Winter 2008 CS567 Stochastic Linear/Integer Programming
Guest Lecturer: Xu, Huan

Class 1: Some Examples on Modeling

1 Farmer's Problem

Consider a farmer, who has 500 acres of land, and raise grain, corn and sugar beet. In winter he wants to decide how much land he should devote for each crop in order to get maximal profit in the next autumn.

The farmer need $200T$ of wheat and $240T$ of corn to feed his cattle. These amounts can be either raised by himself or bought from an external wholesaler. If more than this amount is produced, the exceeded part will be sold. Selling prices are 170$ and 150$ per ton of wheat and corn respectively. The purchase prices are 40% more than the selling price.

Sugar beet sells at 36$ per ton. However, there is a quota on sugar beet production, any amount in excess of the quota can be sold only at 10$ per ton. The quota this year is $6000T$.

Based on past experience, the farmer knows that the mean yield on his land is roughly $2.5T$, $3T$ and $10T$ per acre for wheat, corn and sugar beets, respectively. And the planting costs are 150$, 230$ and 260$ per acre respectively.

Table 1: Data for farmer's problem

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Panting cost($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000T
			10 above 6000T
Purchase price ($/T)	238	210	-
Minimum requirement (T)	200	240	-
Total available land :500 acres			

Let $x_{1,2,3}$ be the acres of land devoted to wheat, corn and sugar beets respectively. $w_{1,2}$ be the tones of wheat and corn sold. $y_{1,2}$ be the tons of wheat and corn purchased. w_3 be the tons of sugar beets sold at the favorable price, and w_4 be the tons of sugar beets sold at the lower price. We formulate the problem as the following Linear Program

$$\begin{aligned}
\text{minimize:} \quad & 150x_1 + 230x_2 + 260x_3 + 238y_1 - 170w_1 + 210y_2 - 150w_2 - 36w_3 - 10w_4 \\
\text{such that:} \quad & x_1 + x_2 + x_3 \leq 500, \\
& 2.5x_1 + y_1 - w_1 \geq 200, \\
& 3x_2 + y_2 - w_2 \geq 240, \\
& w_3 + w_4 \leq 20x_3, \\
& w_3 \leq 6000, \\
& x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0.
\end{aligned}$$

If we solve this LP, we get the optimal solution based on the expected yield:

Table 2: Optimal solution based on expected yields

Culture	Wheat	Corn	Sugar Beets
Surface (acre)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	-	6000
Purchase (T)	-	-	-
Overall Profit :$118,600			

This optimal solution is easy to understand. The farmer devotes enough land to sugar beet to reach the quota of $6000T$. He then devotes enough land to wheat and corn to meet his feeding requirement. The rest land is devoted to wheat production for selling. That is, the optimal solution follows a simple rule according to decreasing profit per acre. In this example, the order is sugar beets at a favorable price, wheat for feeding, corn for feeding, wheat for selling, corn for selling, and sugar beets at the lower price.

One problem of this formulation is that we are assuming to get the expected yield with 100% probability. In fact, yields can vary above/below their mean. Now we want to study:

- does this variability affect the optimality of our solution;
- if it does, how to find the optimal solution under the variability.

We study two possible representation of this variability. One approach is a scenario based representation, and the other one is a continuous representation.

1.1 A scenario representation

Assume that the yields for different crops are correlated and all depends on the whether. Say that we have three possible whether condition: "good", "average" or "bad". Here "good" and "bad" stands for 20% beyond or below the mean yield for all crops.

The LP for the "good" scenario is

$$\begin{aligned}
\text{minimize:}\quad & 150x_1 + 230x_2 + 260x_3 + 238y_1 - 170w_1 + 210y_2 - 150w_2 - 36w_3 - 10w_4 \\
\text{such that:}\quad & x_1 + x_2 + x_3 \leq 500, \\
& 3x_1 + y_1 - w_1 \geq 200, \\
& 3.6x_2 + y_2 - w_2 \geq 240, \\
& w_3 + w_4 \leq 24x_3, \\
& w_3 \leq 6000, \\
& x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0.
\end{aligned}$$

And its solution is

Table 3: Optimal solution for the good scenario

Culture	Wheat	Corn	Sugar Beets
Surface (acre)	183.33	66.67	250
Yield (T)	550	240	6000
Sales (T)	350	-	6000
Purchase (T)	-	-	-
Overall Profit :$167,677			

The LP for the "bad" scenario is

$$\begin{aligned}
\text{minimize:} \quad & 150x_1 + 230x_2 + 260x_3 + 238y_1 - 170w_1 + 210y_2 - 150w_2 - 36w_3 - 10w_4 \\
\text{such that:} \quad & x_1 + x_2 + x_3 \leq 500, \\
& 2x_1 + y_1 - w_1 \geq 200, \\
& 2.4x_2 + y_2 - w_2 \geq 240, \\
& w_3 + w_4 \leq 16x_3, \\
& w_3 \leq 6000, \\
& x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0.
\end{aligned}$$

And its solution is

Table 4: Optimal solution for the bad scenario

Culture	Wheat	Corn	Sugar Beets
Surface (acre)	100	25	375
Yield (T)	200	60	6000
Sales (T)	-	-	6000
Purchase (T)	-	180	-
Overall Profit :$59,950			

Therefore, we see that, knowing the whether condition beforehand is very helpful for the farmer to make his decision. Unfortunately, weather conditions can not be accurately predicted six months ahead. The farmer must make up his mind without perfect information on yields. Assume all three scenarios may happen with probability 1/3, and the farmer is interested in maximizing his expected utility, let's see how this can be done. First, we notice that among all the variables, (x_1, x_2, x_3) have to be chosen before realizing the weather, where the decisions on sales and purchases depend on the yields. So we index those decision by a scenario index $s = 1, 2, 3$ corresponding to "good", "average" and "bad" weather respectively. Now the farmer's decision can be formed as the following LP:

$$
\begin{aligned}
\text{minimize:}\quad & 150x_1 + 230x_2 + 260x_3 \\
& + 1/3(238y_{11} + -170w_{11} + 210y_{21} - 150w_{21} - 36w_{31} - 10w_{41}) \\
& + 1/3(238y_{12} + -170w_{12} + 210y_{22} - 150w_{22} - 36w_{32} - 10w_{42}) \\
& + 1/3(238y_{13} + -170w_{13} + 210y_{23} - 150w_{23} - 36w_{33} - 10w_{43}) \\
\text{such that:}\quad & x_1 + x_2 + x_3 \leq 500, \\
& 3x_1 + y_{11} - w_{11} \geq 200, \\
& 3.6x_2 + y_{21} - w_{21} \geq 240, \\
& w_{31} + w_{41} \leq 24x_3, \\
& w_{31} \leq 6000, \\
& 2.4x_1 + y_{12} - w_{12} \geq 200, \\
& 3x_2 + y_{22} - w_{22} \geq 240, \\
& w_{32} + w_{42} \leq 20x_3, \\
& w_{32} \leq 6000, \\
& 2x_1 + y_{13} - w_{13} \geq 200, \\
& 2.4x_2 + y_{23} - w_{23} \geq 240, \\
& w_{33} + w_{43} \leq 16x_3, \\
& w_{33} \leq 6000, \\
& x, y, w \geq 0.
\end{aligned}
$$

Such a model of a stochastic decision program is known as the extensive form of the stochastic program because it explicitly describe the second stage decision variable for all scenarios. And its solution is

Table 5: Optimal solution for the bad scenario

		Wheat	Corn	Sugar Beets
First Stage	Surface (acre)	170	80	250
Good Weather	Yield (T)	510	288	6000
	Sales (T)	310	48	6000
	Purchase (T)	-	-	-
Average Weather	Yield (T)	425	240	5000
	Sales (T)	225	-	5000
	Purchase (T)	-	-	-
Bad Weather	Yield (T)	340	192	4000
	Sales (T)	140	-	4000
	Purchase (T)	-	48	-
Overall Profit :$108,390				

Suppose we know the scenario before hand, the mean profit is thus $1/3(\$118,600 + \$167,677 + \$59,950) = \$115,406$. Hence, we have a gap $\$7,016$ between the solution of knowing and not knowing the scenario beforehand. This gap, is called *the expected value of perfect information* (EVPI).

Another approach the farmer may have is to assume expected yields ad allocate the optimal planting surface according to these yields (so-called *expected value solution*). This will lead to an expected profit

\$107,240. The loss by not considering the randomness is the difference between this and the stochastic model profit. This value, \$108,390 − \$107,240 = \$1,150 is the *value of the stochastic solution* (VSS).

VSS tells us whether we should take a stochastic approach, and EVPI tells us whether we should dig more information.

1.2 General Model

This example illustrate the general formulation of a stochastic problem. We have a set of decision to be taken without full information on some random events. These decisions are called first-stage decisions, represented as x. Later, full information is received on the realization of some random vector $\boldsymbol{\xi}$. Then, second stage or corrective actions $\mathbf{y}$ are taken. In mathematical programming terms, this defines the so-called two-stage stochastic program with recourse of the form

$$\begin{aligned} \text{Minimize:} \quad & c^\top x + \mathbb{E}_{\boldsymbol{\xi}} Q(x, \boldsymbol{\xi}) \\ \text{Such that:} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

where $Q(x, \boldsymbol{\xi}) = \min\{\mathbf{q}^\top \mathbf{y} | W\mathbf{y} = \mathbf{h} - \mathbf{T}x, y \geq 0\}$, $\boldsymbol{\xi}$ is the vector formed by the components of $\mathbf{q}^\top$, $\mathbf{h}^\top$ and $\mathbf{T}$. We assumed that W is fixed (*fixed recourese*).

In the farmer example, the random vector is a discrete variable with only three different realizations. The second stage problem can be written as

$$\begin{aligned} Q(x, s) = \min\{238y_1 - 170w_1 + 210y_2 - 150w_2 - 36w_3 - 10w_4\} \\ \text{s.t. } t_1(s)x_1 + y_1 - w_1 \geq 200, \\ t_2(s)x_2 + y_2 - w_2 \geq 240, \\ w_3 + w_4 \leq t_3(s)x_3, \\ w_3 \leq 6000, \\ y_1, y_2, w_1, w_2, w_3, w_4 \geq 0. \end{aligned}$$

This form is called the implicit representation of the stochastic program. A more condensed implicit representation is by defining $\mathcal{Q}(x) = \mathbb{E}_{\boldsymbol{\xi}} Q(x, \boldsymbol{\xi})$ as the *value function* or *recourse function*:

$$\begin{aligned} \text{Minimize:} \quad & c^\top x + \mathcal{Q}(x) \\ \text{Such that:} \quad & Ax = b. \\ & x \geq 0, \end{aligned}$$

1.3 Continuous random variable

We now assume that the yields for different crops are independent and follows a continuous distribution. To be more detailed, assume that the yield for each crop uniformly distributed in $[l_i, u_i]$ where l_i is 80% of the mean, and u_i is 120% of the mean.

Notice in this particular example, the recourse function is separable

$$\mathbb{E}_{\boldsymbol{\xi}} Q(x, \boldsymbol{\xi}) = \sum_{i=1}^{3} \mathbb{E}_{\boldsymbol{\xi}} Q_i(x_i, \boldsymbol{\xi}) = \sum_{i=1}^{3} \mathcal{Q}_i(x_i),$$

where $Q_i(x_i, \boldsymbol{\xi})$ is the optimal second-stage value of purchases and sales of crop i.

Considering Sugar beets:

$$\begin{aligned} Q_3(x_3, \boldsymbol{\xi}) = \min\{ &-36w_3(\boldsymbol{\xi}) - 10w_4(\boldsymbol{\xi})\} \\ \text{s.t. } & w_3(\boldsymbol{\xi}) + w_4(\boldsymbol{\xi}) \leq t_3(\boldsymbol{\xi})x_3, \\ & w_3(\boldsymbol{\xi}) \leq 6000, \\ & w_3(\boldsymbol{\xi}), w_4(\boldsymbol{\xi}) \geq 0. \end{aligned}$$

The optimal solution here is

$$\begin{aligned} w_3(\boldsymbol{\xi}) &= \min[6000,\ t_3(\boldsymbol{\xi})x3] \\ w_4 &= \max[t_3(\boldsymbol{\xi})x_3 - 6000,\ 0]. \end{aligned}$$

Therefore, we have

$$Q_3(x_3, \boldsymbol{\xi}) = -36\min[6000,\ t_3(\boldsymbol{\xi})x3] - 10\max[t_3(\boldsymbol{\xi})x_3 - 6000,\ 0].$$

If x_3 is such that $l_3x_3 \leq 6000 \leq u_3x_3$ we have

$$\begin{aligned} \mathcal{Q}_3(x_3) &= -\int_{l_3}^{6000/x_3} 36 + x_3 f(t)dt - \int_{6000/x_3}^{u_3} (216000 + 10tx - 3 - 60000)f(t)dt \\ &= -36\bar{t}_3x_3 + \frac{13(u_3x_3 - 6000)^2}{x_3(u_3 - l_3)}, \end{aligned}$$

where $\bar{t}_3$ denotes the expected yield for sugar beet.

If x_3 is such that $l_3x_3 > 6000$ then

$$\mathcal{Q}_3(x_3) = -156000 - 10\bar{t}_3x_3.$$

If $u_3x_3 < 6000$ then

$$\mathcal{Q}_3(x_3) = -36\bar{t}_3x_3.$$

If we plot the function, we will find it to be continuous and convex.

The recourse function for the other two crops are similarly obtained. For wheat we have

$$\mathcal{Q}_1(x_1) = \begin{cases} 47600 - 595x_1 & \text{if } x_1 < 200/3 \\ 119(200 - 2x_1)^2/x_1 - 85(200 - 3x_1)^2/x_1 & \text{if } 200/3 \leq x_1 \leq 100 \\ 34000 - 425x_1 & \text{if } x_1 \geq 100, \end{cases}$$

and for corn we have

$$\mathcal{Q}_2(x_2) = \begin{cases} 50400 - 630x_2 & \text{if } x_2 < 200/3 \\ 87.5(240 - 2.4x_2)^2/x_2 - 62.5(240 - 3.6x_2)^2/x_2 & \text{if } 200/3 \leq x_2 \leq 100 \\ 36000 - 450x_2 & \text{if } x_2 \geq 100, \end{cases}$$

The global problem is therefore

$$\begin{aligned} \text{Minimize:} \quad & 150x_1 + 230x_2 + 260x_3 + \mathcal{Q}_1(x_1) + \mathcal{Q}_2(x_2) + \mathcal{Q}_3(x_3) \\ \text{Such that:} \quad & x_1 + x_2 + x_3 \leq 500, \\ & x_{1,2,3} \geq 0 \end{aligned}$$

This is a non-linear program, solve the solution $x_1 = 135.83$, $x_2 = 85.07$ and $x_3 = 279.10$.

2 News Vendor Problem

A news vendor goes to the publisher every morning and buys x newspaper at a price of c per paper. This number is usually bounded above by some limit u. The news vendor then tries to sell as many newspapers as he can at a price $q > c$. Any unsold newspaper can be returned at a price $q < c$. Demand for newspapers varies over days and is described by a random variable $\boldsymbol{\xi}$. Assume the news vendor cannot return to the publisher to buy more newspapers during the day and readers only want the last edition.

To formulate this problem, we use y to represent the effective sales and w to be the number of newspapers returned. Hence, the problem can be formulated as

$$\begin{aligned} &\min cx + \mathcal{Q}(x) \\ &\text{s.t. } 0 \leq x \leq u, \end{aligned}$$

where $\mathcal{Q}(x) = \mathbb{E}_{\boldsymbol{\xi}} Q(x, \boldsymbol{\xi})$, and

$$\begin{aligned} Q(x, \boldsymbol{\xi}) = \min\ & -qy(\boldsymbol{\xi}) - rw(\boldsymbol{\xi}) \\ \text{s.t. } & y(\boldsymbol{\xi}) \leq \boldsymbol{\xi}, \\ & y(\boldsymbol{\xi}) + w(\boldsymbol{\xi}) \leq x, \\ & y(\boldsymbol{\xi}), w(\boldsymbol{\xi}) \geq 0. \end{aligned}$$

It is easy to see that the optimal solutions are $y^*(\boldsymbol{\xi}) = \min(\boldsymbol{\xi}, x)$ and $w^*(\boldsymbol{\xi}) = \max(x - \boldsymbol{\xi}, 0)$. Therefore, the second stage expected value function is

$$\begin{aligned} \mathcal{Q}(x) =& \mathbb{E}_{\boldsymbol{\xi}}[-q \min(\boldsymbol{\xi}, x) - r \max(x - \boldsymbol{\xi}, 0)] \\ =& \int_{-\infty}^{x} (-q\xi - r(x - \xi)) dF(\xi) + \int_{x}^{\infty} -qx dF(\xi) \\ =& -(q - r) \int_{-\infty}^{x} \xi dF(\xi) - rxF(\xi) - qx(1 - F(x)). \end{aligned}$$

Since $\mathcal{Q}(x)$ is convex and differentiable when $\boldsymbol{\xi}$ is a continuous random variable, the optimal x mush satisfy

$$\begin{cases} x^* = 0 & \text{if } c + \mathcal{Q}'(0) > 0, \\ x^* = u & \text{if } c + \mathcal{Q}'(u) < 0, \\ c + \mathcal{Q}'(x^*) = 0 & \text{otherwise.} \end{cases}$$

Integrating by parts, we observe that

$$\int_{-\infty}^{x} \xi dF(\xi) = xF(x) - \int_{-\infty}^{x} F(\xi) d\xi$$

(under some mild technical condition on $F(\xi)$). It follows that

$$\mathcal{Q}(x) = -qx + (q - r) \int_{-\infty}^{x} F(\xi) d\xi.$$

We hence have

$$\mathcal{Q}'(x) = -q + (q - r)F(x).$$

Taking derivative of $cx + \mathcal{Q}(x)$ we get the optimal solution

$$\begin{cases} x^* = 0 & \text{if } (q - c)/(q - r) < F(0), \\ x^* = u & \text{if } (q - c)/(q - r) > F(u), \\ x^* = F^{-1}\left(\frac{q-c}{q-r}\right) & \text{otherwise.} \end{cases}$$

3 Financial Planning

To provide for a child's college education Y years from now. We currently have \$$b$ to invest in any of I investments. After Y years, we will have a wealth that we would like to have exceed a tuition goal \$$G$. Suppose we can change investments every v years, so we have Y/v investment periods. Now suppose exceeding \$$G$ would be equivalent to having an income of $q\%$ of the excess, while not meeting the goal would lead to borrowing for a cost $r\%$ of the amount short. The return of each investment on each period is random. We observe the return of each past period, but can not anticipate. Let $I = 2$ including a stock and a government security. $Y = 15$ and $v = 5$. Therefore we have 3 investment periods. Assume we have 8 possible scenario: independent, equal likelihood of having one of the two cases in each period :(1) 1.25 for stock and 1.14 for bound; (2) 1.06 for stock and 1.12 for bond. We have the following program:

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{s_H} \cdots \sum_{s_1} p(s_1, \cdots, s_H)\big(-rw(s_1, \cdots, s_H) + gy(s_1, \cdots, s_H)) \\
\text{s.t.} \quad & \sum_i x(i, 1) = b \\
& \sum_i -\xi(i, t-1, s1, \cdots, s_{t-1})x(i, t-1, s_1, \cdots, s_{t-2}) + \sum_i x(i, t, s1, \cdots, s_{t-1}) = 0, \\
& \quad t = 2, \cdots, H;\ \forall(s_1, \cdots, s_{t-1}); \\
& \sum_i -\xi(i, H, s1, \cdots, s_H)x(i, H, s_1, \cdots, s_{H-1}) - y(s_1, \cdots, s_H) + w(s_1, \cdots, s_H) = G \\
& \quad \forall(s_1, \cdots, s_H).
\end{aligned}$$

Let $b = 55000$, $G = 80000$, $q = 1$, $r = 4$. The resulting optimal expected value is -1.52. If we used the expected parameter everywhere, we get a solution which invest everything in stocks in each period, and having an expected utility -3.79. The VSS is 2.27 in this case. Besides, the probability of achieving the goal for the stochastic formulation is 87.5% and for the expected parameter approach is 50%.