The Visual Computer

Computing the volume of the union of spheres

David Avis[1],
Binay K. Bhattacharya[2],
and Hiroshi Imai[3]

[1] School of Computer Science, McGill University, 805 Sherbrooke Street West, Montreal, PQ, Canada H3A 2K6
[2] School of Computer Science, Simon Fraser University, Burnaby, BC, Canada V5A 1S6
[3] Department of Computer Science and Communication Engineering, Kyushu University, Fukuoka 812, Japan

On $O(n^2)$ exact algorithm is given for computing the volume of a set of n spheres in space. The algorithm employs the Laguerre Voronoi (power) diagram and a method for computing the volume of the intersection of a simplex and a sphere exactly. We give a new proof of a special case of a conjecture, popularized by Klee, concerning the change in volume as the centres of the spheres become further apart.

Key words: Union of Spheres – Volumes – Laguerre Voronoi diagram – Power diagram

1 Introduction

In this paper we give an $O(n^2)$ exact algorithm for computing the volume of the union of a set of n spheres in space. This problem is of interest in nuclear physics [7, 6] and urban planning [4]. The only other exact method known to the authors is by an application of the inclusion-exclusion principle, giving an exponential running time algorithm. Our approach is to partition space into polygonal cells, with one cell for each sphere, so that the volume of the union can be computed by simply summing the volume of the intersection of each sphere with its corresponding cell. The cell decomposition used is the Laguerre-Voronoi diagram [3, 1], which was used to solve the two dimensional version of the same problem. A critical procedure required is to compute the volume of the intersection of a sphere and three half spaces. Our approach generalized to d-dimensions whenever the measure of the intersection of a hypersphere and d half spaces is computable. This problem is particularly simple for $d=2$, but seems hard for $d \geq 4$.

In the paper, we consider the three-dimensional case in detail. In principle it is possible to obtain a formula for computing the volume of the intersection of a sphere and three half spaces, but such a formula would be extremely complex. We rather adopt the decomposition approach. We decompose the problem, using elementary geometric properties, and show that the volume of the intersection of a sphere and three half spaces can be computed if a formula for computing the volume of the intersection of a sphere and *two* half spaces is available. We also present such a formula, thus giving an exact method of computing the volume of the union of n spheres.

As our model of computation, we adopt the real RAM in Preparata and Shamos [8], in which each word is capable of holding a single real number, and, besides the fundamental arithmetic operations and comparisons, the square root and inverse trigonometric functions are available at unit cost. In computing the volume of the union of spheres, those additional operations seem indispensable.

We also mention an old problem on the volume of the union of spheres, and prove a special case of the conjecture by means of the Laguerre-Voronoi diagram.

2 Laguerre-Voronoi diagram and the measure of the union of spheres

Suppose we are given n hyperspheres S_i in d-dimensional space $\mathbf{R}^d$ with center $\mathbf{x}_i$ and radius $r_i (i=1, \ldots, n)$:

$$S_i=\{\mathbf{x}\in\mathbf{R}^d|(\mathbf{x}-\mathbf{x}_i)^T(\mathbf{x}-\mathbf{x}_i)\leq r_i^2\}.$$

The Laguerre-Voronoi diagram in $\mathbf{R}^d$ for these spheres is defined as follows [3, 1]. We define the distance $d_L(S_i, \mathbf{x})$ from hypersphere S_i to an arbitrary point $\mathbf{x}$ in $\mathbf{R}^d$ by

$$d_L(S_i, \mathbf{x})^2=(\mathbf{x}-\mathbf{x}_i)^T(\mathbf{x}-\mathbf{x}_i)-r_i^2$$

in terms of which we denote the Voronoi region $V(S_i)$ of S_i by

$$V(S_i=\bigcap_j\{\mathbf{x}\,|\,d_L(S_i, \mathbf{x})^2\leq d_L(S_j, \mathbf{x})^2\}.$$

The collection of $V(S_i)$ $(i=1, \ldots, n)$ partitions the space, which will be referred to as the Laguerre-Voronoi diagram. The inequality $d_L(S_i, \mathbf{x})^2\leq d_L(S_j, \mathbf{x})^2$ is obviously linear in $\mathbf{x}$ and determines a half space, and hence every region $V(S_i)$ is a convex polyhedron in $\mathbf{R}^d$. In the two-dimensional case, the boundaries of the regions of the diagram consist of straight line segments, and the diagram can be computed in $O(n\log n)$ time [3]. In d-dimensional case $(d\geq 3)$, the diagram can be constructed in $O(n^{\lfloor(d+1)/2\rfloor})$ time [1, 2]. In the Laguerre-Voronoi diagram, S_i itself may not intersect $V(S_i)$ and some $V(S_i)$ may be empty. Figure 1 depicts a Laguerre-Voronoi diagram for twenty circles.
For $X\subseteq\mathbf{R}^d$, denote by $\mu(X)$ the measure of X.
The reason why the Laguerre-Voronoi diagram is of use in computing the measure of the union of hyperspheres is contained in the following lemma. It says that it suffices to compute the measure of the intersection of each hypersphere with its corresponding Voronoi polyhedron.

Lemma 2.1. $\mu\left(\bigcup_{i=1}^n S_i\right)=\sum_{j=1}^n \mu(S_j\cap V(S_j))$.

Proof. We first show that

$$S_i\cap V(S_j)\subseteq S_j\cap V(S_j) \qquad \text{for any } i,j=1, \ldots, n. \quad (2.1)$$

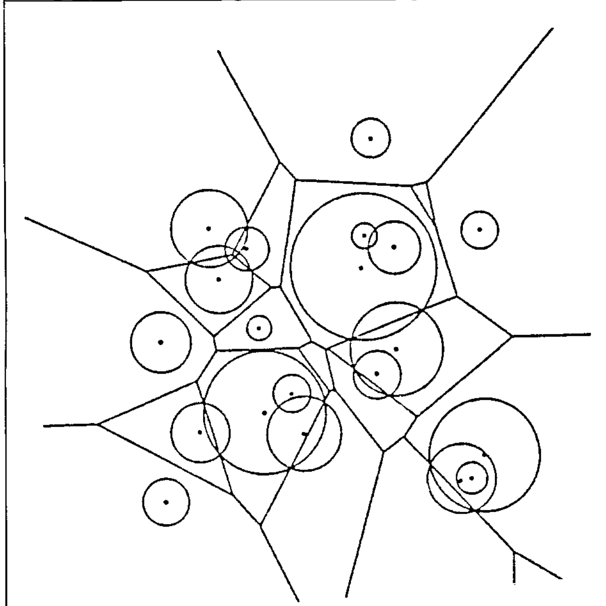


Fig. 1

Indeed,

$$\begin{aligned} S_i\cap V(S_j)&=\bigcap_k\{\mathbf{x}\in\mathbf{R}^d|(\mathbf{x}-\mathbf{x}_j)^T(\mathbf{x}-\mathbf{x}_j)-r_j^2\\ &\quad\leq(\mathbf{x}-\mathbf{x}_k)^T(\mathbf{x}-\mathbf{x}_k)-r_k^2 \text{ and}\\ &\quad\;(\mathbf{x}-\mathbf{x}_i)^T(\mathbf{x}-\mathbf{x}_i)\leq r_i^2\}\\ &\subseteq[\bigcap_k\{\mathbf{x}\in\mathbf{R}^d|(\mathbf{x}-\mathbf{x}_j)^T(\mathbf{x}-\mathbf{x}_j)-r_j^2\\ &\quad\leq(\mathbf{x}-\mathbf{x}_k)^T(\mathbf{x}-\mathbf{x}_k)-r_k^2\}]\\ &\quad\cap\{\mathbf{x}\in\mathbf{R}^d|(\mathbf{x}-\mathbf{x}_j)^T(\mathbf{x}-\mathbf{x}_j)-r_j^2\\ &\quad\leq(\mathbf{x}-\mathbf{x}_i)^T(\mathbf{x}-\mathbf{x}_i)-r_i^2\leq 0\}\\ &\subseteq S_j\cap V(S_j). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mu\left(\bigcup_{i=1}^n S_i\right)&=\sum_{j=1}^n \mu\left(\left(\bigcup_{i=1}^n S_i\right)\cap V(S_j)\right)\\ &\quad[\text{since } V(S_i)\,(i=1, \ldots, n) \text{ partitions the space } \mathbf{R}^d]\\ &=\sum_{j=1}^n \mu\left(\bigcup_{i=1}^n (S_i\cap V(S_j))\right)\\ &=\sum_{j=1}^n \mu(S_j\cap V(S_j)) \text{ [from (2.1)].} \quad \square \end{aligned}$$

Thus, it is seen that, given the Laguerre-Voronoi diagram, the problem of computing the volume of the union of these n spheres is reduced to that of computing $\mu(S_j \cap V(S_j))$. Since $V(S_j)$ is a convex polyhedron, we can easily partition it into simplices, and so we can compute $\mu(S_j \cap V(S_j))$ if the measure of the intersection of a sphere and a simplex can be computed.

We use the following notation in the sequel: given a half space H_k^+, we denote by H_k^0 the hyperplane determining the H_k^+ and define H_k^- to be $\mathbf{R}^d - H_k^+$. Consider a sphere S and a simplex Σ which is the intersection of $d+1$ half spaces H_k^+ $(k=1, \ldots, d+1)$ in $\mathbf{R}^d$. Then we have the following lemma by virtue of a fundamental formula in set theory.

Lemma 2.2. *The measure of the intersection of a sphere and a simplex in* $\mathbf{R}^d$ *can be computed if the measure of the intersection of a sphere and (at most) d half spaces is computable.*

Proof. Since $S \cap \Sigma = S \cap H_1^+ \cap H_2^+ \cap \ldots \cap H_{d+1}^+$, we have

$$\begin{aligned}
&\mu(S \cap \Sigma)\\
&= \mu(S) - \sum_{k=1}^{d+1} \mu(S \cap H_k^-)\\
&\quad + \sum_{1 \le k_1 < k_2 \le d+1} \mu(S \cap H_{k_1}^- \cap H_{k_2}^-)\\
&\quad - \sum_{1 \le k_1 < k_2 < k_3 \le d+1} \mu(S \cap H_{k_1}^- \cap H_{k_2}^- \cap H_{k_3}^-)\\
&\quad + \ldots\\
&\quad + (-1)^d \sum_{1 \le k_1 < k_2 < \ldots < k_d \le d+1} \mu(S \cap H_{k_1}^- \cap H_{k_2}^- \cap \ldots \cap H_{k_d}^-)\\
&\quad + (-1)^{d+1} \mu(S \cap H_1^- \cap H_2^- \cap \ldots \cap H_{d+1}^-) \qquad (2.2)
\end{aligned}$$

In (2.2), since Σ is a simplex, $H_1^- \cap H_2^- \cap \ldots \cap H_{d+1}^-$ is empty, and we obtain the lemma. □

In the two-dimensional case, it is easy to see that the area of the intersection of a circle and two half planes can be computed in a constant time, as is described in Sect. 4. In this case, the Laguerre-Voronoi diagram for n circles can be constructed in $O(n \log n)$ time [3] and a convex polygon can be triangulated in linear time, so that the area of the union of n circles can be computed in $O(n \log n)$ time in total.

In the three-dimensional case, it is not trivial to compute the volume of the intersection of a sphere and three half spaces. A procedure for this will be given in the next section.

3 Computing the volume of the intersection of a sphere and three half spaces

Let S be a sphere and H_1^+, H_2^+, H_3^+ be half spaces in the three-dimensional space such that $C^+ \equiv H_1^+ \cap H_2^+ \cap H_3^+$ is a nonempty cone with apex p. We denote by F_k^+ the face of the cone C^+ on $H_k^0 (k=1, 2, 3)$. We also define a cone C^- to be the closure of $H_1^- \cap H_2^- \cap H_3^-$, and, similarly denote its faces by $F_k^- (k=1, 2, 3)$. We consider the problem of computing the volume of $S \cap C^+$ by simple case analysis.

Case 1. $p \in S$. In this case, let H_4^+ be the half space such that the boundary plane H_4^0 contains all the three points of intersection with the boundary of S and each of three rays of C^+ and such that H_4^+ contains p. Then, we trivially have

$$\mu(S \cap C^+) = \mu(S \cap C^+ \cap H_4^+) + \mu(S \cap C^+ \cap H_4^-).$$

The first term of the right hand side is $\mu\left(\bigcap_{k=1}^{4} H_4^+\right)$ since $C^+ \cap H_4^+ = \bigcap_{k=1}^{4} H_k^+ \subseteq S$. For the second term we need the fact that $S \cap H_4^- \cap H_{k_1}^- \cap H_{k_2}^- = \emptyset$ for $1 \le k_1 \le k_2 \le 3$. To see this, note first that $H_4^- \cap H_{k_1}^- \cap H_{k_2}^-$ is the interior of a simplex pointed at some point p on S with rays $H_4^0 \cap H_{k_1}^0$, $H_4^0 \cap H_{k_2}^0$, $H_{k_1}^0 \cap H_{k_2}^0$ emanating from p (Fig. 2). Since p is on the boundary of S, except at this point, each ray lies entirely outside of S. Hence the simplex intersects S at precisely p, proving the fact. We observe also that $H_4^- \cap H_1^- \cap H_2^- \cap H_3^- = \emptyset$ because $H_4^+ \cap H_1^+ \cap H_2^+ \cap H_3^+ \neq \emptyset$. The second term can therefore be expressed as

$$\begin{aligned}
&\mu((S \cap H_4^-) \cap C^+)\\
&= \mu(S \cap H_4^-) - \sum_{k=1}^{n} \mu(S \cap H_4^- \cap H_k^-)\\
&\quad + \sum_{1 \le k_1 < k_2 \le 3} \mu(S \cap H_4^- \cap H_{k_1}^- \cap H_{k_2}^-)\\
&\quad - \mu(S \cap H_4^- \cap H_1^- \cap H_2^- \cap H_3^-)\\
&= \mu(S \cap H_4^-) - \sum_{k=1}^{n} \mu(S \cap H_4^- \cap H_k^-).
\end{aligned}$$

Thus, in this case, $\mu(S \cap C^+)$ can be computed if the volume of the intersection of a sphere and two half spaces can be computed.

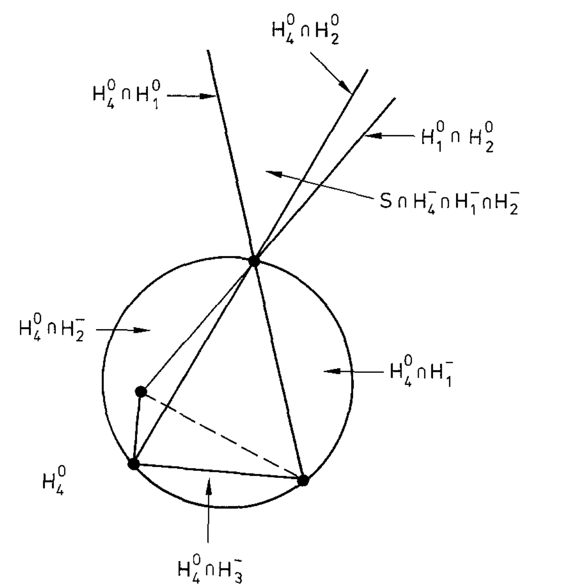


Fig. 2

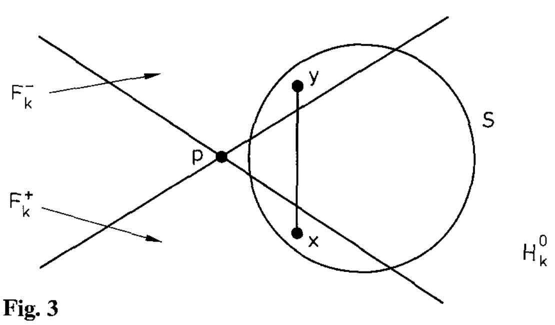


Fig. 3

Case 2. $p \notin S$. If there is a face, say F_1^+, of the cone C^+ that has no intersection with S, we have $\mu(S \cap C^+)=0$ when $S \cap H_1^+ = \emptyset$ and $\mu(S \cap C^+) = \mu(S \cap H_2^+ \cap H_3^+)$ when $S \cap H_1^+ \neq \emptyset$. Therefore, suppose that every face of the cone C^+ intersects S. In this case, we have the following.

Lemma 3.1. *If $p \notin S$ and every face of C^+ intersects S, there is at least one face of the cone C^- having no intersection with S.*

Proof. We first show that if S intersects both F_k^+ and F_k^-, it must intersect at least one of the boundary rays of F_k^+ and one of the boundary rays of F_k^-. Suppose $x \in S \cap F_k^+$ and $y \in S \cap F_k^-$. Then, the line segment $\overline{xy} \subseteq S$ and $p \notin \overline{xy}$ by hypothesis. Therefore, as $\overline{xy} \subseteq H_k^0$, $\overline{xy}$ must intersect a boundary ray of both F_k^+ and F_k^- (Fig. 3).

We may therefore assume that S intersects at least one of the two rays of each face of C^+ and similarly for C^-. Therefore S intersects at least two rays of C^+, which lie on some face F_k^+, at points x and y. S also intersects F_k^- at some point z. But then $p \in \Delta xyz \subset S$, a contradiction (Fig. 4). □

From this proposition, we see that $S \cap H_1^- \cap H_2^- \cap H_3^-$ is the intersection of S and at most two of H_k^- $(k=1, 2, 3)$. Thus, in Case 2, we can compute $\mu(S \cap C^+)$ directly from a formula like (2.2) if the volume of the intersection of a sphere and two half spaces can be computed.

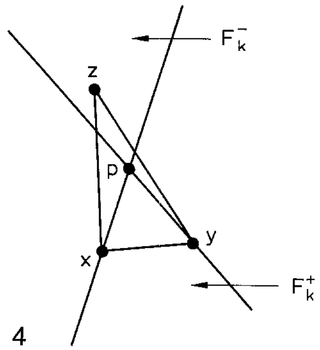
4 Computing the volume of the intersection of a sphere and two half spaces

We begin with how to compute the area of the intersection of a circle with radius r and two distinct half planes. Let A and B be the distances from the center of S to the lines determining the half planes. Let C be the distance from the center of S to the point of intersection of those two lines. We only consider the case with $0 < A \leq B \leq C < r$; other cases are similar or much easier.

The given two lines partition S into four regions. There are two cases 1 and 2 as depicted in Fig. 5. Since the area of a circle and the area of the intersection of a circle and a half plane can be computed easily, we only consider how to calculate the areas $\text{area}_1(r, A, B, C)$ and $\text{area}_2(r, A, B, C)$ of shaded regions in both cases.

We first consider the case 1. Defining two angles θ_A and θ_B and two triangles T_A and T_B as in Fig. 6a, we see that

$$
\begin{aligned}
&\text{area}_1(r, A, B, C)\\
&= \frac{1}{2}(\theta_A + \theta_B)\, r^2 - \mu(T_A) - \mu(T_B)\\
&= \frac{1}{2}\left[\left(\arccos\frac{A}{r} - \arccos\frac{A}{C}\right)\right.\\
&\quad \left. + \left(\arccos\frac{B}{r} - \arccos\frac{B}{C}\right)\right] r^2\\
&\quad - \frac{A}{2}\left(\sqrt{r^2 - A^2} - \sqrt{C^2 - A^2}\right)\\
&\quad - \frac{B}{2}\left(\sqrt{r^2 - B^2} - \sqrt{C^2 - B^2}\right) \qquad (4.1)
\end{aligned}
$$



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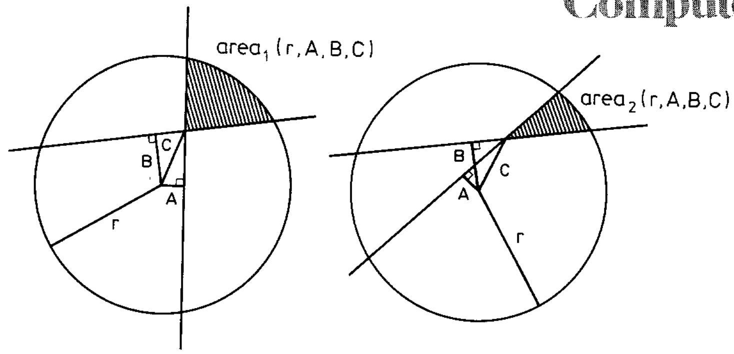
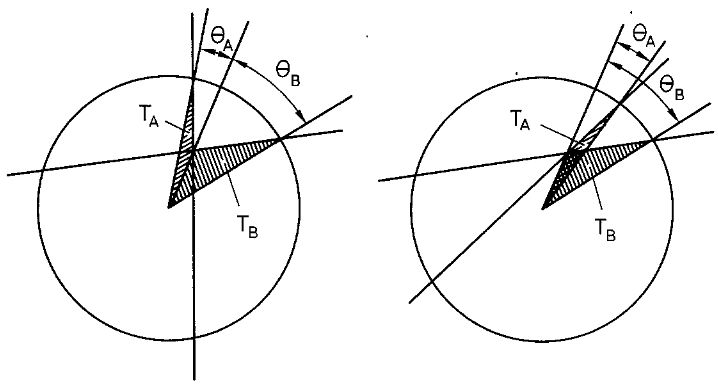


5 A Case 1 B Case 2



6 A Case 1 B Case 2

Figs. 4–6

Supposing that $\text{area}_1(r, A, B, C)$ is redefined by (4.1) as a function of r, A, B and C such that $0<|A|\leq B\leq C<r$, from Fig. 6b, we see that

$$\text{area}_2(r, A, B, C)=\frac{1}{2}(-\theta_A+\theta_B)\,r^2+\mu(T_A)-\mu(T_B)$$
$$=\text{area}_1(r, -A, B, C). \qquad (4.2)$$

We thus can compute the area of the intersection of a circle and two half planes.

We next consider how to compute the volume of the intersection of a sphere S with radius R and two distinct half spaces in three-dimensional space. Let A and B be the distances from the center of S to the two planes determining the half spaces. Let C be the distance from the center of S to the line of intersection of those two planes. We only consider the case with $0<A\leq B\leq C<R$; other cases are similar or easier to handle.

Consider on orthogonal coordinate system such that its origin is the center of S and the x-axis is parallel to the line of intersection of those two planes. Then, the section of the space by the plane $x=\xi$ for ξ with $-\sqrt{R^2-C^2}<\xi<\sqrt{R^2-C^2}$ looks like Fig. 5 with $r=\sqrt{R^2-\xi^2}$. Since we can compute the volume of a sphere and the volume of the intersection of a sphere and a half space easily, and from (4.2), we have only to evaluate

$$\text{volume}_1(R, A, B, C)$$
$$\equiv \int_{-\sqrt{R^2-C^2}}^{\sqrt{R^2-C^2}} \text{area}_1(\sqrt{R^2-x^2}, A, B, C)\,\mathrm{d}x$$
$$=2\int_{0}^{\sqrt{R^2-C^2}} \text{area}_1(\sqrt{R^2-x^2}, A, B, C)\,\mathrm{d}x.$$

To evaluate this, using

$$F(x, \alpha, r)$$
$$\equiv \int\left[(r^2-x^2)\arccos\frac{\alpha}{\sqrt{r^2-x^2}}-\alpha\sqrt{r^2-\alpha^2-x^2}\right]\mathrm{d}x$$
$$=\left(r^2x-\frac{x^3}{3}\right)\arccos\frac{\alpha}{\sqrt{r^2-x^2}}-\frac{2}{3}\alpha x\sqrt{r^2-\alpha^2-x^2}$$

$$+\left(\alpha r^2-\frac{\alpha^3}{3}\right)\arccos\frac{x}{\sqrt{r^2-\alpha^2}}$$
$$-\frac{2}{3}r^3\arccos\frac{\alpha x}{\sqrt{(r^2-\alpha^2)(r^2-x^2)}},$$

we have

$$\begin{aligned}
&\text{volume}_1(R,A,B,C)\\
&=[F(\sqrt{R^2-C^2},A,R)-F(0,A,R)]\\
&\quad+[F(\sqrt{R^2-C^2},B,R)-F(0,B,R)]\\
&\quad-\left(\arccos\frac{A}{C}+\arccos\frac{B}{C}\right)\int_0^{\sqrt{r^2-C^2}}(R^2-x^2)\,dx\\
&\quad+(A\sqrt{C^2-A^2}+B\sqrt{C^2-B^2})\sqrt{R^2-C^2}\\
&=F(\sqrt{R^2-C^2},A,R)+F(\sqrt{R^2-C^2},B,R)\\
&\quad+\frac{\pi}{2}\left(\frac{2}{3}R^3-AR^2+\frac{A^3}{3}\right)+\frac{\pi}{2}\left(\frac{2}{3}R^3-BR^2+\frac{B^3}{3}\right)\\
&\quad-\frac{1}{3}(2R^2+C^2)\sqrt{R^2-C^2}\left(\arccos\frac{A}{C}+\arccos\frac{B}{C}\right)\\
&\quad+(A\sqrt{C^2-A^2}+B\sqrt{C^2-B^2})\sqrt{R^2-C^2}
\end{aligned}$$

In three-dimensional case, the Laguerre-Voronoi diagram for n spheres can be computed in $O(n^2)$ time [2], and each Voronoi polyhedron can be partitioned into simplices in linear time. Also, through Sects. 3 and 4 above, we have shown that the volume of the intersection of a sphere and a simplex can be computed exactly in a constant time. Then, from Lemma 2.1, we have the following theorem.

Theorem 4.1. *The volume of the union of n spheres in three-dimensional space can be found in* $O(n^2)$ *time.* □

5 The union of spheres problem

The following problem is discussed in Klee [5]. For $i=1, \ldots, n$, let S_i be a hypersphere with center $\mathbf{x}_i$, and S'_i a hypersphere with center $\mathbf{x}'_i$ in d-dimensional space such that S_i and S'_i have the same radius and, for any i,j with $1\leq i\leq j\leq n$, $s(\mathbf{x}_i,\mathbf{x}_j)\leq s(\mathbf{x}'_i,\mathbf{x}'_j)$, where $s(.,.)$ is the L_2-distance between two points. Is it true that

$$\mu\left(\bigcup_{i=1}^{n} S_i\right)\leq\mu\left(\bigcup_{i=1}^{n} S'_i\right)?$$

Apparently this problem has only been solved completely in $\mathbf{R}^1$. For a history of the problem the reader is referred to Klee [5]. We can prove a special case of this problem by using the Laguerre-Voronoi diagram. This result also appears implicitly in Lieb and Simon [7] using a different approach.

Theorem 5.1. *If* $s(\mathbf{x}'_i,\mathbf{x}'_j)=\alpha s(\mathbf{x}_i,\mathbf{x}_j)$ *for* $\alpha\geq 1$ *and all* $1\leq i<j\leq n$, *we have* $\mu\left(\bigcup_{i=1}^{n} S_i\right)\leq\mu\left(\bigcup_{i=1}^{n} S'_i\right)$.

Proof. We can suppose $\mathbf{x}_i=0$ for some i, and $\mathbf{x}'_j=\alpha\mathbf{x}_j$ for $j=1, \ldots, n$. Since S_j and S'_j have the same radius and $\alpha\geq 1$, we have $V(S_i)\subseteq V'(S'_i)$, where $V(S_j)$ (resp. $V'(S'_j)$) is the Voronoi region of S_j (resp. S'_j) in the Laguerre-Voronoi diagram for spheres S_j (resp. S'_j) ($j=1, \ldots, n$). Hence

$$\mu(S_i\cap V(S_i))\leq\mu(S'_i\cap V(S'_i)). \tag{5.1}$$

By similar arguments, we can see that (5.1) holds for any $i=1, \ldots, n$, and then from Lemma 2.1, we obtain the proposition. □

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