0 & i \equiv 0 \pmod 2,\ j \equiv 1 \pmod 4 \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

Letting $*$ denote elements than can be zero or one, a generic form of the matrix for m=4, n=8 is:

$$a = \begin{pmatrix} 1 & * & 1 & 0 & 1 & * & 1 & 0 \\ 0 & 1 & * & 1 & 0 & 1 & * & 1 \\ 1 & * & 1 & 0 & 1 & * & 1 & 0 \\ 0 & 1 & * & 1 & 0 & 1 & * & 1 \end{pmatrix}$$

Note that $mn/4$ entries of a may be chosen freely so there are $2^{mn/4}$ of these matrices. Using Theorem 3.5 we can show they generate facets for the m by n toric board. Indeed, by construction, there can be no two consecutive zeroes. The vertices in G_a corresponding to any given row are connected, since no row has 4 consecutive ones. Finally column one connects all rows together, so G_a is connected.

□

4 Skeletons and Diameters of Solitaire Cones

In this Section, after presenting in details more than 50 small dimensional cases, we give some results and conjectures on the combinatorial and geometric properties of the solitaire cone. In particular, we investigate the diameter, edge connectivity, adjacency and incidence relations of the solitaire cone and its dual. Two extreme rays (resp. facets) of a polyhedral cone are *adjacent* if they belong to a face of dimension (resp. codimension) two. The number of rays (resp. facets) adjacent to the ray r (resp. facet F) is denoted A_r (resp. A_F). A ray and a facet are *incident* if the ray belongs to the facet. We denote by I_r (resp. I_F) the number of facets (resp. rays) incident to the ray r (resp. facet F). The diameters of S_B and its dual S_B^* are, respectively, $\delta(S_B)$ and $\delta(S_B^*)$.

As well as in previous sections, for a solitaire game played on a board B a black (respectively white) hole represents a peg (respectively an empty hole) as in see Fig. 4.10 and 4.13. The coordinates of a ray or a facet of S_B are naturally indexed by B as in Fig. 4.12 and 4.16.

4.1 Small dimensional solitaire cones

We first consider a solitaire game played on a rectangular or triangular board as in Fig. 4.10. In Table 4.11 we give for each board B, the number of extreme rays and facets, the minimal and maximal adjacency and incidence of the extreme rays A_r, I_r and of the facets A_F, I_F of the solitaire cone S_B, its diameter $\delta(S_B)$ and the diameter of its dual $\delta(S_B^*)$.

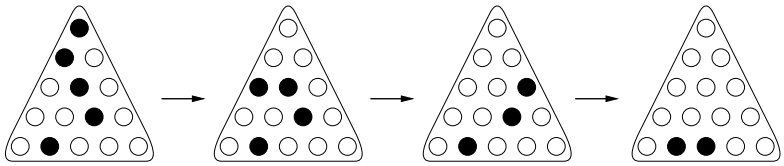


Figure 4.10: Moves on the 15 triangular board $B = \Delta_{15}$

For example, the last row in Table 4.11 means that the 36 extreme rays of the cone $S_{\Delta_{15}}$ belong to at least 6 920 and at most 10 905 of its 21 744 facets. Those facets are of size at least 14, that is, simplices, and at most 30. The unique facet F_0 of maximal incidence $I_{F_0} = 30$ and maximal adjacency $A_{F_0} = 4\,750$ is induced by the inequality: $c \cdot x \leq 0$, where $c =$[1 0 0 1 0 1 0 0 0 0 1 0 1 0 1], see Fig. 4.12. More than half of the facets of $S_{\Delta_{15}}$ are simplices and its diameter $\delta(S_{\Delta_{15}}) = 2$.

Board	$\#rays$	I_r	A_r	$\#facets$	I_F	A_F	$\delta(S_B)$	$\delta(S_B^*)$
3×3	12	11∼17	10∼11	18	8∼11	8∼17	2	2
4×4	32	1 584∼2 109	29∼31	3 531	15∼26	15∼695	2	≤ 3
3×4	20	50∼82	17∼19	107	11∼28	11∼79	2	≤ 3
3×5	28	421∼856	25∼27	1 277	14∼24	14∼429	2	≤ 3
Δ_{10}	18	87∼105	16∼17	182	9∼14	9∼79	2	≤ 3
Δ_{15}	36	6 920∼10 905	33∼35	21 744	14∼30	14∼4 750	2	≤ 3

Figure 4.11: Small rectangular and triangular boards

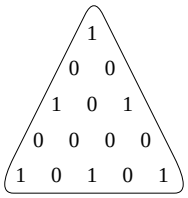


Figure 4.12: The unique facet F_0 of maximal size and adjacency of $S_{\Delta_{15}}$

Then, we consider a solitaire game played on a toric rectangular or triangular board as in Fig. 4.13. The adjacency and incidence relations and the diameters are given in Table 4.14. For example, the 16-dimensional cone generated by the 64 moves of the 4 by 4 toric board has 95 444 facets of which almost half are simplices.

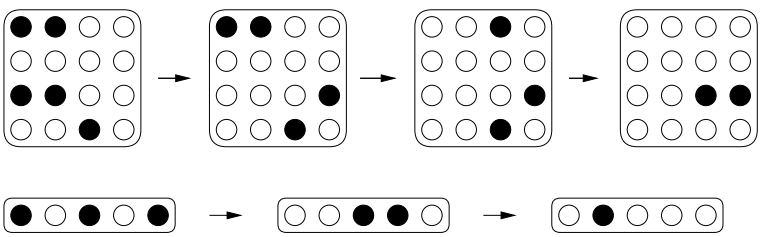


Figure 4.13: Moves on the 4 by 4 and 1 by 5 toric boards

4.2 Skeletons and diameters

The following results and conjectures are stated in terms of rectangular toric board but require only minor modifications for solitaire games played on toric or non-toric boards of

Board	$\#rays$	I_r	A_r	$\#facets$	I_F	A_F	$\delta(S_B)$	$\delta(S_B^*)$
1×3	3	2	2	3	2	2	1	1
1×4	8	3	3	6	4	4	3	2
1×5	10	7	6	15	4∼6	4∼6	2	3
1×6	12	9	9	17	6∼8	6∼12	2	2
1×7	14	32	12	70	6∼8	6∼13	2	
1×8	16	42	14	86	7∼10	7∼21	2	
1×9	18	119	16	255	8∼12	8∼36	2	
1×10	20	214	18	447	9∼12	9∼34	2	
1×11	22	508	20	1 078	10∼14	10∼51	2	
1×12	24	964	22	2 013	11∼16	11∼84	2	
3×3	18	11	15	15	12∼14	12∼14	2	2
4×4	64	25 348	58	95 444	15∼48	15∼8 195	2	
3×4	36	190∼233	30∼33	498	13∼26	14∼166	2	
3×5	45	12 963∼13 438	40∼42	39 060	14∼34	14∼3 404	2	

Figure 4.14: Small rectangular toric boards

any shapes, see Remark 4.7. We consider the solitaire cone $S_{m\times n}$ induced by a game played on a m by n rectangular toric board with $n \geq m \geq 1$ and $n \geq 3$. The number of extreme rays of $S_{m\times n}$, that is, the moves of the solitaire game is:

$f_o(S_{m\times 3}) = 3, 6, 18$ for $m = 1, 2, 3$.
$f_o(S_{m\times n}) = 2mn$ for $n \geq 4$ and $m \leq 2$.
$f_o(S_{3\times n}) = 3mn$ for $n \geq 4$.
$f_o(S_{m\times n}) = 4mn$ for $n \geq m \geq 4$.

The coordinates and coefficients of the extreme rays and facets of $S_{m\times n}$ are naturally indexed by the $m \times n$ board. For example, the extreme ray r =[0 -1 -1 1 0] of $S_{1\times 5}$ and the corresponding start and finish positions are represented in Fig. 4.15. The *support* of r is the set $\sigma_r = \{2, 3, 4\}$ of nonzero coordinates of r.

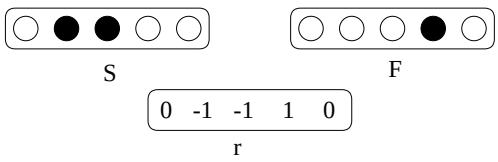


Figure 4.15: The extreme ray r of $S_{1\times 5}$ corresponding to the move from S to F

In the next lemma we give a characterization of the adjacency of the extreme rays of the solitaire cone induced by an 1 by n board. As corollary, we obtain the adjacency, edge-connectivity, diameter and the number of 2-faces of this cone.

Lemma 4.1 *Any pair of extreme rays of $S_{1\times 3}$ are adjacent and, for $n \geq 4$, two distinct extreme rays u and v of $S_{1\times n}$ are non-adjacent if and only if:*

1. $u \cdot v \in \{0, -2\}$ *for* $n = 4$;
2. $u \cdot v = -2$ *or* $u \cdot v = -1$ *and* $|\sigma_u \cap \sigma_v| = 1$ *for* $n = 5$;
3. $u \cdot v = -2$ *or* $u_{i+3 \bmod 6} = v_i$ $i : 1, \ldots, 6$ *for* $n = 6$;
4. $u \cdot v = -2$ *for* $n \geq 7$.

Proof. Given in Appendix 7. □

Corollary 4.2 *The skeleton of $S_{1\times n}$ satisfies*

1. *the adjacency of an extreme ray* $k_{1\times n} = 2, 3, 6, 9, 2(n-1)$ *for* $n = 3, 4, 5, 6, \geq 7$;
2. *the diameter* $\delta(S_{1\times n}) = 1, 3, 2$ *for* $n = 3, 4, \geq 5$;
3. $S_{1\times n}$ *has exactly* $3, 12, 30, 54, 2n(n-1)$ 2*-faces for* $n = 3, 4, 5, 6, \geq 7$;
4. *the edge connectivity* $c_e(S_{1\times n}) = 2, 3, 6, 9, 2(n-1)$ *for* $n = 3, 4, 5, 6, \geq 7$.

Proof. (1), (2) The adjacency and the diameter are straightforward. (3) The number of 2-faces of a cone is half of the total adjacency of its skeleton. (4) We recall the following result of Plesník [22]: the edge connectivity of a graph of diameter 2 equals its minimum degree. Then, since for $n \geq 5$ the cone $S_{1\times n}$ has diameter 2 and since $c_e(S_{1\times n}) = 2, 3$ for $n = 3, 4$, we have $c_e(S_{1\times n}) = k_{1\times n}$ for $n \geq 3$. □

Clearly, for the solitaire cone induced by a 2 by n board, two rays laying in the same row have the same adjacency relations as in $S_{1\times n}$ and two rays laying lying in different rows are always adjacent. In Theorem 4.3 we generalize the adjacency relations of $S_{1\times n}$ and $S_{2\times n}$ to $S_{m\times n}$ for any $n \geq m \geq 3$.

Theorem 4.3 [extreme rays adjacency characterization]
Any pair of extreme rays of $S_{m\times n}$ with distinct support are adjacent and, for $m \geq 3$, the adjacency relation of any pair u, v of extreme rays of $S_{m\times n}$ is given by:

1. *either σ_u and σ_v do not belong to the same row or column, then u and v are adjacent;*
2. *or σ_u and σ_v belong to the same row or column of length n, then u and v have the same adjacency relations as in $S_{1\times n}$, except that u and v are not adjacent if $\sigma_u = \sigma_v$.*

PROOF. Given in Appendix 7. □

Corollary 4.4 *For $n \geq m \geq 7$, the skeleton of a $S_{m\times n}$ satisfies*

1. *the adjacency of an extreme ray $k_{m\times n} = 4mn - 3$;*
2. *the diameter $\delta(S_{m\times n}) = 2$;*
3. *$S_{m\times n}$ has exactly $2mn(4mn - 3)$ 2-faces;*
4. *the edge connectivity $c_e(S_{m\times n}) = 4mn - 3$.*

PROOF. Same as for Corollary 4.2. □

Two rays are called *strongly conflicting* (respectively *conflicting*) in there exist 2 pairs i, j and k, l (respectively a pair i, j) such that the two rays have nonzero coordinates of distinct signs at positions i, j and k, l (respectively i, j). For example, the two adjacent rays of $S_{4\times 4}$ given in Fig. 4.2 are conflicting at the position $2, 2$ but not strongly conflicting.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 4.16: Two rays conflicting as the position $2, 2$

Remark 4.5 *While for $n \geq m \geq 7$ a pair of extreme rays of the solitaire cone $S_{m\times n}$ are adjacent if and only if they are not strongly conflicting, for $n \geq 4$ two extreme rays of the dual metric cone M_n^* are adjacent if and only if they are not conflicting, see [9].*

The following conjectures are based on Remark 4.5 and other similarities between the solitaire cone and the dual metric cone investigated in Section 2.

Conjecture 4.6

1. *For $m \geq 3$, the $\{0,1\}$-valued facets of the solitaire cone form a dominating set in the skeleton of its dual, that is, each facet of $S_{m\times n}$ is adjacent to a $\{0,1\}$-valued facet. This holds for $S_{3\times 4}$. This conjecture is false for $m \leq 2$. The smallest 1 by n toric board for which the conjecture fails is $B = 1 \times 7$.*

2. *For m, n large enough, at least one facet of $S_{n\times m}$ is a simplex (that is, the number of rays contained in the facet equals the dimension of the cone minus one). If true, this would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of $S^*_{m\times n}$ are equal to $mn-1$. This holds for the cones presented in Fig. 4.14 except for $S_{3\times i} : i = 3, 4$ and $S_{1\times i} : i = 4, 6$.*

Remark 4.7 *For the non-toric boards given in Table 4.11, a pair of extreme rays of the solitaire cone S_B are adjacent if and only if they are not strongly conflicting and their supports do not lay entirely on the boundary of the board.*

5 Related Polyhedra

In the same way as for the dual metric cone M_n^*, for which the $\{0,1\}$-relaxation is the dual of the well-studied cut cone, we consider the $\{0,1\}$-relaxation of the solitaire cone and some other related polyhedra.

5.1 The boolean solitaire cone

We consider the relaxation of the solitaire cone S_B by its facets which coordinates are, up to a constant multiplier, $\{0,1\}$-valued. This cone is called *boolean solitaire cone* and denoted BS_B. The adjacency and incidence of the facets and extremes rays of small toric rectangular boolean cones are given in Fig. 5.17.

Board	$\#rays$	I_r	A_r	$\#facets$	I_F	A_F
1×3	3	2	2	3	2	2
1×4	8	3	3	6	4	4
1×5	15	4∼5	4∼5	10	6∼8	5∼7
1×6	30	5∼6	5∼6	11	12∼16	8∼9
1×7	42	8∼11	7∼10	21	18∼20	11∼13
1×8	72	10∼16	9∼14	30	24∼36	14∼18
1×9	126	9∼25	9∼19	48	32∼54	18∼26
1×10	200	14∼36	12∼24	67	40∼80	16∼33
1×11	231	11∼58	11∼26	110	50∼86	24∼45
1×12	516	12∼85	12∼51	159	60∼172	21∼62
3×3	18	11	15	15	12∼14	12∼14
4×4	930 048	15∼168	15∼?	340	?	?
3×4	1 284	11∼(30 34)	11∼369	54	194∼506	35∼52
3×5	101 444	14∼(118 129)	14∼14 607	240	?	?

Figure 5.17: Small boolean toric boards

Theorem 5.1 *The extreme rays of the solitaire cone, that is, the moves, are extreme rays of the boolean solitaire cone.*

Proof. Given in Appendix 7. □

Corollary 5.2 *The boolean solitaire cone is full dimensional.*

Out of the 930 048 extreme rays of $BS_{4\times4}$, the 64 extreme rays of $S_{4\times4}$, that is, the moves, reached the highest incidence $I_r^{max} = 168$ which is almost three times larger that the second highest incidence $I_r^{submax} = 57$. Similarly, out of the 101 444 extreme rays of $BS_{3\times5}$, the 15 vertical moves of $S_{3\times5}$, reached the highest adjacency $A_r^{max} = 14\ 607$ while the average adjacency is $A_r^{ave} \simeq 33.16$. Those computational results and other similarities with the metric cone, see Section 2, lead us to the following conjectures:

Conjecture 5.3

1. *For $m \geq 3$, the moves form a dominating set in the skeleton of the boolean solitaire cone $BS_{m\times n}$. This holds for $BS_{3\times4}$. This conjecture is false for $m \leq 2$. The smallest 1 by n board for which the conjecture fails is $B = 1 \times 10$.*

2. *The incidence of the moves is maximal in the skeleton of $BS_{m\times n}$. This holds for all cones presented in Fig. 5.17. This conjecture is false is we replace the incidence by the adjacency as, for example, for $BS_{3\times5}$.*

3. *For m, n large enough, at least one ray of $BS_{m\times n}$ is simple, (that is, the incidence equals the dimension of the cone minus one). If true, this would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of $BS_{m\times n}$ are equal to $nm - 1$. This holds for $BS_{3\times i} : i = 4, 5$ and $BS_{4\times4}$.*

5.2 The trellis solitaire cone

We defined the *trellis solitaire cone* TS_B as the relaxation of the boolean solitaire cone BS_B by its *trellis facets*. A vector c of length mn induced a trellis facet $F : c \cdot x \leq 0$ of $S_{m\times n}$ if F is a facet of $BS_{m\times n}$ and $c_{ij} = f_i \cdot g_j$ for $i = 1, \ldots m$, $j = 1, \ldots n$ where f (respectively g) is a $\{0, 1\}$-valued string of length m (respectively n). See item 2 of Corollary 3.6 for an easy construction of trellis facets. For example, among the two following facets of $BS_{3\times5}$, only the right one is a trellis facet.

1	1	0	1	0
1	0	1	0	1
0	1	1	1	1

1	0	1	0	1
0	0	0	0	0
1	0	1	0	1

Conjecture 5.4 *The boolean trellis solitaire cone is full dimensional.*

5.3 The complete solitaire cone

The *complete solitaire cone* CS_B is induced by a variation of the Solitaire game. To the classical moves we add the moves which consist of removing two pegs surrounding an empty hole and placing one peg in this empty hole as showed in Fig. 5.18. The incidence and adjacency of small dimensional complete solitaire cones are presented in Fig. 5.19.

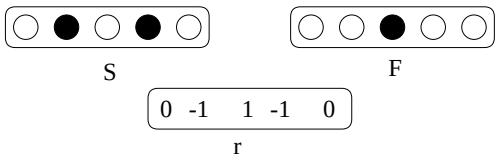


Figure 5.18: The extreme ray r of $CS_{1\times 5}$ corresponding to the move from S to F

Board	$\#rays$	I_r	A_r	$\#facets$	I_F	A_F	$\delta(CS_B)$
1×7	21	26∼33	16∼17	91	6∼12	6∼32	2
1×8	24	119∼130	19∼20	404	7∼12	7∼44	2
3×3	18	11	15	15	12∼14	12∼14	2
3×4	48	904∼1 192	39∼45	3 576	11∼33	11∼738	2

Figure 5.19: Small complete toric boards

We have $CS_{3\times 3} = S_{3\times 3}$ and, by analogy with the classical solitaire cone case, we conjecture:

Conjecture 5.5 *For $n \geq m \geq 7$ a pair of extreme rays of $CS_{m\times n}$ are adjacent if and only if they are not strongly conflicting. If true, this would imply that $\delta(CS_{m\times n}) = 2$.*

5.4 The boolean complete solitaire cone

In the same way as we did for the solitaire cone, we consider the relaxation of the complete solitaire cone CS_B by its facets which coordinates are, up to a constant multiplier, $\{0, 1\}$-valued. The incidence and adjacency of small dimensional complete boolean solitaire cones are presented in Fig. 5.20.

Board	$\#rays$	I_r	A_r	$\#facets$	I_F	A_F	$\delta(BCS_B)$
1×7	7	6	6	7	6	6	1
1×8	16	9	10	12	12	9∼10	2
3×3	18	11	15	15	12∼14	12∼14	2
3×4	72	14∼28	15∼58	36	30∼45	30∼35	2

Figure 5.20: Small boolean complete toric boards

6 Conclusions

The solitaire cone shares many similar combinatorial properties with the dual metric cone M_n^*. In particular:

1. Their extreme rays have similar adjacency relations, see Remark 4.5,

2. Both cones have diameter 2, see Deza and Deza [9] and Corollary 4.4,

3. Their numbers of facets are bounded above and below by an exponential in the dimension, see Lemma 3.7 and Theorem 3.8 and, for the metric cone, Avis [2] (lower bound) and Graham, Yao and Yao [15] (upper bound).

4. Their extremes rays are also extremes rays of their relaxations by their $\{0,1\}$-valued facets, see Theorem 5.1.

5. While the extremes rays of the solitaire cone are conjectured to be of maximum incidence in its relaxation by its $\{0,1\}$-valued facets, the corresponding result is proved for the dual metric cone, see item 2 of Conjecture 5.3 and Deza and Deza [10],

6. We have $M_n^* = CS_{L_n}$ where L_n is the line graph of the complete graph on n nodes, see Theorem 2.3.

7. The 0-1 facets of the flow cone are the incidence vectors of cuts in the complete graph. The cone generated by these facets is the dual of the well studied cut cone, see Deza and Laurent [12]. Papernov [21] gave a complete characterization of multiflow problems for which the metric cone F_n in Theorem 2.2 can be replaced by the dual of the cut cone. For example, single commodity flow problems are in this class, and the corresponding theorem is the celebrated max flow/min cut theorem. It would be interesting to see if any analogous relaxation of Theorem 1.4 can be found.

8. So far we have not yet found analogue of the *hypermetric* facets of the metric cone M_n, that is, a "nice" family of $\{0,-1,1\}$-valued extreme rays of the boolean solitaire cone BS_B. Another open question is the determination of a tighter relaxation of the solitaire

cone S_B by some analogues of the *cuts*. The trellis solitaire cone TS_B is a candidate as well as the relaxation by the $\{0,1\}$-valued facets with the minimal number of ones. For $S_{4\times 4}$ and $S_{3\times i} : i = 3,4,5$, those facets have maximal incidence and adjacency in the skeleton of $S^*_{m\times n}$.

7 Appendix

In this section, the dual problem being easier to state, we always consider the dual solitaire cone S_B^* which extreme rays (respectively facets) are the r_i (respectively induced by $c_j \cdot x \leq 0$) where $r_i \cdot x \leq 0$ induces a facet of S_B (respectively c_j is an extreme ray of S_B). clearly, a pair of extreme rays r', r'' are adjacent in S_B if and only if their corresponding facets F', F'' are adjacent in S_B^*, that is, if the codimension of $F' \cap F''$ is two.

7.1 Proof of Lemma 4.1 and Theorem 4.3

7.1.1 Proof of Lemma 4.1

All facets of $S^*_{1\times n}$ being equivalent up to a scrolling and a reversing of the board, it is enough to find all the neighbours of the facet F_u : [-1 -1 1 0 ... 0]. The adjacency relations were checked by computer up to $n = 8$, so we can assume that $n \geq 9$. We first prove that the facet F_v : [0 1 -1 -1 0 ... 0] is not adjacent to F_u and then that all other facets are adjacent to F_u. Any extreme ray r belonging to $F_u \cap F_v$ satisfies:

$$\begin{cases} r_1 + r_2 - r_3 = 0 \\ r_2 - r_3 - r_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} r_1 = r_4 = 0 \\ r_2 = r_3 \end{cases}$$

(since $0 \leq r_i$ for $1 \leq i \leq n$)

$$\Rightarrow \begin{cases} r_3 = r_5 \\ r_4 = 0 \end{cases}$$

(since $r_3 - r_4 - r_5 \leq 0$ and $-r_3 - r_4 + r_5 \leq 0$)

which implies that $F_u \cap F_v \subset F'_v$ where $v' =$ [0 0 1 -1 -1 0 ... 0] and therefore $F_u \cap F_v$, being an intersection of more than 3 facets, is of codimension at least 3, that is, F_u and F_v are not adjacent.

Then, to prove that all other facets are adjacent to F_u, we consider any facet $F_a \neq F_v$ and show that for any third facet F_b we can find an extreme ray r of $S^*_{1\times m}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, $codim(F_u \cap F_a) = 2$. First, let assume that the supports of F_u, F_a and F_b do not overlap and that the gaps between σ_b and σ_u and between σ_b and σ_a are not equal to 0 or 2, as in the following example:

$$\underbrace{\overbrace{-1,-1,1}^{\sigma_u},0,\ldots,0,\overbrace{1,-1,-1}^{\sigma_a},0,\ldots,0,\overbrace{1,-1,-1}^{\sigma_b},0,\ldots,0}_{n}$$

Consider the following $\{0,1\}$-valued ray:

$$r = \underbrace{\overbrace{1,0,1}^{\sigma_u},0,1,0\ldots,0,1,0\overbrace{1,0,1}^{\sigma_a},0,1,0\ldots,1,0,\overbrace{1,1,1}^{\sigma_b},0,1\ldots,1,0}_{n}$$

This ray r is an extreme ray of $S^*_{1\times n}$ if and only if its 1-graph G_r is connected (see Theorem 3.5). According to the parity of the gaps between the supports σ_u and σ_a, σ_a and σ_b and σ_b and σ_u, we can fill those gaps by 0, 1-valued strings: 0 1 0 1 ... 0 1 0 or 0 1 1 0 1 0 1 ... 0 1 0 such that the graph G_r obtained is connected. Therefore r is an extreme ray of $S^*_{1\times n}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, F_u and F_a are adjacent.

If F_u, F_a and F_b do not overlap but the gaps between σ_b and σ_u or between σ_b and σ_a equal 0 or 2, then we can use the same technique considering one of the following $\{0,1\}$-valued rays:

$$r^1 = \underbrace{\overbrace{1,0,1}^{\sigma_u},0,1,0\ldots,0,1,0\overbrace{1,0,1}^{\sigma_a},\overbrace{0,1,0}^{\sigma_b}}_{n}$$

$$r^2 = \underbrace{\overbrace{1,0,1}^{\sigma_u},0,1,0\ldots,0,1,0\overbrace{1,0,1}^{\sigma_a},0,1,\overbrace{0,1,1}^{\sigma_b},0,1\ldots,1,0}_{n}$$

When the supports of F_u, F_a and F_b overlap, we can, by essentially projecting on the joint support $\sigma_u \cup \sigma_a \cup \sigma_b$, use the same technique and generalize what happens for $n \leq 8$). For example, take $n = 11$ and F_u, F_a and F_b given by:

$$\overbrace{-1,-1,1}^{\sigma_u},0,0,0,0,0,0,0,0,0 \qquad \overbrace{1,-1,-1}^{\sigma_a},0,0,0,0,0,0,0,0,0 \qquad 0,0,\overbrace{1,-1,-1}^{\sigma_b},0,0,0,0,0,0$$

the desired extreme ray of $S^*_{1\times 11}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$ is:

$$r = \overbrace{1,0,1,1,1}^{\sigma_u\cup\sigma_a\cup\sigma_b},0,1,0,1,0,1$$

which means that F_u and F_a are adjacent.

7.1.2 Proof of Theorem 4.3

We first consider a pair of facets F_u and F_v such that σ_u and σ_v belong to the same row or column of length d. For $d = 3$, we have $\sigma_u = \sigma_v$ and one can easily check that F_u and F_v are never adjacent. For $d \geq 4$ and $\sigma_u \neq \sigma_v$, we can in the same way as for the proof of Lemma 4.1, find a third facet F'_v containing $F_u \cap F_v$ if F_u and F_v, seen as facet of $S_{1\times n}$, are non-adjacent. For example, with:

$$F_u : \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \end{bmatrix} \qquad F_v : \begin{bmatrix} 0 & 1 & -1 & -1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \end{bmatrix}$$

the following facet:

$$F_{v'} : \begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

satisfies $F_u \cap F_v \subset F_{v'}$, that is, F_u and F_v are non-adjacent. The only difference between $S_{m\times n}$ and $S_{1\times n}$ is that F_u and the facet F_w with $\sigma_u = \sigma_w$ are not adjacent. We have:.

$$F_u : \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad F_w : \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

and the following facet :

$$F_{w'} : \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

satisfies $F_u \cap F_w \subset F_{w'}$, that is, F_u and F_w are non-adjacent. Then, to prove that all other facets which support belongs to the same row or column as σ_u are adjacent to F_u, we can apply the same technique as for the proof of Lemma 4.1.
We then consider F_u and F_a with σ_u and σ_a not in the same row or column. As for Lemma 4.1, we show that for any third facet F_b, we can find an extreme ray r of $S^*_{m\times n}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, $codim(F_u \cap F_a) = 2$. First let assume that the supports of F_u, F_a and F_b do not overlap as in the following example:

$$F_a : \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & -1 & -1 & 1 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad F_b : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

Consider the following $\{0,1\}$-valued ray

$$r: \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

This ray r is an extreme ray of $S^*_{m\times n}$ if and only if its 1-graph G_r is connected (see Theorem 3.5). According to the parity of the gaps between the supports σ_u and σ_a, σ_a and σ_b and σ_b and σ_u, we can fill those gaps by the following $0,1$-valued matrices (or their transposes) such that the graph G_r obtained is connected.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

Therefore r is an extreme ray of $S^*_{m\times n}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, F_u and F_a are adjacent.

When the supports of F_u, F_a and F_b overlap, once again, we use the same technique which amounts to a tedious but easy case by case study and completes the proof. For example, take the following facets of $S^*_{4\times 4}$:

$$F_u: \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_a: \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_b: \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the desired extreme ray of $S^*_{1\times 11}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$ is:

$$r: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

which means that F_u and F_a are adjacent.

7.2 Proof of Theorem 5.1

As for the previous Theorem 4.3, we consider the dual boolean solitaire cone $BS^*_{m\times n}$, that is, we want to prove that the inequality $c\cdot r \leq 0$ where c is an extreme ray of $S_{m\times n}$ induced a facet of $BS^*_{m\times n}$. By definition of $BS_{m\times n}$, $c\cdot x \leq 0$ is valid over $BS^*_{m\times n}$. To show that $f = \{r : r \in BS^*_{m\times n} \text{ and } c\cdot r = 0\}$ is of codimension 1, that is, $c\cdot r \leq 0$ induced a facet, we prove that the only hyperplane containing f is $H : c\cdot r = 0$. Let G be an hyperplane containing f.

7.2.1 Case $m \leq 2$

First, take $m = 1$. All extreme rays of $S_{1\times n}$ being equivalent up to a scrolling and a reversing, we can assume that $c =$ [-1 -1 1 0 ... 0]. Consider the following two $\{0,1\}$-rays x_i and $\bar{x}_i$ where the boxed value is the i^{th} coordinate

$$x_i = \underbrace{\overbrace{1,0,1}^{\sigma_c},0,1,0\ldots,0,1,0,1,\boxed{1},1,0,1,0\ldots,1,0}_{n}$$

$$\bar{x}_i = \underbrace{\overbrace{1,0,1}^{\sigma_c},0,1,0\ldots,0,1,0,1,\boxed{0},1,0,1,0\ldots,1,0}_{n}$$

Those rays satisfy $c\cdot r = 0$ and are in f if and only if their 1-graphs G_{x_i} and $G_{\bar{x}_i}$ are connected (see Theorem 3.5). For $i \geq 4$ we can, in exactly the same way as in 7.1.1, complete x_i and $\bar{x}_i$ such that both G_{x_i} and $G_{\bar{x}_i}$ are connected. Then, any hyperplane $G : \sum_{k=1}^n g_k\cdot r_k \leq 0$ containing f should satisfy $G(x_i) = G(\bar{x}_i)$ for $i = 4\ldots n$ where $G(r) = \sum_{k=1}^n g_k\cdot r_k$. This implies $g_k = 0$ for $k = 4\ldots n$. Moreover, the following two rays

$$x_0 = \underbrace{\overbrace{0,1,1}^{\sigma_c},0,1,0,\ldots,1}_{n} \qquad \bar{x}_0 = \underbrace{\overbrace{1,0,1}^{\sigma_c},0,1,0\ldots,0}_{n}$$

also belong to f; which implies: $G(x_0) = G(\bar{x}_0) = 0$, that is, $g_2 + g_3 = g_1 + g_3 = 0$. The hyperplane G is therefore induced by $g\cdot r = 0$ with $g = (-g_3, -g_3, g_3, 0, \ldots, 0)$. In other words, $G = H$; which completes the proof. Since the case $m = 2$ is almost equivalent to the case $m = 1$, we next consider case $m = 3$.

7.2.2 Case $m = 3$

The two cases

$$c_1 = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \qquad c_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

being essentially the same, we can assume that $c = c_1$. We consider the following ray x_{ij} where the boxed value is the ij^{th} coordinate.

$$x_{ij} = \begin{bmatrix} 1 & 0 & 1 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 1 & 0 & 1 & \boxed{1} & 1 & 0 & 1 & 1 & 0 & 1 & \dots \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \dots \end{bmatrix}$$

Up to a rotation along the axis $i = 2$, we can assume that x_{ij} satisfies $c \cdot x_{ij} = 0$. Then, the graph $G_{x_{ij}}$ being connected, x_{ij} is an extreme ray of $S_{n\times 3}$ and therefore belongs to f. Then, the following ray $\bar{x}_{ij}$: $\bar{x}_{kl} = x_{kl}$ for $\{kl\} \neq \{ij\}$ and $\bar{x}_{ij} = 0$

$$\bar{x}_{ij} = \begin{bmatrix} 1 & 0 & 1 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 1 & 0 & 1 & \boxed{0} & 1 & 0 & 1 & 1 & 0 & 1 & \dots \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \dots \end{bmatrix}$$

also belongs to f for the same reasons. Therefore, any hyperplane G containing f should satisfy $G(x_{ij}) = G(\bar{x}_{ij})$ for $ij \notin \sigma_c$, that is, $g_{ij} = 0$ for $ij \notin \sigma_c$. The values of the remaining 3 coordinates: $g_{1,1}$, $g_{1,2}$ and $g_{1,3}$ are given by the following two extreme rays belonging to f:

$$x_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & \dots \end{bmatrix} \qquad \bar{x}_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & \dots \end{bmatrix}$$

This implies $g_{1,1} = g_{1,2} = -g_{1,3}$, that is, $G = H$; which completes the proof.

7.2.3 Case $m \geq 4$

This case is an almost direct generalization of the previous cases. Assume that $c_{ij} = 0$ except $c_{1,1} = c_{1,2} = -c_{1,3} = -1$ and consider the ray x_{ij} where the boxed value is the ij^{th} coordinate.

$$x_{ij} = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & \boxed{1} & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

By filling the gaps with the following $\{0,1\}$-valued matrices (or their transposes), x_{ij} can satisfy both $G_{x_{ij}}$ is a connected graph and $c \cdot x_{ij} = 0$, that is, $x_{ij} \in f$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

Then, the following ray $\bar{x}_{ij}$: $\bar{x}_{kl} = x_{kl}$ for $\{kl\} \neq \{ij\}$ and $\bar{x}_{ij} = 0$

$$\bar{x}_{ij} = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & \boxed{0} & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

also belongs to f by construction. Therefore, any hyperplane G containing f should satisfy $G(x_{ij}) = G(\bar{x}_{ij})$ for $i, j \neq k, l$ for $k = 1, 2, m$ and $l = 1, 2, 3$, that is, $g_{ij} = 0$ for $i, j \neq k, l$ for $k = 1, 2, m$ and $l = 1, 2, 3$. Of the 9 remaining values, the g_{ij} for $ij \notin \sigma_c$ are set to 0 by six pairs of extreme rays, such as the following one, which belong to f.

$$x_{2,2} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & \boxed{1} & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \end{bmatrix} \qquad \bar{x}_{2,2} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & \boxed{0} & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \end{bmatrix}$$

The values of the remaining 3 coordinates: $g_{1,1}$, $g_{1,2}$ and $g_{1,3}$ are given by the following two extreme rays belonging to f:

$$x_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad \bar{x}_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

This implies $g_{1,1} = g_{1,2} = -g_{1,3}$, that is, $G = H$; which completes the proof.

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