On a conjecture of Baïou and Balinski

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Abstract

We disprove a conjecture of Baïou and Balinski concerning a variation on the Birkhoff-von Neumann theorem.

Let $\mathcal{BB}_n$ denote the set of all integer $n \times n$ matrices X that satisfy

$$\sum_{j=1}^{n} X(i,j) \leq 1 \qquad \text{for all } i = 1, 2, \ldots, n \tag{1}$$

$$\sum_{i=1}^{n} X(i,1) \leq 1 \tag{2}$$

$$\sum_{i=1}^{n} X(i,j) \leq \sum_{i=1}^{n} X(i,j-1) \qquad \text{for all } j = 2, 3, \ldots, n \tag{3}$$

$$X(i,j) \geq 0 \qquad \text{for all } i \text{ and } j; \tag{4}$$

to put it differently, $\mathcal{BB}_n$ is the set of all matrices that arise from $n \times n$ permutation matrices by first choosing an integer k such that $0 \leq k \leq n$ and then overwriting all entries in the last k columns with zeros. This set of matrices has been introduced by Mourad Baïou and Michel Balinski. At first (Baïou and Balinski (1998)), they asserted that an arbitrary $n \times n$ matrix X belongs to the convex hull of $\mathcal{BB}_n$ if and only if it satisfies (1), (2), (3), (4) as well as all inequalities

$$\sum_{i \in I} \sum_{j=1}^{n} X(i,j) \leq \sum_{i=1}^{n} \sum_{j=1}^{|I|} X(i,j) \tag{5}$$

where $I \subset \{1, 2, \ldots, n\}$. Later (Baïou and Balinski (1998)), they realized that this assertion is false — matrix

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}$$

is an extreme point of the polytope of all X in $\mathbf{R}^{n \times n}$ that satisfy (1), (2), (3), (4), and so it constitutes a counterexample — and conjectured that an arbitrary $n \times n$ matrix X belongs to the convex hull of $\mathcal{BB}_n$ if and only if it satisfies (1), (2), (3), (4) as well as all inequalities

$$\sum_{i \notin I} \sum_{j \in J} X(i,j) \geq \sum_{i \in I} \sum_{j=k+1}^{n} X(i,j) \tag{6}$$

where $k \in \{1, 2, \ldots, n-1\}$, $I \subseteq \{1, 2, \ldots, n\}$, $J \subseteq \{1, 2, \ldots, k\}$, and $|I| = |J|$. The purpose of our note is to point out that this conjecture, too, is false.

Theorem 1 *For all positive integers n greater than four, every X in $\mathcal{BB}_n$ satisfies*

$$\sum_{i=4}^{n} X(i,1) + \sum_{i=2}^{n} X(i,2) - X(1,3) + \sum_{i=4}^{n} X(i,4) - \sum_{i=1}^{3}\sum_{j=5}^{n} X(i,j) \geq 0 \qquad (7)$$

and inequality (7) induces a facet of the convex hull of $\mathcal{BB}_n$.

Proof. First, consider an arbitrary X in $\mathcal{BB}_n$. We have

$$\begin{aligned}
\sum_{i=4}^{n} X(i,1) + \sum_{i=2}^{n} X(i,2) - X(1,3) + \sum_{i=4}^{n} X(i,4) - \sum_{i=1}^{3}\sum_{j=5}^{n} X(i,j) &\geq \\
\sum_{i=1}^{n}(X(i,1) + X(i,2) + X(i,4)) - \sum_{j=1}^{n}(X(1,j) + X(2,j) + X(3,j)) &\geq \\
\sum_{i=1}^{n}(X(i,1) + X(i,2) + X(i,4)) - 3,&
\end{aligned}$$

and so (7) holds as long as $\sum_{i=1}^{n} X(i,4) = 1$. If $\sum_{i=1}^{n} X(i,4) = 0$, then

$$\begin{aligned}
\sum_{i=4}^{n} X(i,1) + \sum_{i=2}^{n} X(i,2) - X(1,3) + \sum_{i=4}^{n} X(i,4) - \sum_{i=1}^{3}\sum_{j=5}^{n} X(i,j) &= \\
\sum_{i=4}^{n} X(i,1) + \sum_{i=2}^{n} X(i,2) - X(1,3) &\geq \\
\sum_{i=1}^{n} X(i,2) - \sum_{j=1}^{n} X(1,j) &\geq \\
\sum_{i=1}^{n} X(i,2) - 1,&
\end{aligned}$$

and so (7) holds as long as $\sum_{i=1}^{n} X(i,2) = 1$. If $\sum_{i=1}^{n} X(i,2) = 0$, then

$$\begin{aligned}
\sum_{i=4}^{n} X(i,1) + \sum_{i=2}^{n} X(i,2) - X(1,3) + \sum_{i=4}^{n} X(i,4) - \sum_{i=1}^{3}\sum_{j=5}^{n} X(i,j) &= \\
\sum_{i=4}^{n} X(i,1) &\geq 0,
\end{aligned}$$

and so (7) holds again.

Next, let $\mathcal{S}_n$ denote the set of all X in $\mathcal{BB}_n$ that satisfy

$$\sum_{i=4}^{n} X(i,1) + \sum_{i=2}^{n} X(i,2) - X(1,3) + \sum_{i=4}^{n} X(i,4) - \sum_{i=1}^{3}\sum_{j=5}^{n} X(i,j) = 0 \qquad (8)$$

and consider an arbitrary equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A(i,j)X(i,j) = b \qquad (9)$$

that is satisfied by all X in $\mathcal{S}_n$. To prove that (7) induces a facet of the convex hull of $\mathcal{BB}_n$, we will show that (9) is a multiple of (8).

Since the all zeros matrix belongs to $\mathcal{S}_n$, we have

$$b = 0.$$

For each $i = 1, 2, 3$, the X in $\mathcal{BB}_n$ defined by $X(i,1) = 1$ and $X(r,s) = 0$ whenever $s > 1$ belongs to $\mathcal{S}_n$; it follows that

$$A(1,1) = A(2,1) = A(3,1) = 0.$$

The X in $\mathcal{BB}_n$ defined by $X(2,1) = X(1,2) = 1$ and $X(r,s) = 0$ whenever $s > 2$ belongs to $\mathcal{S}_n$; it follows that

$$A(1,2) = 0.$$

For every choice of i such that $2 \le i \le n$, there is an i' in $\{2,3\} - \{i\}$; the X in $\mathcal{BB}_n$ defined by $X(i',1) = X(1,2) = X(i,3) = 1$ and $X(r,s) = 0$ whenever $s > 3$ belongs to $\mathcal{S}_n$; it follows that

$$A(i,3) = 0 \;\text{ whenever }\; 2 \le i \le n.$$

For either choice of i in $\{2,3\}$, there is an i' in $\{2,3\} - \{i\}$; the X in $\mathcal{BB}_n$ defined by $X(i',1) = X(1,2) = X(4,3) = X(i,4) = 1$ and $X(r,s) = 0$ whenever $s > 4$ belongs to $\mathcal{S}_n$; it follows that

$$A(2,4) = A(3,4) = 0.$$

For every choice of i and j such that $4 \leq i \leq n$ and $5 \leq j \leq n$, there are an i' in $\{4,5\} - \{i\}$ and an X in $\mathcal{S}_n$ such that $X(2,1) = X(1,2) = X(i',3) = X(3,4) = X(i,j) = 1$ and $X(r,s) = 0$ whenever $s > j$; it follows by induction on j that

$$A(i,j) = 0 \quad \text{whenever } 4 \leq i \leq n \text{ and } 5 \leq j \leq n.$$

For every choice of i such that $2 \leq i \leq n$, there is an i' in $\{2,3\} - \{i\}$; the X in $\mathcal{BB}_n$ defined by $X(i',1) = X(i,2) = X(1,3) = 1$ and $X(r,s) = 0$ whenever $s > 3$ belongs to $\mathcal{S}_n$; it follows that

$$A(i,2) = -A(1,3) \quad \text{whenever} \quad 2 \leq i \leq n.$$

For every choice of i in $\{1,2,3\}$ and j such that $5 \leq j \leq n$, there are an i' in $\{1,2,3\} - \{i\}$ and an i'' in $\{2,3\} - \{i\}$, such that $i' \neq i''$; every X in $\mathcal{BB}_n$ such that $X(i',1) = X(4,2) = X(5,3) = X(i'',4) = X(i,j) = 1$ belongs to $\mathcal{S}_n$; it follows that

$$A(1,j) = A(2,j) = A(3,j) = A(1,3) \quad \text{whenever} \quad 5 \leq j \leq n.$$

For every choice of i such that $4 \leq i \leq n$, there is an i' in $\{4,5\} - \{i\}$; every X in $\mathcal{BB}_n$ such that $X(i,1) = X(1,2) = X(i',3) = X(2,4) = X(3,5) = 1$ belongs to $\mathcal{S}_n$; it follows that

$$A(i,1) = -A(1,3) \quad \text{whenever} \quad 4 \leq i \leq n.$$

Every X in $\mathcal{BB}_n$ such that $X(2,1) = X(4,2) = X(5,3) = X(1,4) = X(3,5) = 1$ belongs to $\mathcal{S}_n$; it follows that

$$A(1,4) = 0.$$

For every choice of i such that $4 \leq i \leq n$, there is an i' in $\{4,5\} - \{i\}$; every X in $\mathcal{BB}_n$ such that $X(2,1) = X(1,2) = X(i',3) = X(i,4) = X(3,5) = 1$ belongs to $\mathcal{S}_n$; it follows that

$$A(i,4) = -A(1,3) \quad \text{whenever} \quad 4 \leq i \leq n.$$

References

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M. Baïou and M. Balinski (1999), "A Conjecture", manuscript dated 9 February 1999.