Stochastic Programming: introduction and examples

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Outline

- What is Stochastic Programming?
- Why should we care about Stochastic Programming?
    - The farmer’s problem
- General formulation of two-stage stochastic programs with recourse

Introduction

- Mathematical Programming, alternatively Optimization, is about decision making

- Decisions must often be taken in the face of the unknown or limited knowledge (uncertainty)
  - Market related uncertainty
  - Technology related uncertainty (breakdowns)
  - Weather related uncertainty….

How to deal with uncertainty?

- Ignore it?
    - The uncertain “factors” might **interact with** our **decision** in a **meaningful way…**

- “Carefully” determine the problem parameters?
    - No matter how careful we are, we cannot get rid of the inherent randomness…

Stochastic Programming is the way !

What is Stochastic Programming?

- Mathematical Programming, alternatively Optimization, is about decision making
- Stochastic Programming is about decision making **under uncertainty**
- Can be seen as **Mathematical Programming with random parameters**

Reference

Birge, J. R., and F. Louveaux, 1997

Introduction to stochastic programming

Springer-Verlag, New York

Why should we care about Stochastic Programming? An example…

The farmer’s problem (from Birge and Louveaux, 1997)

- Farmer Tom can grow **wheat**, **corn**, and **sugar beets** on his 500 acres.

How much land to devote to each crop?

What Farmer Tom knows about wheat and corn

- He requires 200 tons of wheat and 240 tons of corn to feed his cattle
  - These can be grown on his land or bought from a wholesaler
  - Any production in excess of these amounts can be sold for $170/ton (wheat) and $150/ton (corn)
  - Any shortfall must be bought from the wholesaler at a cost of $238/ton (wheat) and $210/ton (corn)

What Farmer Tom knows about sugar beets

- He can also grow sugar beets on his 500 acres
  - Sugar beets sell at $36/ton for the first 6000 tons
  - Due to economic quotas on sugar beets, sugar beets in excess of 6000 tons can only be sold at $10/ton

What Farmer Tom knows about his land

- Based on **experience**, the **mean yield** is roughly:
  - Wheat: 2.5 tons/acre
  - Corn: 3 tons/acre
  - Sugar beets: 20 tons/acre

- And the planting costs are:
  - Wheat: $150/acre
  - Corn: $230/acre
  - Sugar beets: $260/acre

The data

	Wheat	Corn	Sugar Beets
Yield (tons/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling Price ($/ton)	170	150	36 (<= 6000 T) 10 (> 6000 T)
Purchase Price ($/ton)	238	210	-
Minimum Requirement (ton)	200	240	-
500 acres available for planting			

Linear Programming (LP) formulation

Decision variables

1. $x_{1,2,3}$: acres of wheat, corn, sugar beets planted (x_1: wheat, x_2: corn, x_3: sugar beets)
2. $w_{1,2,3}$: tons of wheat, corn, sugar beets sold at favorable price
3. w_4: tons of sugar beets sold at lower price
4. $y_{1,2}$: tons of wheat, corn purchased (y_1: wheat, y_2: corn)

LP formulation

Objective function

Maximize

$170\ w_1 - 238\ y_{\mathbf{1}} - 150\ x_1$ ← **Wheat**

$+ 150\ w_2 - 210\ y_2 - 230\ x_2$ ← **Corn**

$+ 36\ w_3 + 10\ w_4 - 260\ x_3$ ← **Sugar beets**

Equivalent to

Min $150x_1 + 230x_2 + 260x_3 + 238y_{\mathbf{1}} - 170w_1 + 210y_2$

$-150w_2 - 36w_3 - 10w_4$

Constraints

- Acres available for planting

  $x_1+x_2+x_3 <= 500$

- Minimum requirement for wheat

  $\underline{2.5}x_1+ y_1 - w_1 >= 200$

- Minimum requirement for corn

  $\underline{3}x_2+ y_2-w_2 >= 240$

- Maximum that can be sold at favorable price (sugar B)

  $w_3 <= 6000$

- Logical link (production sugar beets)

  $\underline{20}x_3 >= w_3 + w_4$

- Non-negativity

  $x_1,x_2,x_3,y_1,y_2, w_1,w_2,w_3,w_4>=0$

Putting it all together

Maximize

$-150x_1-230x_2-260x_3-238y_{\mathbf{1}}+170w_1-210y_2+150w_2+36w_3+10w_4$

Subject to

$x_1+x_2+x_3 \le 500$

$\underline{2.5}x_1+y_1-w_1 \ge 200$

$\underline{3}x_2+y_2-w_2 \ge 240$

$\underline{20}x_3-w_3-w_4 \ge 0$

$w_3 \le 6000$

$x_1,x_2,x_3,\ ,y_1,y_2,w_1,w_2,w_3,w_4 \ge 0$

Solution with expected yields (mean yields)

Culture	Wheat	Corn	Sugar Beets
Plant area (acres) Production (tons) Sales (tons) Purchase (tons)	120 300 100 0	80 240 0 0	300 6000 6000 0
Profit:$118,600			

Solution corresponds to Tom's intuition!

- Plant land necessary to grow sugar beets up the quota limit
- Plant land to meet the production requirements for wheat and corn
- Plant remaining land with wheat and sell the excess.

But the weather…

- The **mean yield** is roughly
  - Wheat: 2.5 tons/acre
  - Corn: 3 tons/acre
  - Sugar beets: 20 tons/acre

- But farmer Tom knows that his yields aren't that precise

- Two scenarios
  - **Good weathe**r: **1.2 Yield**
  - **Bad weather: 0.8 Yield**

Formulation - Good Weather

Maximize

$-150x_1 - 230x_2 - 260x_3 - 238y_{\mathbf{1}} + 170w_1 - 210y_2$

$+150w_2 + 36w_3 + 10w_4$

Subject to

$x_1 + x_2 + x_3 <= 500$

3 $x_1 + y_1 - w_1 >= 200$ (2.5 * 1.2 = 3)

3.6 $x_2 + y_2 - w_2 >= 240$ (3 * 1.2 = 3.6)

24 $x_3 - w_3 - w_4 >= 0$ (20 *1.2 = 24)

$w_3 <= 6000$

$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4, >= 0$

Solution if Good Weather

Culture	Wheat	Corn	Sugar Beans
Plant area (acres)	183.33	66.67	250
Production (ton)	550	240	6000
Sales (ton)	350	0	6000
Purchase (ton)	0	0	0
Profit:$167,667			

Formulation - Bad Weather

Maximize

$-150x_1-230x_2-260x_3 - 238y_{\mathbf{1}}+170w_1- 210y_2$

$+150w_2+ 36w_3+ 10w_4$

Subject to

$x_1+ x_2+ x_3<=500$

2 $x_1+ y_1- w_1>=200$ (2.5 * 0.8 = 2)

2.4 $x_2+ y_2-$ w2>=240 (3 * 0.8 = 2.4)

16 $x_3- w_3- w_4>=0$ (20 * 0.8 = 16)

$w_3<=6000$

$x_1,x_2,x_3,y_1,y_2, w_1,w_2,w_3,w_4 >= 0$

Solution if Bad Weather

Culture	Wheat	Corn	Sugar Beans
Plant area (acres)	100	25	375
Production (ton)	200	60	6000
Sales (ton)	0	0	6000
Purchase (ton)	0	180	0
Profit:$59,950			

What should Tom do?

- The optimal solution is very sensitive to change on the weather and the respective yields. The overall profit ranges from $59,950 to $167,667
- Main issue **sugar beets** production: without knowing the weather, he cannot determine how much land to devote to this crop?
  - Large surface: Might have to sell some at the unfavorable price
  - Small surface: Might miss the opportunity to sell the full quota at the favorable price

What should Tom do?

- Long term weather forecasts would be very helpful: If only he can predict the weather conditions 6 months ahead….

- Tom realizes that it is **impossible** to make **a perfect decision**: The **planting decisions** must be made **now**, but **purchase and sales decisions** can be made **later**.

Maximizing the Expected Profit (long-run profit, risk-neutral decisions)

- Assume three scenarios occur with equal probability
  We use a scenario subscript 1, 2, 3 to represent **good weather**, **average weather** and **bad weather**, respectively, and add it to each of the purchase and sale variables.

  For example,
  w_{32}: the amount of sugar beet sold @ favorable price if yields is average.

  w_{21}: the amount of corn sold @ favorable price if yields is above average.

  w_{13}: the amount of wheat sold @ favorable price if yields is below average.

The objective function

Tom's expected profit can be expressed as follows:

$$-150x_1-230x_2-260x_3$$
$$+1/3(170w_{11}+150w_{21}+36w_{31}+10w_{41}-238y_{11}-210y_{21})$$
$$+ 1/3(170w_{12}+150w_{22}+36w_{32}+10w_{42}-238y_{12}-210y_{22})$$
$$+ 1/3(170w_{13}+150w_{23}+36w_{33}+10w_{43}-238y_{13}-210y_{23})$$

The constraints

$x_1 + x_2 + x_3 <= 500$

$3x_1 + y_{11} - w_{11} >= 200$; $2.5x_1 + y_{12} - w_{12} >= 200$; $2x_1 + y_{13} - w_{13} >= 200$

$3.6x_2 + y_{21} - w_{21} >= 240$; $3x_2 + y_{22} - w_{22} >= 240$; $2.4x_{23} + y_{23} - w_{23} >= 240$

$24x_{31} - w_3 - w_4 >= 0$; $20x_{32} - w_3 - w_4 >= 0$; $16x_{33} - w_3 - w_4 >= 0$

$w_{31}, w_{32}, w_{33} <= 6000$

All variables >=0

Solution of the resulting model

		Wheat	Corn	Sugar Beets
First Stage	Area (Acres)	170	80	250
S=1 Above	Production (t) Sales (t) Purchase (t)	510 310 0	288 48 0	375 6000(Favor.price) 0
S=2 Average	Production (t) Sales (t) Purchase	425 225 0	240 0 0	5000 5000(Favor.price) 0
S=3 Below	Production (t) Sales (t) Purchase (t)	340 140 0	192 0 48	4000 4000(Favor.price) 0
Expected Profit = $108,390				

Top line: planting decisions which must be determined before knowing the weather **(now)** are called ***first stage decisions***. Production, sales, and purchases decisions for the three scenarios are termed the ***second stage decisions (later)***

What is this solution telling us?

- Allocate land for sugar beets to always avoid having to sell them at the unfavorable price (the 3 scenarios)
- Plant the corn so that to meet the production requirement in the average scenario
- Plant the remaining land with wheat. This area is large enough to cover minimum requirement and sales always occur
- **The solution is not ideal under all scenarios (it is impossible to find one). The solution is "hedged/balanced" against the various scenarios**

Expected Value of Perfect Information

- Now assume yields vary over the years, but on a random basis. If the farmer gets the information on the yields before planting (HFT), he will choose one of the following solutions.

  Good yields: (183.33, 66, 67, 250) or Profit: $167,667

  Average yields: (120, 80, 67, 300) or Profit: $118,600

  Bad yields: (100,25,375) or Profit: $59,950

- In the long run, if each yield is realized one third of the years (each of the scenarios occurs with probability 1/3), Tom's average profit would be $115,406.
- As we all know, the farmer doesn't get prior information on the yields. The best he can do in the long run is take the solution as given in the last table, and this case he would have an expected profit of $108,390.

Expected Value of Perfect Information (EVPI)

- The difference
  $115,406 - $108,390 = $7,016
  is called *expected value of perfect information*

- It represents how much farmer Tom would be willing to pay for the perfect information

Expected Value of Perfect Information (EVPI)

- **EVPI = how much it is "worth" to invest in better or perfect forecasting technology**
- **What is the "value" of including the uncertainty?**

The Value of the Stochastic Solution (VSS)

Another approach farmer may have is to assume expected yields and allocate the optimum planting surface according to this yields. *Would we get the same expected profit?*

- Solve the "mean value" problem to get a first stage solution x or "a policy"
    - Mean yields: (2.5, 3, 20)
    - Solution: x_1:120, x_2:80, x_3:300.
- Fix the first stage solution at that value x, and then solve all the scenarios to see farmer's profit in each

Profits based on Mean Value

Yield	Profit($)
Good Average Bad	148,000 118,600 55,120

- If Tom implements the **policy based on using only the average yields**, in the long run, he would expect to make an average profit of:

1/3(148,000)+1/3*(118,600)+1/3*(55,120)=$107,240

- If Tom implements the **policy based on the solution of the stochastic programming problem** (x_1=170, x_2= 80, x_3=250), he would expect to make **$108,390**.

The value of the Stochastic Solution (VSS)

- The difference of the values \$108,390-\$107,240=**\$1,150 is the *value of the stochastic solution.***

- If Tom uses the stochastic solution rather than the mean value solution, he would get **\$1,150 more every season!**

Thursday

- Mine production scheduling with uncertain mineral supply

It is worth to model uncertainty!

General Model Formulation

- We have a set of **decisions to be taken without full information on some random events**, which we call **first-stage decisions** (*x*)
- Later, **full information is received** on the realization of some random vector ξ, and **second-stage or corrective actions (recourse)** *y* are taken
- We assume that the probabilistic property of ξ is known a priori

Two-stage stochastic program with recourse – Implicit form

$$\min c^T x + E_\xi Q(x,\xi)$$
$$Ax = b,$$
$$x \geq 0$$

Minimum cost way to "correct" so that the constraints hold again

First stage deviation

Where $Q(x,\xi) = \min\{q^T y \mid Wy = h - Tx, y \geq 0\}$, ξ is the vector formed by the components of q^T, h^T, and T and E_ξ denote the mathematical expectation with respect to ξ

Back to the farmer's example

- The random vector is a discrete variable with only three different values (the three scenarios)
- A second stage problem for one particular scenario s can be written as:

$$Q(x,s) = \min\{238y_1-170w_1+210y_2-150w_2-36w_3-10w_4\}$$

$$\text{s.t. } t_1(s)X_1+y_1-w_1 \geq 200,$$

$$t_2(s)X_2+y_2-w_2 \geq 240,$$

$$w_3+w_4 \leq t_3(s)X_3,$$

$$w_3 \leq 6000,\ y,\ w >= 0$$

Two-stage stochastic program with recourse – Explicit form

- Less condensed

- Associate one decision vector in the second-stage to each possible realization of the random vector

Two-stage stochastic program with recourse – Explicit form

Farmer's problem

Minimize $150x_1+230x_2+260x_3+1/3(-170w_{11}-150w_{21}-36w_{31}-10w_{41}+238y_{11}+210y_{21})+1/3(-170w_{12}-150w_{22}-36w_{32}-10w_{42}+238y_{12}+210y_{22})+1/3(-170w_{13}-150w_{23}-36w_{33}-10w_{43}+238y_{13}+210y_{23})$

Subject to

$3x_1+y_{11}-w_{11}>=200$; $2.5x_1+y_{12}-w_{12}>=200$; $2x_1+y_{13}-w_{13}>=200$

$3.6x_2+y_{21}-w_{21}>=240$; $3x_2+y_{22}-w_{22}>=240$; $2.4x_{23}+y_{23}-w_{23}>=240$

$24x_{31}-w_3-w_4>=0$; $20x_{32}-w_3-w_4>=0$; $16x_{33}-w_3-w_4>=0$

$x_1+x_2+x_3<=500$; $W_{31},w_{32},w_{33}<=6000$; All variables >=0

Solution methods

- The ease of solving the problem depends on the properties of $Q(x) = E_{\xi}[Q(x,\xi)]$, known as **the recourse function or the value function**

- Problems where some variables (x and/or y) are integer (Stochastic Integer Programming), are generally more difficult to solve