

On the binary solitaire cone

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Abstract

The *solitaire cone* S_B is the cone of all feasible fractional Solitaire Peg games. Valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. The link with the well studied dual metric cone and the similarities between their combinatorial structures - see (3) - leads to the study of a dual cut cone analogue; that is, the cone generated by the $\{0,1\}$ -valued facets of the solitaire cone. This cone is called *binary solitaire cone* and denoted BS_B . We give some results and conjectures on the combinatorial and geometric properties of the binary solitaire cone. In particular we prove that the extreme rays of S_B are extreme rays of BS_B strengthening the analogy with the dual metric cone whose extreme rays are extreme rays of the dual cut cone. Other related cones are also considered.

1 Introduction and Basic Properties

1.1 Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes - see Fig. 1 - although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column respectively) two consecutive pegs are adjacent

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to an empty hole in the same row (column respectively), we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of BEASLEY (4). The book mentions an engraving of BEREY, dated 1697, of a lady with a Solitaire board. The modern mathematical study of the game dates to the 1960s at Cambridge University. The group was led by CONWAY who has written a chapter in (5) on various mathematical aspects of the subject. One of the problems studied by the Cambridge group is the following basic *feasibility* problem of peg solitaire:

For a given board B , starting configuration c and finishing configuration c' , determine if there is a legal sequence of moves from c to c' .

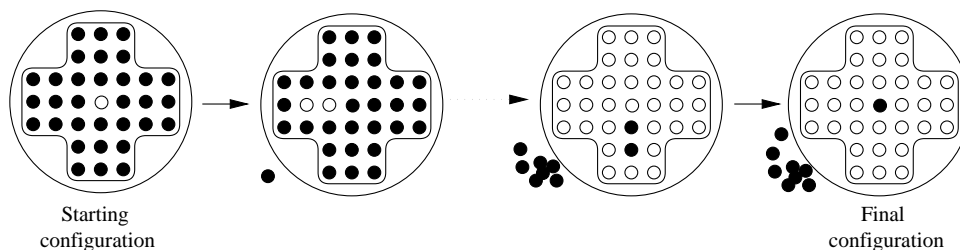


Fig. 1. A feasible English solitaire peg game with possible first and last moves

The complexity of the feasibility problem for the game played on a n by n board was shown by UEHARA AND IWATA (11) to be NP-complete, so easily checked necessary and sufficient conditions for feasibility are unlikely to exist. One of the tools used to show the infeasibility of certain starting and finishing configurations is a polyhedral cone called the *solitaire cone* S_B , corresponding to some given board B .

1.2 Basic properties

For ease of notation, we will mostly be concerned with rectangular boards which we represent by 0-1 matrices. A zero represents an empty hole and a one represents a peg. For example, let $c = [1 0 1 1]$ and $c' = [0 0 1 0]$ be starting and finishing positions for the 1 by 4 board. This game is *feasible*, involving two moves and the intermediate position $[1 1 0 0]$. For any move on an m by n board B we can define an m by n *move matrix* which has 3 nonzero entries: two entries of -1 in the positions from which pegs are removed

and one entry of 1 for the hole receiving the new peg. The two moves involved in the previous example are represented by $m_1 = [0 \ 1 \ -1 \ -1]$ and $m_2 = [-1 \ -1 \ 1 \ 0]$. Clearly $c' = c + m_1 + m_2$. By abuse of language, we use the term *move* for both the move itself and the move matrix. In general it is easily seen that if c, c' define a feasible game of k moves there exist move matrices m_1, \dots, m_k such that

$$c' - c = \sum_{i=1}^k m_i. \quad (1)$$

Clearly, Equation 1 is necessary but not sufficient for the feasibility of a peg game. For example, take $c = [1 \ 1 \ 1 \ 1]$ and $c' = [0 \ 0 \ 0 \ 1]$. We have $c' - c = [-1 \ -1 \ 1 \ 0] + [0 \ 1 \ -1 \ -1] + [0 \ -1 \ -1 \ 1]$, but c, c' do not define a feasible game; in fact there are no legal moves! Let us relax the conditions of the original peg game to allow a fractional (positive or negative) number of pegs to occupy any hole. We call this game the *fractional game*, and call the original game the *0-1 game* (in a 0-1 game we require that in every position of the game a hole is either empty or contains a single peg). A *fractional move matrix* is obtained by multiplying a move matrix by any positive scalar and is defined to correspond to the process of adding a move matrix to a given position. For example, let $c = [1 \ 1 \ 1]$, $c' = [1 \ 0 \ 1]$. Then $c' - c = [0 \ -1 \ 0] = \frac{1}{2} [-1 \ -1 \ 1] + \frac{1}{2} [1 \ -1 \ -1]$ is a feasible fractional game and can be expressed as the sum of two fractional moves, but is not feasible as a 0-1 game.

Let B be a board and n_B the total number of possible moves on the board. The *solitaire cone* S_B is the set of all non-negative combinations of the n_B corresponding move matrices. Thus $c' - c \in S_B$ if:

$$c' - c = \sum_{i=1}^{n_B} y_i m_i, \quad y_i \geq 0, \quad i = 1, \dots, n_B. \quad (2)$$

In the above definition it is assumed that the h_B holes in the board B are ordered in some way and that $c' - c$ and m_i are h_B -vectors. When B is a rectangular m by n board $B_{m,n}$ it is convenient to display $c' - c$ and m_i as m by n matrices, although of course all products should be interpreted as dot products of the corresponding mn -vectors. For $n \geq 4$ or $m \geq 4$, the solitaire cone $S_{m \times n}$ associated to the m by n board is a pointed full-dimensional cone and the moves of the solitaire cone are extreme rays; see (3) for a detailed study of the solitaire cone. The following result obtained in 1961 is credited to BOARDMAN (who apparently has not published anything on the subject) by BEASLEY (4), page 87. We identify $c' - c$ with the fractional game defined by c and c' .

Proposition 1 *Equation 2 ($c' - c \in S_B$) is necessary and sufficient for the*

feasibility of the fractional game; that is, the solitaire cone S_B is the cone of all feasible fractional games.

The condition $c' - c \in S_B$ is therefore a necessary condition for the feasibility of the original peg game and, more usefully, provides a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for S_B which is violated by $c' - c$. According to (4), page 71, these inequalities “were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway...”. They are also known as *resource counts*, and are discussed in some detail in CONWAY (5). The strongest such inequalities are induced by the facets of S_B .

Other tools to show the infeasibility of various peg games include the so-called *rule-of-three* which simply amounts to color the board by diagonals of α, β and γ (in either direction). Then, with $\#\alpha$ ($\#\beta, \#\gamma$ resp.) denoting the number of pegs in an α -colored (β, γ resp.) holes, one can check that the parity of $\#\alpha - \#\beta, \#\beta - \#\gamma$ and $\#\gamma - \#\alpha$ is an invariant for the moves. The rule-of-three was apparently first exposed in 1841 by SUREMAIN DE MISSERY; see BEASLEY’s book (4) for a detailed historical background. Another necessary condition generalizing the rule-of-three - the *solitaire lattice criterion* - is to check if $c' - c$ belong to the *solitaire lattice* generated by all integer linear combinations of moves, that is:

$$c' - c = \sum_{i=1}^{n_B} y_i m_i, \quad y_i \in \mathbb{Z}, \quad i = 1, \dots, n_B$$

While the lattice criterion is shown to be equivalent to the rule-of-three for the classical English 33-board and French 37-board as well as for any $m \times n$ board, the lattice criterion is stronger than the rule-of-three for games played on more complex boards. In fact, for a wide family of boards the lattice criterion exponentially outperforms the rule-of-three, see DEZA AND ONN (7).

The solitaire cone is generated by a set of extreme rays, each of which is all zero except for three nonzero components which are 1,-1,-1. In AVIS AND DEZA (3), the solitaire cone is related to another cone with the same property, the *flow cone* which is dual to the much studied *metric cone* which arises in the study of multicommodity flows; see, for example, (1; 6; 8; 10).

2 Facets of the Solitaire Cone

For simplicity we consider rectangular boards and, to avoid the special effects created by the boundary, we study their toric closures which are simply called

toric boards. In other words, the toric m by n board is an m by n rectangular board with additional jumps which traverse the boundary. Note that the associated toric solitaire cone $S_{m \times n}$ is pointed and full-dimensional for $m \geq 3$ or $n \geq 3$. Let B be a rectangular m by n board, with $m \geq 3$ or $n \geq 3$. Using the notation described following equation 2, we will represent the coefficients of the facet inducing inequality

$$az \leq 0 \tag{3}$$

by the m by n array $a = [a_{i,j}]$. Inequality 3 holds for every $z \in S_{m \times n}$. It is a convenient abuse of terminology to refer to a as a *facet* of $S_{m \times n}$. Three consecutive row or column elements of an m by n array are denoted by (t_1, t_2, t_3) and called a *consecutive triple* of row or column indices. For example both $t_1 = i, j, t_2 = i, j + 1, t_3 = i, j + 2$ and $t_1 = i + 2, j, t_2 = i + 1, j, t_3 = i, j$ are consecutive triples. Using this notation we see that a move matrix for B is an m by n matrix that is all zero except for elements of some consecutive triple which take the values 1,-1,-1. Each consecutive triple defines a *triangle inequality*

$$a_{t_1} \leq a_{t_2} + a_{t_3} \tag{4}$$

The definition of *consecutive triple* is extended by allowing row indices to be taken modulo m and column indices to be taken modulo n . For example, for a 4 by 4 toric board both $t_1 = 2, 3, t_2 = 2, 4, t_3 = 2, 1$ and $t_1 = 1, 3, t_2 = 4, 3, t_3 = 3, 3$ are consecutive triples. Similarly we extend the definition of a consecutive string of entries to include strings that traverse the boundary.

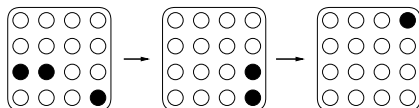


Fig. 2. Moves on the 4 by 4 toric board

The $\{0, 1\}$ -valued facets the solitaire cone are considerably more complex than the $\{0, 1\}$ -valued facets of the dual metric cone, which are generated by cuts in the complete graph. For a toric board B , a complete characterization of $\{0, 1\}$ -valued facets of $S_{m \times n}$ was given in (3). Let a be an m by n 0-1 matrix. We define the 1-graph G_a on a as follows: vertices of G_a correspond to the ones, and two ones are adjacent if the corresponding coefficients are in some consecutive triple where the remaining coefficient is zero. Note that in fact there must be at least two such triples since if (t_1, t_2, t_3) is such a triple then so is (t_3, t_2, t_1) .

Theorem 2 *Let B be the m by n toric board. A m by n 0-1 matrix a is a facet of $S_{m \times n}$ if and only if (i) no nonzero row or column contains two consecutive zeroes, and (ii) the 1-graph G_a is connected.*

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Fig. 3. Two pagoda functions of $S_{4 \times 4}$, only the first one being a facet

Theorem 2 is useful for proving large classes of 0-1 matrices are facets. Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ be two vectors. We say the m by n matrix a is the *product* of x and y if for all $1 \leq i \leq m$ and $1 \leq j \leq n$ $a_{i,j} = x_i y_j$. A simple application of Theorem 2 gives:

Corollary 3

- (1) Let $B_{1,n}$ be the 1 by n toric board. A $\{0, 1\}$ -valued n -vector is a facet of $S_{1 \times n}$ if and only if it has no pair of consecutive zeroes, no string of five or more ones, and at most one string of four ones.
- (2) Let $B_{m,n}$ be the m by n toric board. The product of two $\{0, 1\}$ -valued facets of $S_{1 \times m}$ and $S_{1 \times n}$ gives a $\{0, 1\}$ -valued facet of $S_{m \times n}$.

Proposition 4 Let $h(n)$ be the number of $\{0, 1\}$ -valued facets of $S_{1 \times n}$. For $n \geq 7$, $(n + 18)1.46^{n-8} \leq h(n) \leq (n + 19)1.47^{n-6}$.

PROOF. The formula can be verified directly for $n \leq 11$ by referring to the 5th column of Table 1. We define an f -vector to be a $\{0, 1\}$ -valued vector of length n with no 2 consecutive zeroes, no string of 4 or more ones and starting and ending with a one. We first count $f(n)$, the number of f -vectors. Direct calculation shows that: $f(2) = 1, f(3) = 2, f(4) = 2, f(5) = 4$. For $n \geq 6$, a f -vector has the form $[1 \ 0 \ 1 \ \dots \ 1]$, $[1 \ 1 \ 0 \ 1 \ \dots \ 1]$ or $[1 \ 1 \ 1 \ 0 \ 1 \ \dots \ 1]$, where the string $1 \ \dots \ 1$ is an f -vector. In other words, we have $f(n) = f(n - 2) + f(n - 3) + f(n - 4)$. It is easy to show by induction that for $n \geq 6$,

$$1.46^{n-2} \leq f(n) \leq 1.47^n. \tag{5}$$

Now, for $n \geq 12$, by Item (1) of Corollary 3, the number $h(n)$ of $\{0, 1\}$ -valued facets of $S_{1 \times n}$ is the number of toric $\{0, 1\}$ -valued vectors of length n with no 2 consecutive zeroes, no string of 5 or more ones, and at most one string of 4 ones. Call such vectors h -vectors. If an h -vector has no string of 4 ones, then it either starts with $[0 \ 1 \ \dots \ 1]$, $[1 \ 0 \ 1 \ \dots]$, $[1 \ 1 \ 0 \ 1 \ \dots \ 1 \ 0]$, $[1 \ 1 \ 0 \ 1 \ \dots \ 1 \ 0 \ 1]$ or $[1 \ 1 \ 1 \ 0 \ 1 \ \dots \ 1 \ 0]$ where the string $1 \ \dots \ 1$ is an f -vector. In other words, we have $2f(n - 1) + f(n - 4) + 2f(n - 5) = 2f(n - 3) + 3f(n - 4) + 4f(n - 5)$ h -vectors without a string of 4 ones. We have $nf(n - 6)$ h -vectors with one string of 4 ones as each is of the form $[\dots \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ \dots]$. Therefore, the

total number of h -vectors for $n \geq 8$ is given by

$$h(n) = 2f(n - 3) + 3f(n - 4) + 4f(n - 5) + nf(n - 6). \quad (6)$$

The proposition follows by substituting the asymptotic bounds for f obtained above in this equation.

3 The Binary Solitaire Cone and Other relatives

The link with the dual metric cone and the similarities between their combinatorial structures - see (3) - leads to the study of a dual cut cone analogue; that is, the *binary solitaire cone* BS_B generated by the $\{0, 1\}$ -valued facets of the solitaire cone. We give some results and conjectures on the combinatorial and geometric properties of the binary solitaire cone. In particular we prove that the extreme rays of S_B are extreme rays of BS_B strengthening the analogy with the dual metric cone, for which the extreme rays are also extreme rays of the dual cut cone. Other related cones are also considered.

3.1 The binary solitaire cone

The dual cut cone is generated by the $\{0, 1\}$ -valued facets of the dual metric cone. Similarly, we consider the cone generated by the $\{0, 1\}$ -valued facets of the solitaire cone. This cone is called *binary solitaire cone* and denoted BS_B .

We present in details some small dimensional cases and give some results and conjectures on the combinatorial and geometric properties of the binary solitaire cone. In particular, we investigate the diameter, adjacency and incidence relationships of the binary solitaire cone $BS_{m \times n}$ and its dual $BS_{m \times n}^*$. Two extreme rays (resp. facets) of a polyhedral cone are *adjacent* if they belong to a face of dimension (resp. codimension) two. The number of rays (resp. facets) adjacent to the ray r (resp. facet F) is denoted A_r (resp. A_F). A ray and a facet are *incident* if the ray belongs to the facet. We denote by I_r (resp. I_F) the number of facets (resp. rays) incident to the ray r (resp. facet F). The diameter of BS_B (its dual BS_B^* resp.), that is, the smallest number δ such that any two vertices can be connected by a path with at most δ edges, is $\delta(BS_B)$ ($\delta(BS_B^*)$ resp.); see Table 1.

Finding all extreme rays of the cone BS_B (such as the 930 048 rays of $BS_{4 \times 4}$) is an example of a *convex hull* or *vertex enumeration* problem, for which various computer programs are available. The computational results in this paper were obtained using the double description method *cdd* implemented by

Table 1
Small binary toric boards

Board	#rays	I_r	A_r	#facets	I_F	A_F	$\delta(BS_B)$	$\delta(BS_B^*)$
1×3	3	2	2	3	2	2	1	1
1×4	8	3	3	6	4	4	3	2
1×5	15	4~5	4~5	10	6~8	5~7	3	2
1×6	30	5~6	5~6	11	12~16	8~9	4	2
1×7	42	8~11	7~10	21	18~20	11~13	3	2
1×8	72	10~16	9~14	30	24~36	14~18	3	2
1×9	126	9~25	9~19	48	32~54	18~26	3	2
1×10	200	14~36	12~24	67	40~80	16~33	4	2
1×11	231	11~58	11~26	110	50~86	24~45	4	3
1×12	516	12~85	12~51	159	60~172	21~62	4	3
3×3	18	11	15	15	12~14	12~14	2	2
4×4	930 048	15~168	15~?	340	?	?	?	?
3×4	1 284	11~(30 34)	11~369	54	194~506	35~52	3	2
3×5	101 444	14~(118 129)	14~14 607	240	?	?	?	?

FUKUDA (9), and the reverse search method *lrs* implemented by AVIS (2). The diameters of cones were computed using *graphy* implemented by FUKUDA (9).

Theorem 5 *The extreme rays of the solitaire cone, that is, the moves, are extreme rays of the binary solitaire cone.*

PROOF. *Given any extreme ray (move) c of $S_{m \times n}$, let Φ be the intersection of all the facets of $BS_{m \times n}$ containing c . We want to prove that any vector $r \in \Phi$ is a scalar multiple of c .*

Case $m \leq 2$. *First, take $m = 1$. For $n = 3, \dots, 12$ Theorem 5 was checked by computer so we can assume that $n \geq 13$. All extreme rays of $S_{1 \times n}$ being equivalent up to scrolling and reversing, we can assume that $c = [-1 \ -1 \ 1 \ 0 \ \dots \ 0]$. For $j = 4, \dots, n$, consider the two inequalities defined by $f_{1j}^1 r \leq 0$ and $f_{1j}^0 r \leq 0$ as given bellow where the boxed value is the j^{th} coordinate*

$$f_{1j}^1 = 1, 0, 1, 0, 1, 0, \dots, 0, 1, 0, 1, \boxed{1}, 1, 0, 1, 0, \dots$$

$$f_{1j}^0 = 1, 0, 1, 0, 1, 0, \dots, 0, 1, 0, 1, \boxed{0}, 1, 0, 1, 0, \dots$$

Since the associated 1-graphs $G_{f_{1j}^1}$ and $G_{f_{1j}^0}$ are connected, Theorem 2 gives that those inequalities induce 2 facets F_{1j}^1 and F_{1j}^0 of $BS_{1 \times n}$. As clearly $f_{1j}^1 c = f_{1j}^0 c = 0$, we have $F_{1j}^1 \cap F_{1j}^0 \subset \Phi$. Therefore, any vector $r \in \Phi$ satisfies $f_{1j}^1 r = f_{1j}^0 r = 0$ for $j = 4 \dots n$. This implies $r_j = 0$ for $j = 4 \dots n$. Moreover, the two

inequalities defined by $f_{1,1}r \leq 0$ and $f_{1,2}r \leq 0$ as given bellow

$$f_{1,1} = 0, 1, 1, 0, 1, 0, 1, \dots \quad f_{1,2} = 1, 0, 1, 0, 1, 0, 1, \dots$$

clearly induce 2 facets also belonging to Φ . It implies $f_{1,1}r = f_{1,2}r = 0$, that is, $r_2 + r_3 = r_1 + r_3 = 0$. In other words $r = r_3 \times c$; which completes the proof. Since the case $m = 2$ is almost equivalent to the case $m = 1$, we next consider case $m = 3$.

Case $m = 3$. For $n = 3, 4, 5$ Theorem5 was checked by computer so we can assume that $n \geq 6$. The two cases $c'_{1,1} = c'_{1,2} = -c'_{1,3} = -1$ and $c''_{1,1} = c''_{2,1} = -c''_{3,1} = -1$ being essentially the same, we can assume that $c = c'$. For $i = 1, 2, 3$ and $j = 4, \dots, n$, consider the inequalities defined by $f_{ij}^1 r \leq 0$ ($f_{ij}^0 r \leq 0$ resp.) as given bellow where the boxed value is the ij^{th} coordinate. The coordinates of f_{ij}^0 differ from f_{ij}^1 only for the ij^{th} coordinate which is set to 0.

$$f_{ij}^1 = \begin{bmatrix} 1 & 0 & 1 & 1 & \dots & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 1 & 0 & 1 & \boxed{1} & 1 & 0 & 1 & 1 & 0 & 1 & \dots \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \dots \end{bmatrix}$$

Similarly to the case $m \leq 2$, the associated 1-graphs $G_{f_{ij}^1}$ and $G_{f_{ij}^0}$ are connected and, up to a rotation along the axis $i = 2$, we have $f_{ij}^1 c' = f_{ij}^0 c' = 0$; that is, the induced facets satisfy $F_{ij}^1 \cap F_{ij}^0 \subset \Phi$. Therefore, any vector $r \in \Phi$ satisfies $f_{ij}^1 r = f_{ij}^0 r = 0$ for $i = 1, 2, 3$ and $j = 4, \dots, n$. This implies $r_{ij} = 0$ for $i = 1, 2, 3$ and $j = 4, \dots, n$. Slightly modified f_{ij}^1 and f_{ij}^0 for $i = 2, 3$ and $j = 1, 2, 3$ give $r_{ij} = 0$ for $i = 2, 3$ and $j = 1, 2, 3$. Moreover, the two inequalities defined by $f_{1,1}r \leq 0$ and $f_{1,2}r \leq 0$ as given bellow

$$f_{1,1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & \dots \end{bmatrix} \quad f_{1,2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots & 1 & 0 & 1 & \dots \end{bmatrix}$$

implies $r_{1,1} = r_{1,2} = -r_{1,3}$, that is, $r = r_3 \times c'$; which completes the proof.

Case $m \geq 4$. Theorem5 was checked by computer for $n = 4$ so we can assume that $n \geq 5$. We can take $c_{ij} = 0$ except $c_{1,1} = c_{1,2} = -c_{1,3} = -1$. For $i = 3, \dots, m - 1$ or $j = 4, \dots, n$, consider the inequalities defined by $f_{ij}^1 r \leq 0$ ($f_{ij}^0 r \leq 0$ resp.) as given bellow where the boxed value is the ij^{th} coordinate. The coordinates of f_{ij}^0 differ from f_{ij}^1 only for the ij^{th} coordinate which is set

to 0.

$$f_{ij}^1 = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & \boxed{1} & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

Clearly, we have $f_{ij}^1 c = f_{ij}^0 c = 0$. By filling the gaps with the following $\{0, 1\}$ -valued matrices (or their transposes), the associated 1-graphs $G_{f_{ij}^1}$ and $G_{f_{ij}^0}$ are both connected. Therefore, any vector $r \in \Phi$ satisfies $r_{ij} = 0$ for $i = 3, \dots, m-1$ or $j = 4, \dots, n$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

Slightly modified f_{ij}^1 and f_{ij}^0 for $i = 2, m$ and $j = 1, 2, 3$ give $r_{ij} = 0$ for $i = 2, m$ and $j = 1, 2, 3$. For example, the following inequalities set $r_{2,2}$ to 0

$$f_{2,2}^1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & \boxed{1} & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \quad f_{2,2}^0 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & \boxed{0} & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

Finally, the following two inequalities set $r_{1,1} = r_{1,2} = -r_{1,3}$,

$$f_{1,1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \quad f_{1,2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

that is, $r = r_3 \times c$; which completes the proof.

Corollary 6 *The binary solitaire cone is full-dimensional.*

Out of the 930 048 extreme rays of $BS_{4 \times 4}$, the 64 extreme rays of $S_{4 \times 4}$, that is, the moves, reached the highest incidence $I_r^{max} = 168$ which is almost three times larger than the second highest incidence $I_r^{submax} = 57$. Similarly, out of the 101 444 extreme rays of $BS_{3 \times 5}$, the 15 vertical moves of $S_{3 \times 5}$, reached the highest adjacency $A_r^{max} = 14\,607$ while the average adjacency is $A_r^{ave} \simeq 33.16$. These computational results and other similarities with the metric cone - see (3) - lead us to the following conjectures:

Conjecture 7

- (1) *For $n \geq 3$ and $m \geq 3$, the moves form a dominating set in the skeleton of $BS_{m \times n}$.*
- (2) *The incidence of the moves is maximal in the skeleton of $BS_{m \times n}$.*
- (3) *For m, n large enough, at least one ray r of $BS_{m \times n}$ is simple, (that is, $I_r = mn - 1$).*

Item (1) of Conjecture 7 holds for $BS_{3 \times 4}$ and is false for $m \leq 2$. The smallest 1 by n board for which the conjecture fails is the 1 by 10 board. Item (2) holds for all cones presented in Table 1 and is false if we replace the incidence by the adjacency as, for example, for $BS_{3 \times 5}$. If true, Item (3) would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of $BS_{m \times n}$ are equal to $nm - 1$. This holds for $BS_{3 \times 4}$, $BS_{3 \times 5}$ and $BS_{4 \times 4}$.

3.2 *The trellis solitaire cone*

The $\{0, 1\}$ -valued facets of the solitaire cone have much less structure than the set of cut metrics. In fact, the cut metrics are related to products of vectors of length n . This motivates the next definition. Let f and g be $\{0, 1\}$ -valued

vectors of length m and n respectively, and let $c_{ij} = f_i \cdot g_j$ for $i = 1, \dots, m$, $j = 1, \dots, n$. If $c \cdot x \leq 0$ defines a facet of $BS_{m \times n}$, we call it a *trellis facet*. The *trellis solitaire cone* TS_B is generated by all of the *trellis facets* of the binary solitaire cone BS_B . See Item (2) of Corollary 3 for an easy construction of trellis facets. For example, among the two following facets of $BS_{3 \times 5}$, only the right one is a trellis facet.

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

Fig. 4. A facet and a trellis facet of $BS_{3 \times 5}$

Conjecture 8 *The binary trellis solitaire cone is full-dimensional.*

3.3 The complete solitaire cone

The *complete solitaire cone* CS_B is induced by a variation of the Solitaire game. To the classical moves we add the moves which consist of removing two pegs surrounding an empty hole and placing one peg in this empty hole as shown in Fig. 5. The incidence and adjacency relationships and diameters of small dimensional complete solitaire cones are presented in Table 2. Two rays

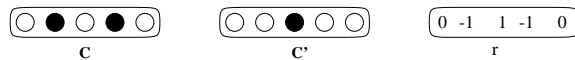


Fig. 5. The extreme ray r of $CS_{1 \times 5}$ corresponding to the move from c to c'

Table 2

Small complete toric boards

Board	#rays	I_r	A_r	#facets	I_F	A_F	$\delta(CS_B)$	$\delta(CS_B^*)$
1×7	21	26~33	16~17	91	6~12	6~32	2	4
1×8	24	119~130	19~20	404	7~12	7~44	2	5
3×3	18	11	15	15	12~14	12~14	2	2
3×4	48	904~1 192	39~45	3 576	11~33	11~738	2	?

are called *strongly conflicting* if there exist two pairs i, j and k, l such that the two rays have nonzero coordinates of distinct signs at positions i, j and k, l (respectively i, j). We have $CS_{3 \times 3} = S_{3 \times 3}$ and, by analogy with the classical solitaire cone case, we conjecture:

Conjecture 9 *For $n \geq 7$ and $m \geq 7$ a pair of extreme rays of $CS_{m \times n}$ are adjacent if and only if they are not strongly conflicting.*

If true, Conjecture 9 would imply that $\delta(CS_{m \times n}) = 2$.

3.4 The binary complete solitaire cone

In the same way as we did for the solitaire cone, we consider the cone generated by the facets of the complete solitaire cone CS_B whose coordinates are, up to a constant multiplier, $\{0, 1\}$ -valued. This cone is called *complete binary solitaire cone* and denoted BCS_B . The incidence and adjacency relationships of small dimensional complete binary solitaire cones BCS_B are presented in Table 3.

Table 3
Small binary complete toric boards

Board	#rays	I_r	A_r	#facets	I_F	A_F	$\delta(BCS_B)$	$\delta(BCS_B^*)$
1×7	7	6	6	7	6	6	1	1
1×8	16	9	10	12	12	9~10	2	2
3×3	18	11	15	15	12~14	12~14	2	2
3×4	72	14~28	15~58	36	30~45	30~35	2	2

4 Conclusion

Theorem 5 strengthens the analogy of the solitaire cone with the dual metric cone, for which the extreme rays are also extreme rays of the dual cut cone. On the other hand, so far we have not yet found an analogue of the *hypermetric* facets of the metric cone M_n , that is, a “nice” family of $\{0, -1, 1\}$ -valued extreme rays of the binary solitaire cone BS_B . Another open question is the determination of a tighter relaxation of the solitaire cone S_B by some *cuts analogue*. The trellis solitaire cone TS_B is a candidate as well as the cone generated by the $\{0, 1\}$ -valued facets with the minimal number of ones. For $S_{4 \times 4}$ and $S_{3 \times i} : i = 3, 4, 5$, these facets have maximal incidence and adjacency in the skeleton of $S_{m \times n}^*$.

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