Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **48**, 2009

What is the Worst Case Behavior of the Simplex Algorithm?

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ABSTRACT. The examples published by Klee and Minty in 1972 do not preclude the existence of a pivot rule which will make the simplex method, at worst, polynomial. In fact, the continuing success of Dantzig's method suggests that such a rule does exist.

A study of known examples shows that

(a) those which use "selective" pivot rules require exponentially large coefficients, and
(b) none of the examples' pivot rules are typically used in practice, either because of computational requirements or due to a lack of even-handed movement through the column set.

In all "bad" problems, certain improving columns are entered $\approx 2^{m-2}$ times before other improving columns are entered once. This is done by making the unused columns "appear" to yield small objective function improvement.

The purpose of this paper is to explain the Klee–Minty and Jeroslow constructions, show how they can be modified to be pathological with small integral coefficients, and then suggest a "least entered" pivot rule which forces an improving column to be entered before any other column is entered for the second time. This rule seems immune to the "deformed product construction" which is the essence of all known exponential counterexamples.

1. Introduction

The simplex method has been solving linear programs with m constraints in m to $3m$ pivots for over twenty years. In 1972, Klee and Minty demonstrated the existence of linear programs with m inequality constraints in m non-negative variables which require $2^m - 1$ pivots when any improving column may enter and when the standard "$\max c_j - z_j$" rule is followed. Applying their construction for the standard rule leads to coefficients in excess of 3^m.

2000 *Mathematics Subject Classification.* 90C05.

Prepared under National Science Foundation Grant ENG 76-12266 and also partially supported by the Office of Naval Research Contract N00014-75-C-0493.

The author would like to thank George B. Dantzig for numerous enlightening conversations and Arthur F. Veinott, Jr. for suggesting a study of alternative pivot rules which ultimately led to the "least entered" rule.

I would like to thank Louise Letendre of the CRM for typesetting the original manuscript.

Reprint of Technical report no. 27, Department of Operations Research, Stanford University, May 1st, 1980.

This is the final form of the paper.

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In 1973, Jeroslow published a modification of a second Klee and Minty construction. His modification is pathological for the "maximum increase" rule. An unrefined application of this construction also yields exponential coefficients.

Other examples involving large coefficients were subsequently published by Zadeh [16] for minimum cost network flow problems, Avis–Chvátal [1] for Bland's rule (first positive), Murty [14] and Fathi [6] for complementary pivot algorithms, and Goldfarb–Sit [7] for a "gradient" selection rule. An example due to Edmonds for shortest path computations is also known [16].

The above examples may be viewed as "deformed product constructions." Given a polytope P^m requiring $\approx 2^m$ pivots with a polynomial number of dimensions, a new polytope P^{m+1} is constructed by deforming a product $P^m \times V$, where V is some polytope usually of low dimension. In the first Klee–Minty construction P^{m+1} differed from P^m by one dimension and two facets (V has one dimension and two facets). In the Klee–Minty–Jeroslow construction, P^{m+1} differed from P^m by two dimensions and roughly $4k$ facets, where k is some positive integer. In the network constructions [16], P^{m+1} differed from P^m by $2m$ dimensions and $2m+4$ facets.

We show that any linear program with rational coefficients may be expressed with coefficients 0, 1, -1, and 2. Modifications of the Klee–Minty and Jeroslow constructions are given with integral coefficients no greater than four. The Klee–Minty examples are shown to be equivalent to resource allocation problems with non-negative coefficients in which all bases have determinants of ± 1.

In all "bad" examples, the coefficients are chosen so that the best columns price out moderately, and are not entered until other columns have been entered exponentially many times. Roughly speaking, for a deformed product $P^{m+1} \approx P^m \times V^m$, this means that the simplex method performs a 2^m step pivot sequence for P^m before entering any of the new variables associated with V^m. The pivot sequence for P^m is then performed again in the reverse order.

Geometrically, the simplex method stays on a lower P^m face of $P^m \times V^m$ for $\approx 2^m$ pivots, then moves through the added V^m dimensions to an "upper" P^m face where it spends another 2^m pivots "undoing" pivots performed on the lower face.

Entering variables from V^m early causes a permanent move away from the lower face, killing the exponential growth.

The following rule forces movements away from faces irrespective of the level or rate of improvement. It was considered primarily for theoretical purposes after a thought provoking conversation with Arthur F. Veinott, Jr.

2. Least entered rule

Enter the improving variable which has been entered least often.

The above rule is easy to implement, and when used in conjunction with the standard or "max increase" rules speeds up both. It is unlikely to cycle (the cycle must contain *all* improving columns). It is our hope that the rule will prove to have a worst case bound proportional to $m \times n$, where m is the number of rows and n is the number of columns.[1] Examples of maximum flow problems requiring $\approx m \times n$ pivots using this rule will be given in a forthcoming paper.

[1]This is similar to the old conjecture $\Xi(d, n) \approx (d-1)+1$ of Klee [10] which was proven false by Klee and Minty for the standard rule.

Other rules similar to the "least entered" rule which have been suggested [4] are the Least Recently Considered (LRC) rule of Cunningham and the Least Recently Basic (LRB) rule of E. L. Johnson. Both methods were apparently designed for shortest path computations in networks but have obvious extensions to general linear programming which would kill the exponential growth of known counterexamples.

Unfortunately, polynomial proofs for the above rules, if they exist, might be extremely hard, as they would reduce the current best bound for the diameters of polytopes from $\frac{1}{3} \times 2^{d-2}\left(n - d + \frac{5}{2}\right)$ to a polynomials in n and d, where n is the number of facets and d, the dimension.[2]

3. The Klee–Minty construction

The first Klee–Minty construction creates from an n-dimensional polytope P^n with $2n$ faces requiring $2^n - 1$ pivots when any improving column may enter a polytope P^{n+1} with two more faces requiring $2^{n+1} - 1$ pivots.

The construction is illustrated in Fig. 1. The path of vertices visited in P^n is denoted $p_0, p_1, \ldots, p_{2^n-1}$. The first polytope P^1 has two faces $(x_1 \geq 0, x_1 \leq 1)$ and requires one pivot. The second polytope P^2 is obtained from P^1 by adding two additional constraints $-x_1/3 + x_2 \geq 0$ and $x_1/3 + x_2 \leq 1$, involving one additional variable.

It is convenient to think of the pivot sequence for P^2 in terms of the slack variables associated with the various faces. The initial point $p_0 = (0, 0)$ is determined by s_2, s_4 basic, s_1, s_3 non-basic. The sequence p_0, p_1, p_2, p_3 corresponds to entering s_1 then s_3, and then s_2. The variables s_2, s_4 and s_1 are respectively deleted.

P^3 is obtained from P^2 by adding two more constraints involving one additional variable. Note in Fig. 1 that the pivot sequence for P^3 is essentially the pivot sequence for P^2, plus a movement from the lower face, followed by the sequence for P^2 in the reverse order. We express this phenomenon in general by writing $\overrightarrow{P}^{n+1} = \overrightarrow{P}^n,\ s_{2n+1},\ \overleftarrow{P}^n$. In terms of entering slack variables, $\overrightarrow{P}^3 = s_1 s_3 s_2\ \ s_5\ \ s_1 s_4 s_2$.

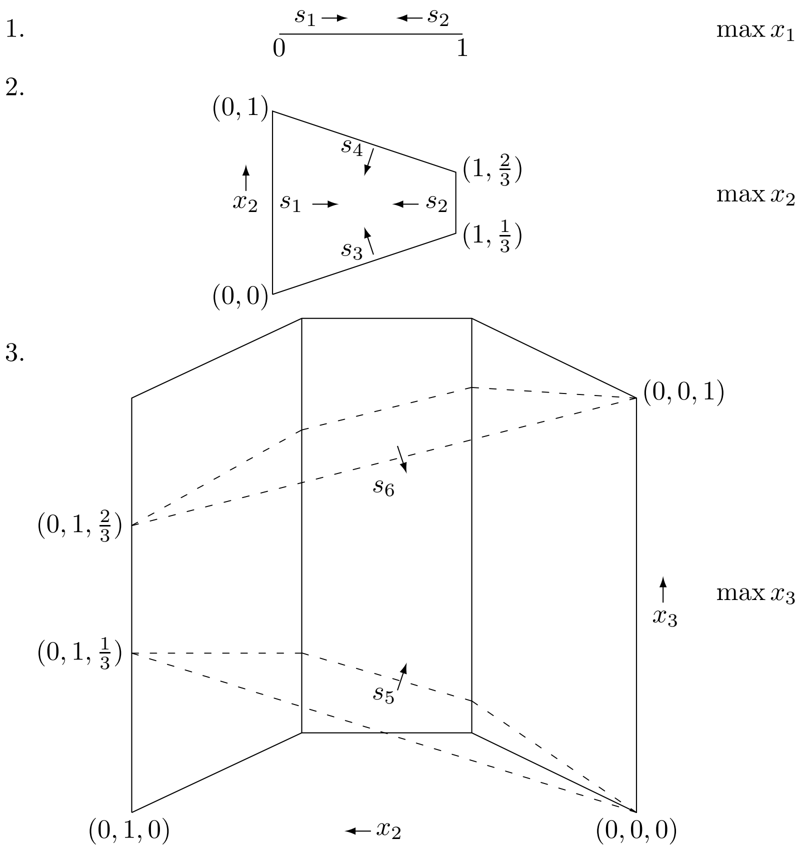
4. Fooling the standard rule

The examples in Fig. 1 take one pivot to solve when the standard $\max c_j - z_j$ rule is employed. To fool this rule, Klee and Minty scale the variables so that a much larger change in the entering slack variable is required to achieve the same objective function change, or equivalently, to move to the same adjacent vertex.

As an illustration, let $\bar{c}(s_i)$ denote the relative cost factor for s_i. If Δf_i denotes the change in the objective when s_i is entered, then $\bar{c}(s_i) = \Delta f_i / \Delta s_i$. At $p_0 = (0, 0, 0)$ in Fig. 1, $\bar{c}(s_1) = \frac{1}{9}$, $\bar{c}(s_2) = \frac{1}{3}$, and $\bar{c}(s_5) = 1$. The standard rule would enter s_5, moving from $(0, 0, 0)$ to $(0, 0, 1)$, the optimum, in one pivot. However, if s_5 were replaced by $s_5/16$, it would take a 16 unit change in s_5 to move from $(0, 0, 0)$ to $(0, 0, 1)$, and $\bar{c}(s_5)$ would be $\frac{1}{16}$. A similar replacement of s_2 by $s_2/4$ would case the standard rule to enter s_1 and follow the same sequence as before.

The right hand side of Table 1 gives a scaling which will make the standard rule exponential. Note that the coefficients grow at a rate of 4^m.

[2]Eds. note: A subexponential bound on the diameter of a polytope was proved in 1992 by G. Kalai and D. J. Kleitman.



Pivot sequence:

No. 2	$s_1\,s_3\,s_2$							
No. 3	$s_1\,s_3\,s_2$	s_5	$s_1\,s_4\,s_2$					
No. 4	$s_1\,s_3\,s_2$	s_5	$s_1\,s_4\,s_2$	s_7	$s_1\,s_3\,s_2$	s_6	$s_1\,s_4\,s_2$	

FIGURE 1. An example of the Klee–Minty construction.

5. Examples with small integer coefficients

The large coefficients in expression like $s_8/64$, or more generally, $s_{2n}/4^{n-1}$, may be eliminated adding $n-1$ additional variables and constraints. For the case $s_8/64$, we replace s_8 by s_8' with the additional constraints $4s_8' - s_8'' = 0$, $4s_8'' - s_8''' = 0$, $4s_8''' - s_8 = 0$, s_8', s_8'', $s_8''' \geq 0$, as done in Table 1. To construct P^m in this fashion using coefficients no greater than 4, $m(m-1)$ constraints and non-negative variables must be added.

It should be noted that such a "coefficient reduction" can always be performed, but the "reduction" is cleanest when the large coefficients in each column are multiples of a fixed power of two, for example,

$$\begin{pmatrix} 3 \times 2^{74} \\ -1 \times 2^{74} \\ 2 \times 2^{74} \end{pmatrix}$$

Theorem 5.1. *Let $\mathcal{L}$ be a linear program with rational coefficients whose representation requires a polynomial number of digits. Then $\mathcal{L}$ may be expressed using integral coefficients of* 2, 1, -1, *and* 0 *with a polynomial number of variables and constraints.*

PROOF. The b_i may be made to be 0 or 1 by suitably multiplying each row. With this change, let d_j denote the least common multiple of the divisors of elements in column j. Then column j may be written as

$$\frac{x_j}{d_j} \times \begin{pmatrix} c_j \\ a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix},$$

where $d_j, c_j, a_{1j}, \ldots, a_{mj}$ are integers. Let $\sum_k d_j^{(k)} 2^k$ denote the binary representation of d_j and let

$$q_j = \max_{1=1,\ldots,m} \{\lfloor \log_2 d_j \rfloor, \lfloor \log_2 a_{ij} \rfloor\}.$$

Note that $d_j^{(k)} = 0$ or 1 for every k, j. Define a new variable $\overline{x}_j = x_j/d_j$ by adding new variables $\overline{x}_j^{(k)}$, $k = 0, 1, 2, \ldots, q_j$, $\left(\overline{x}_j^{(k)} = 2^k \overline{x}_j\right)$ and constraints $\left(\sum_k d_j^{(k)} \overline{x}_j^{(k)}\right) - x_j = 0$ and $-\overline{x}_j^{(k)} + 2\bar{x}_j^{(k-1)} = 0$, $k = 1, \ldots, q_j$. Let $\sum_k d_j^{(k)} 2^k$ be the binary representation of a_{ij}. Now the term $x_j a_{ij}/d_j$ may be expressed as $\sum_k a_{ij}^{(k)} \overline{x}_j^{(k)}$. All coefficients are 0, ± 1, or 2. The above construction requires $\sum_j (q_j + 1)$ additional variables and constraints. $\square$

When applying the simplex method to the above problems, care must be taken to ensure that initial pivots eliminate $\bar{x}_j^{(k)}$ variables and retain x_j. If x_j is eliminated

TABLE 1. Example of the original Klee–Minty construction (upper left), a scaling of the slacks to fool the standard rule (upper right), and the addition of $m(m-1)$ variables and constraints to yield integral coefficients ≤ 4 (below).

Any improving column	Standard rule
$\max x_4$	
$x_1 - s_1 = 0$	replace
$x_1 + s_2 = 1$	slacks by
$-x_1/3 + x_2 - s_3 = 0$	$-s_3/4$
$x_1/3 + x_2 + s - 4 = 1$	$s_4/4$
$-x_2/3 + x_3 - s_5 = 0$	$-s_5/16$
$x_2/3 + x_3 + s_6 = 1$	$s_6/16$
$-x_3/3 + x_4 - s_7 = 0$	$-s_7/64$
$x_3/3 + x_4 + s_8 = 1$	$s_8/64$

Small Coefficients

Replace a quantity like $s_8/64$ by a variable s_8', along with the constraints $4s_8' - s_8'' = 0$, $4s_8'' - s_8''' = 0$, $4s_8'''' - s_8 = 0$, all variables ≥ 0.

and replaced by $\overline{x}_j^{(\ell)}$, a rescaling of variables has occurred which will change relative cost factors and may affect the pivot sequence.

The following theorem notes some similarities between the Klee–Minty construction and the "bad" complementary pivot example due to Murty, and explains how the Avis–Chvátal examples was obtained.

Theorem 5.2. *Let $\mathcal{L}^n$ denote the nth problem constructed on the left side of Table 1, with s_{2i}, respectively, s_{2i-1} replaced by $s_{2i}/3^{i-1}$, respectively, $s_{2i-1}/3^{i-1}$.*

Then $\mathcal{L}^n$ is equivalent to a resource allocation problem with non-negative integral coefficients, equal objective coefficients, and basis matrices whose determinants are 1 or -1.

Proof. Solving the triangular system

$$\begin{array}{lllllllll} x_1 & & & & - s_1 & & & & = 0 \\ -\dfrac{x_1}{3} + x_2 & & & & & -\dfrac{s_3}{3} & & & = 0 \\ & -\dfrac{x_2}{3} + x_3 & & & & & -\dfrac{s_5}{9} & & = 0 \\ & & -\dfrac{x_3}{3} + x_4 & & & & & -\dfrac{s_7}{27} & = 0 \\ & & \cdots\cdots\cdots & & & & & & \end{array}$$

for $x_1, \ldots, x_n$ yields

$$\begin{aligned} x_1 &= s_1, \\ x_2 &= \frac{s_1 + s_3}{3}, \\ x_3 &= \frac{s_1 + s_3 + s_5}{9}, \\ x_4 &= \frac{s_1 + s_3 + s_5 + s_7}{27}, \\ &\cdots\cdots\cdots \end{aligned}$$

Substituting for x_i in the remaining equations produces the equivalent problem

$$\begin{array}{llll} \text{maximize} & \dfrac{1}{3^{n-1}}\big(s_1 + s_3 + s_5 + s_7 + \cdots + s_{2n-1}\big) & & = 1 \\ \text{subject to} & s_1 & + s_2 & = 1 \\ & 2s_1 + s_3 & + s_4 & = 3 \\ & 2s_1 + 2s_3 + s_5 & + s_6 & = 9 \\ & 2s_1 + 2s_3 + 2s_5 + s_7 & + s_8 & = 27 \\ & \vdots & \ddots & \\ & 2s_1 + 2s_3 + 2s_5 + 2s_7 + \cdots + s_{2n-1} & + s_{2n} & = 3^{n-1}, \\ & & s_i \geq 0. & \end{array}$$

The constraint matrix is of the form $(L \mid I)$ where L is a lower triangular matrix with ones on the diagonal. This gives the result. $\square$

The above problem can yield the same pivot sequence as the nth scaled problem in Table 1 because all relative cost factors will be 0 or $\pm\frac{1}{3}^{n-1}$ at every vertex (there will be many ties). To *insure* that the same sequence is followed s_{2i}, respectively, s_{2i-1} must be replaced by

$$\frac{s_{2i}}{k^{i-1}}, \text{ respectively, } \frac{s_{2i-1}}{k^{i-1}} \quad \text{with} \quad k > 3,$$

in which case the constraint matrix would change but would remain lower triangular.

An example of Avis and Chváatal, which for $m = 3$ with a rearranging of indices is

$$\begin{aligned} &\text{maximize } 10^2 s_1 + 10 s_3 + s_5 \\ &\text{subject to } s_1 + s_2 = 10^2 \\ &\qquad 20 s_1 + s_3 + s_4 = 10^4 \\ &\qquad 200 s_1 + 20 s_3 + s_5 + s_6 = 10^6, \end{aligned}$$

It may be obtained from Table 1 by replacing the 3's by 10's and taking $k = 10^2$.

The following assertion notes that a bounded pathological example can always be transformed into one with all a_{ij}, b_i, and $c_j \geq 0$.

Assertion 1. Let $\mathcal{L}$ be a linear program with a finite optimal solution. Then $\mathcal{L}$ may be transformed to an equivalent program $\mathcal{L}'$ in which all coefficients are positive (non-negative).

PROOF. Affix the constraint $\Sigma x_i + s_{m+1} = M$ for sufficiently large M. Then add suitable multiples of this constraint to each row until all coefficients are positive. The objective function will have a constant term involving $-M$ which may be disregarded. □

6. Bland's rule (first improving column)

Table 2 lists the sequence of relative cost factors $\bar{c}(s_i)$ associated with the vertices $p_0, \ldots, p_7$ of P^3. Notice that the variables s_{2i} and s_{2i-1} are complementary, i.e., $s_{2i} \cdot s_{2i-1} = 0\ \forall i$, as are their relative cost factors $\bar{c}(s_{2i}) \cdot \bar{c}(s_{2i-1}) = 0\ \forall i$.

TABLE 2. Relative cost factors associated.

	s_1	s_2	s_3	s_4	s_5	s_6
0	$\frac{1}{9}$	0	$\frac{1}{12}$	0	$\frac{1}{16}$	0
1	0	$-\frac{1}{9}$	$-\frac{1}{12}$	0	$\frac{1}{16}$	0
2	0	$\frac{1}{9}$	0	$-\frac{1}{12}$	$\frac{1}{16}$	0
3	$-\frac{1}{9}$	0	0	$-\frac{1}{12}$	$\frac{1}{16}$	0
4	$\frac{1}{9}$	0	0	$\frac{1}{12}$	0	$-\frac{1}{16}$
5	0	$-\frac{1}{9}$	0	$\frac{1}{12}$	0	$-\frac{1}{16}$
6	0	$\frac{1}{9}$	$-\frac{1}{12}$	0	0	$-\frac{1}{16}$
7	$-\frac{1}{9}$	0	$-\frac{1}{12}$	0	0	$-\frac{1}{16}$

The following theorem notes that examples given in Table 1 are pathological for Bland's Rule. A similar statement can be made for the forthcoming Jeroslow modification, and for network examples in [16].

Theorem 6.1. *The examples in Table* 1 *follow the same pivot sequence with Bland's rule.*

Outline of Proof. It suffices to show that the first improving column prices out best. Let ϕ denote the objective function. For every n, $\phi(p_0) = 0$, $\phi(p_{2^n-1} = 1$, and the jump in ϕ between lower and upper fase is $\frac{1}{3}$. Let $p_i^1 = (p_i, \phi(p_i)/3)$ and $p_i^2 = (p_1, 1 - \phi(p_i)/3)$ for $0 \le i \le 2^n - 1$. Then the vertex sequence of P^{n+1} is

$$\underbrace{p_0^1, \quad p_1^1, \quad \dots, \quad p_{2^n-1}^1}_{\text{lower case}}, \quad \underbrace{p_{2^n-1}^2, \quad \dots, \quad p_1^2, \quad p_0^2}_{\text{upper case}}$$

For each increase in n, the objective change between successive points on lower (upper) faces decreases by a factor of three. Because the vertices for P^{n+1} are obtained from the vertices for P^n by adding an extra dimension (the objective value), the change in the entering slack required to move from p_i to p_{i+1} on the lower (upper) face remains the same. This implies that relative cost factors for old slacks are decreased in absolute value by a factor of three for each increase in n. The new slack variables (with the highest indices) are scaled to price out worse than the other variables. This observation and its predecessor imply that the lowest indexed variables, when profitable, price out best. The exact formula, for $\bar{c}(s_{2i}) > 0$, is $\bar{c}(s_{2i}) = \frac{4}{3^n}\left(\frac{3}{4}\right)^i$, which decreases by a factor of three for each increase in n. $\square$

7. The maximum increase rule

This rule enters the column yielding the maximum objective increase. A sequence of "bad" polytopes, $\mathcal{P}^1, \dots, \mathcal{P}^n$, will be constructed recursively. $\mathcal{P}^1$ is shown at the top of Fig. 2. It has two dimensions, four faces, and requires two pivots starting from $(0,0)$ when the objective is maximize x_1. The two "lower faces" are dotted for the purposes of identification

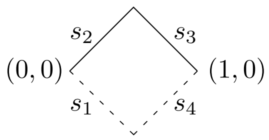
The second polytope $\mathcal{P}^2$, is four dimensional and appears below $\mathcal{P}^1$. $\mathcal{P}^2$ is a deformed product of $\mathcal{P}^1$ with V^1, the two-dimensional polytope shown in the upper right.

$\mathcal{P}^2$ is best appreciated by imagining that one is looking down at the top of a mountain. The shaded edges of $\mathcal{P}^2$ correspond to the upper faces of $\mathcal{P}^1$ crossed with V^1. The dotted edges of $\mathcal{P}^2$ correspond to the bottom faces of $\mathcal{P}^1$ crossed with V^1 and are not all shown. $\mathcal{P}^1$ corresponds to the two dimensional polytope determined by $(0,0)$ and points a and b. Figure 2 is essentially an approximate projection of $\mathcal{P}^2$ onto the V^1 coordinates, which are denoted x_3 and x_4.

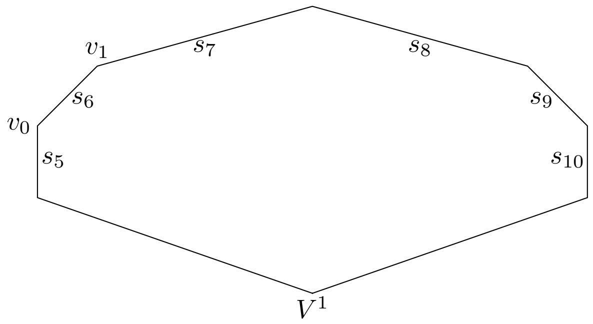
$\mathcal{P}^2$ was designed so that, starting at $(0,0)$, and maximizing the x_3 or "x" coordinate, one first performs the pivot sequence for $\mathcal{P}^1$; executes several pivots involving V^1 variables; "reverses" the sequence for $\mathcal{P}^1$; and ends at $(1,0)$.

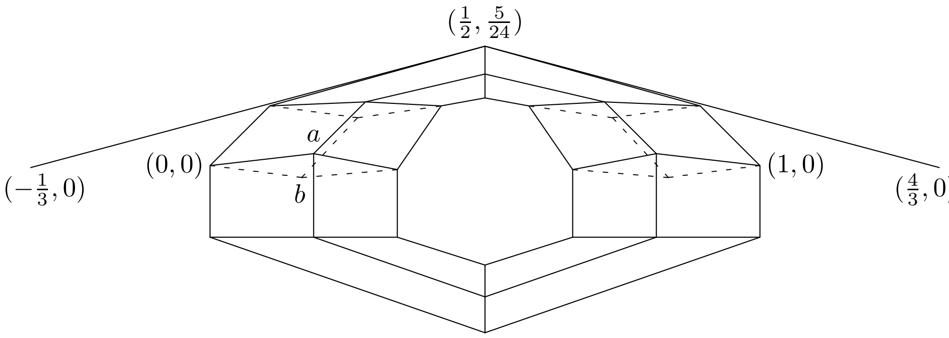
In terms of entering slack variables, the forward pivot sequence p_0 to p_8 shown in Fig. 2 may be expressed as

$$\underset{\overrightarrow{\mathcal{P}}}{s_1 s_2} \quad \underset{V^1}{s_5 s_6 s_7} \quad \underset{\overleftarrow{\mathcal{P}}}{s_3 s_4} \quad \underset{V^1}{s_8}$$



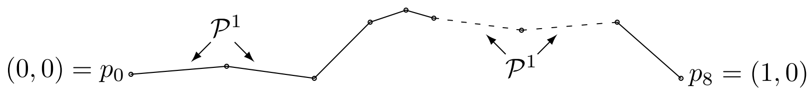
Initial polytope $\mathcal{P}^1$



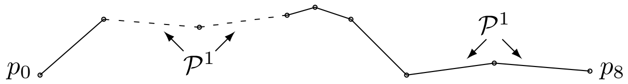


$\mathcal{P}^2 \cong \mathcal{P}^1 \times V^1$

Coordinates shown are x_3 and x_4, the coordinates of V^1.



Forward pivot sequence p_0 to p_8 $\max x_3$, 8 pivots.



Reverse pivot sequence p_8 to p_0, $\max x_3$, 8 pivots.

Figure 2. A modification of the Klee–Minty–Jeroslow construction.

$\mathcal{P}^2$ is a "reversible" polytope, in the sense that eight pivots are also required if one starts at $(1,0)$ and minimizes x_3. The reverse pivot sequence from $(1,0)$ to $(0,0)$ is shown at the bottom of Fig. 2.

To insure that the pivot sequence for $\mathcal{P}^1$ is performed before variables in V^1 are entered, the difference in x coordinates between $v_0 = (0,0)$ and $v_1 = \left(\frac{1}{9}, \frac{1}{9}\right)$ is chosen smaller than the difference in x coordinates between $(0,0)$ and vertex a. This ensures that pivots involving variables of $\mathcal{P}^1$ are formed first as long as such pivots are profitable.

8. Construction of $\mathcal{P}^3$

$\mathcal{P}^3$ is constructed as a deformed product of $\mathcal{P}^2 \times V^2$. V^2 is the same as V^1 except that the slopes of the lines through $\left(-\frac{1}{3}, 0\right)$, $\left(1/2, 5/24\right)$ and $\left(1/2, 5/24\right)$,

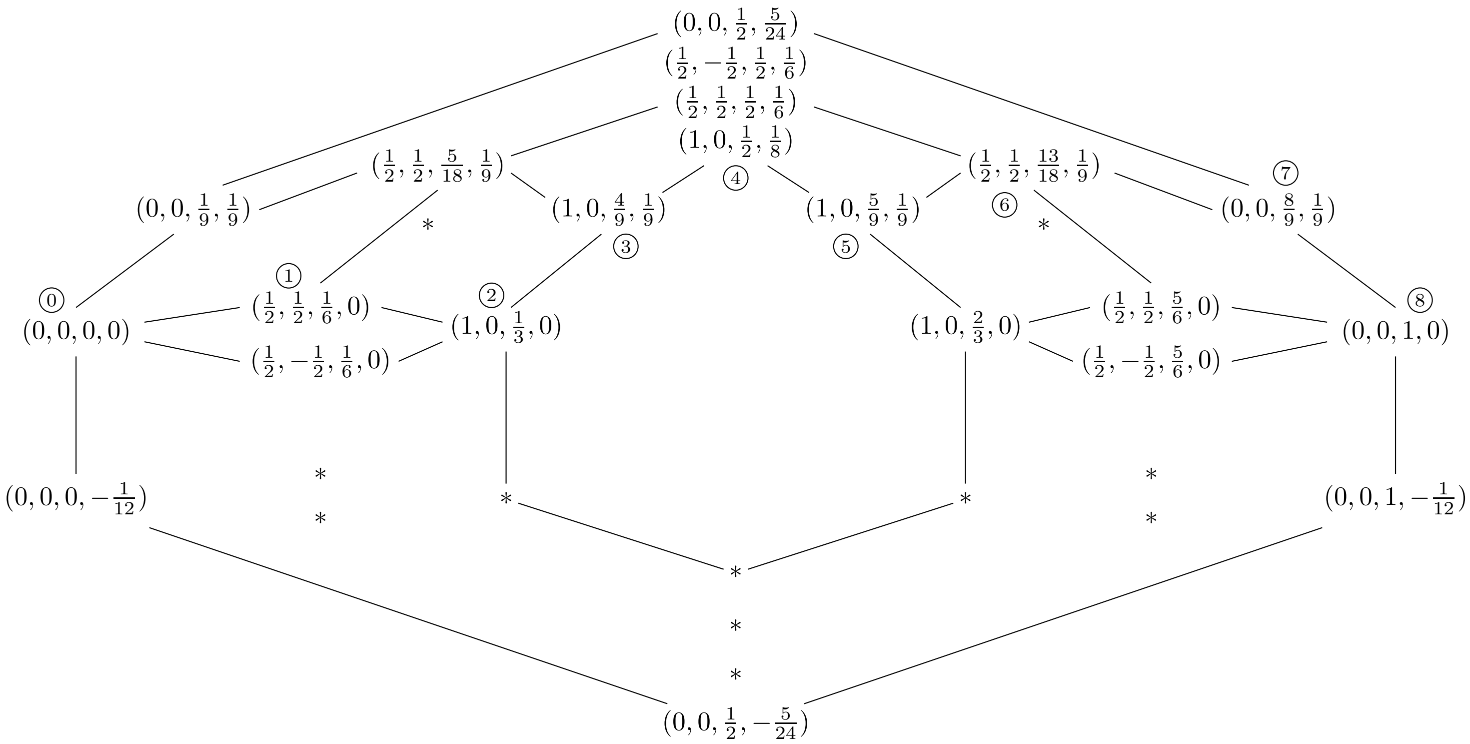


FIGURE 3. Relevant extreme points of $\mathcal{P}^2$, the pathological four-dimensional polytope for the "max increase" rule. Vertices not shown are starred. Pivot sequence when maximizing x_3 is represented by the vertices labelled 0, 1, 2, 3, 4, 5, 6, 6, 7, 8.

$\left(\frac{4}{3}, 0\right)$ are decreased in absolute value by a factor of 4. This effectively squashes the top half of $\mathcal{P}^3$ so that the difference in x coordinates between v_0 and v_1 is $1/45$[3] Variables of $\mathcal{P}^2$ are now more "profitable" than variables of V^2, so the whole pivot sequence for $\mathcal{P}^2$ is performed before variables of V^2 are entered.

Denoting the relevant slacks of V^2 corresponding to s_5, s_6, s_7, s_8, s_9, s_{10}, in V^1 by s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, the forward pivot sequence for $\mathcal{P}^3$ in terms of entering slacks is

$$\underbrace{s_1 s_2\ s_5 s_6 s_7\ s_3 s_4\ s_8}_{\overrightarrow{P}^2}\ s_{11} s_{12} s_{13}\ \underbrace{s_1 s_2\ s_{10} s_9 s_8\ s_3 s_4\ s_7}_{\overleftarrow{P}^2}\ s_{14}$$

In general, $\mathcal{P}^n$ is constructed as a deformed product of $\mathcal{P}^{n-1}$ and V^{n-1}, where V^{n-1} is the same as V^1 except the lines through $\left(-\frac{1}{3}, 0\right)$, $\left(1/2, 5/24\right)$ and $\left(1/2, 5/24\right)$, $\left(\frac{4}{3}, 0\right)$ have their slopes decreased in absolute value by a factor of $1/4^{n-2}$

9. Examples with Small Coefficients

Constraints with small integral coefficients defining $\mathcal{P}^1$, $\mathcal{P}^2$, and $\mathcal{P}^3$ are shown in Table 3. The system for $\mathcal{P}^n$ is generated by taking the system for $\mathcal{P}^{n-1}$ and adding the constraints determining V^{n-1}, with x_{2n-1} replaced by $x_{2n-1} - (x_{2n-3}/3)$ for facets on the left of the line $x_{2n-1} = \frac{1}{2}$ and x_{2n-1} replaced by $x_{2n-1} + (x_{2n-3}/3)$ for facets on the right of $x_{2n-1} = \frac{1}{2}$. This yields the deformation, or tilting of the product. Note that, aside from a translation of subscripts, the set of constraints for V^2 differs from that for V^1 only in the first two inequalities, where a variable x_6'' (representing $16x_6$) has replaced a variable x_4' (representing $4x_4$). This corresponds to reducing the slope of the top two facets by a factor of four.

10. Testing the problems

To run the problems it is recommended that the x variables be eliminated and replaced by slacks. The starting basis then consists of those slacks which are positive at the point $(0, 0, 0, \ldots, 0)$. For $\mathcal{P}^2$ the starting basis would be s_3, s_4, s_7, s_8, s_9, s_{10}, and the slacks for the bottom two faces of V^1.

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[3] v_1 is determined by the intersection of lines $y = x$ and $y = x/16 + 1/48$.

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TABLE 3. x_4, x_4', x_6, x_6', x_6'', x_8, x_8', x_8'', x_8''' unrestricted.

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{r}
\max\limits_{x_1} -x_1 + x_2 \le 0 \\
x_1 + x_2 \le 1 \\
-x_1 - x_2 \le 0 \\
x_1 - x_2 \le 1
\end{array}\right\}\text{2 pivots} \\
\\
\begin{array}{r}
\max\limits_{x_3} x_1 + 3x_3 + 3x_4' \le 4 \\
x_1 - 3x_3 + 3x_4' \le 1 \\
4x_4 - x_4' = 0 \\
x_1 - 3x_3 + 3x_4 \le 0 \\
x_1 + 3x_3 + 3x_4 \le 3 \\
x_1 - 3x_3 \le 0 \\
x_1 + 3x_3 \le 3 \\
x_1 - 3x_3 - 3x_4' \le 1 \\
x_1 + 3x_3 - 3x_4' \le 4
\end{array}
\end{array}\right\} 2 \times 2 + 4 = 8 \text{ pivots} \\
\\
\begin{array}{r}
\max\limits_{x_5} x_3 + 3x_5 3x_6'' \le 4 \\
x_3 - 3x_5 + 3x_6'' \le 1 \\
4x_6 - x_6' = 0 \\
4x_6' - x_6'' = 0 \\
x_3 - 3x_5 + 3x_6 \le 0 \\
x_3 + 3x_5 + 3x_6 \le 3 \\
x_3 - 3x_5 \le 0 \\
x_3 + 3x_5 \le 3 \\
x_3 - 3x_5 - 3x_6' \le 1 \\
x_3 + 3x_5 - 3x_6' \le 4
\end{array}
\end{array}\right\}\text{20 pivots}
$$

$$
\begin{array}{r}
\max\limits_{x_7} x_5 + 3x_7 + 3x_7''' \le 4 \\
x_5 - 3x_7 + 3x_7''' \le 1 \\
\cdots\cdots\cdots\cdots
\end{array}
$$

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