Operations Research Letters 23 (1998) 35–40

The complexity of cover inequality separation[1]

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Received 1 November 1995; received in revised form 1 June 1998

Abstract

Crowder et al. (Oper. Res. 31 (1983) 803–834) conjectured that the separation problem for cover inequalities for binary integer programs is polynomially solvable. We show that the general problem is NP-hard but a special case is solvable in linear time. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Integer programming; Cover inequalities; Separation problem

1. Introduction

Separation problems for linear programs with an exponential number of constraints arise in polyhedral approaches to combinatorial optimization problems. These separation problems frequently are special cases of NP-hard problems such as knapsack or max-cut. The special cases arise because of the restricted inputs, which come from the solutions of relaxed linear programs. An example is the separation problem for cover inequalities for binary integer programs introduced by Crowder et al. [2] more than 10 years ago. The following is a quote from their paper that enlightens the separation problem:

> "We start by solving the linear program $\max\{cx\colon Ax \leqslant d,\ 0 \leqslant x \leqslant 1\}$ and obtain an optimal solution x^*. If x^* is a zero-one solution, we stop: x^* solves the binary integer program. Otherwise we solve the following problem.
>
> Given x^* find a minimal cover inequality that chops off x^*, if such an inequality exists. … We conjecture that violated minimal cover inequalities can be identified by a polynomially bounded algorithm".

We resolve the conjecture by showing that the cover inequality separation problem is NP-hard, even if the binary integer program is a knapsack problem. However, there is some validity to the conjecture, since if the optimal linear programming solution is an extreme point and the number of rows is a constant or grows polylogarithmically, then the separation problem is polynomially solvable.

It has been pointed out to us, after this note first appeared as a technical report, that Ferreira resolved this conjecture in his thesis [3] which is also cited in [4]. However, Ferreira's input for the separation problem is decoupled from the solution of the linear

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[1] This research was supported by US Army Research Office DAAH04-94-G-0017 and NSF Grant No. DMI-9700285.

PII: S 0167-6377(98)00025-X

program that provides the point to be separated and his argument does not work if the LP solution does not satisfy the original knapsack constraint.

Binary integer programming (BIP) is the problem

$\max\{cx:\ Ax \leqslant d,\ x \in B^n\}$,

where $A \in \mathbb{Z}_+^{m\times n}$ is an integer matrix, $d \in \mathbb{Z}_+^m$ is an integer vector, $c \in \mathbb{R}^n$ and B stands for the set $\{0, 1\}$. BIP is NP-hard. Branch-and-cut is one of the most successful approaches to BIP (for details see e.g. [5–8]). An important subroutine is to generate cuts from a single knapsack constraint.

Suppose a is a row of the matrix A and let b be the corresponding coordinate of the right-hand side d. To generate valid inequalities for BIP, we can generate valid inequalities for the polytope

$P = \mathrm{conv}\{ax \leqslant b,\ x \in B^n\}$,

where 'conv' is the abbreviation for convex hull.

Let C be a minimal subset of $N = \{1, \ldots, n\}$ such that $\sum_{i\in C} a_i > b$. Then

$$\sum_{i\in C} x_i \leqslant |C| - 1$$

is a valid inequality for P. Inequalities of this type are called *cover inequalities*. Let x^* be a solution to the LP relaxation of BIP, i.e. x^* is a solution to

$$\max\{cx:\ Ax \leqslant d,\ 0 \leqslant x \leqslant 1\}. \tag{1}$$

We can assume that Eq. (1) is feasible since otherwise BIP is infeasible. If $x^* \in B^n$, then we are done. So suppose that $x^* \notin B^n$. In this case we would like to, if possible, cut off x^* from P using cover inequalities. It is reasonable to find the most violated cover inequality, i.e. one that minimizes $|C| - 1 - \sum_{i\in C} x_i^*$, since if this minimum is nonnegative, we know that there are no violated cover inequalities.

We introduce new variables z_i for each $i \in N$ such that $z_i = 1$ if $i \in C$ and $z_i = 0$ otherwise. Then we can rewrite $\sum_{i\in C} a_i \geqslant b+1$ as $\sum_{i\in N} a_i z_i \geqslant b+1$ and $|C| - 1 - \sum_{i\in C} x^*$ as $\sum_{i\in N} (1 - x_i^*)z_i - 1$. So our goal is to solve

$$\min\left\{\sum_{i\in N} (1 - x_i^*)z_i:\ \sum_{i\in N} a_i z_i \geqslant b + 1,\ z \in B^n\right\}, \tag{2}$$

which is known as *the cover inequality separation problem for BIP*.

Although the knapsack problem is NP-hard, (2) is not a general knapsack problem since the vector x^* is not arbitrary. x^* is the output of an LP relaxation and hence it must be in the polyhedron $\{x \in \mathbb{R}^n:\ Ax \leqslant d,\ 0 \leqslant x \leqslant 1\}$. In particular, it has to satisfy the condition $\sum_{i\in N} a_i x_i^* \leqslant b$.

Instead of Eq. (2) we can pose a weaker question: Does there exist a violated cover inequality?

Problem DSP

Input: (A, d, c) and an optimal solution x^* to Eq. (1).

Output: Is there a row (a, b) of (A, d) such that $\min\{\sum_{i\in N} (1 - x_i^*)z_i:\ \sum_{i\in N} a_i z_i \geqslant b+1,\ z \in B^n\} < 1$, i.e. is there a violated cover inequality for the given LP solution x^*?

Note that DSP cannot be harder then the problem where a row index is specified as part of the input since solving the latter problem m times solves DSP. Thus if DSP is NP-complete, then the version of DSP with a specified row is NP-hard. It is reasonable to assume that the size of the input is the size of (A, d, c) since we can always find an x^* whose size is polynomially bounded by the size of (A, d, c) (see e.g. [6]). By standard methods we can show that DSP is in NP. Also, the NP-completeness of DSP implies that Eq. (2) is NP-hard. We show in Section 2 that DSP is NP-complete even for a single constraint. We do this by first showing that a restricted version of the subset sum problem is NP-complete and then we reduce this restricted version to DSP.

However, if we have an extreme point optimal solution to Eq. (1) and the number of constraints is constant, we show in Section 3 that the separation problem is polynomially solvable. We also show that if the number of constraints is not a constant, then the problem remains NP-complete. Hence we see that imposing structure on x^* changes the behavior of the problem.

2. The separation problem DSP is NP-hard

We will first show that problems RPP and RSSP given below are NP-complete.

Problem RPP

Input: $r \in \mathbb{Z}_+^n$, $m \in \mathbb{Z}_+$ such that $\sum_{i \in N} r_i = (2^{m+1} - 1)$.

Output: Is there a subset $T \subset N$ such that $\sum_{i \in T} r_i = 2^{m+1} - 1$ or $\sum_{i \in T} r_i = 2^{m+1} - 2$?

Proposition 1. *Problem RPP is NP-complete.*

Proof. We do a reduction from the subset sum problem, which is known to be NP-complete, see e.g. [9].

Subset sum problem SSP

Input: $a \in \mathbb{Z}_+^n$ such that $\sum_{i \in N} a_i$ is even.

Output: Is there a subset $S \subset N$ such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$?

Let $a \in \mathbb{Z}_+^n$ be the data for SSP. Define the following data for RPP. Let m be the smallest integer such that $\sum_{i \in N} a_i < 2^m$. Note that m can be easily computed and its size is polynomial in the size of the input of SSP. Let

$$r_i = 2a_i, \quad i \in N,$$
$$r_{n+1} = r_{n+2} = 2^{m+1} - \sum_{i \in N} a_i - 1,$$

which is a valid input for RPP.

Suppose that S is such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$. Then it is easy to check that $T = S \cup \{n+1\}$ yields a yes answer to RPP.

Now let $T \subset \{1, \ldots, n+2\}$ be such that $\sum_{i \in T} r_i = 2^{m+1} - 1$ or $\sum_{i \in T} r_i = 2^{m+1} - 2$. Suppose that $n+1 \in T$ and $n+2 \in T$. Then

$$2^{m+1} - 1 \geqslant \sum_{i \in T} r_i \geqslant 2^{m+2} - 2 - 2\sum_{i \in N} a_i$$

contradicts the choice of m.

Hence, without loss of generality, we can assume that $n+1 \in T$ and $n+2 \notin T$. Let $S = T - \{n+1\}$. Then

$$2^{m+1} - 1 \geqslant \sum_{i \in S} r_i + 2^{m+1} - \sum_{i \in N} a_i - 1 \geqslant 2^{m+1} - 2.$$

It easily follows that

$$\frac{\sum_{i \in N} a_i}{2} \geqslant \sum_{i \in S} a_i \geqslant \frac{\sum_{i \in N} a_i}{2} - \frac{1}{2}$$

and hence S is the desired set for SSP. □

We say that a rational number is *encoded in binary* if it is represented in base 2 as a single string, e.g. '101.1011'.

Problem RSSP

Input: Rational $f \in [0,1]^n$ encoded in binary.

Output: Is there a subset $T \subset N$ such that $\sum_{i \in N} f_i - \sum_{i \in N} f_i^2 < \sum_{i \in T} f_i < 1$?

Proposition 2. *The problem RSSP is* NP-*complete.*

Proof. We do a reduction from RPP.

Let $r \in \mathbb{Z}_+^n$ and $m \in Z_+$ be a valid input for RPP. Define the input for RSSP by

$$f_i = 2^{-m^2} r_i, \quad i \in N,$$
$$f_{n+1} = \tfrac{1}{2},$$
$$f_{n+2} = \tfrac{1}{2} - 2^{-m^2+m+1},$$
$$f_{n+3} = \tfrac{1}{2},$$
$$f_{n+4} = \tfrac{1}{2} - 2^{-(m+1)(m-2)/2}.$$

A straightforward calculation shows that

$$A = \sum_{i \in N} f_i - \sum_{i \in N} f_i^2 = 1 - 2^{-m^2+1} - 2^{-2m^2+2m+2} - 2^{-2m^2} \sum_{i \in N} r_i^2.$$

Since $\sum_{i \in N} r_i^2 \leqslant (\sum_{i \in N} r_i)^2 \leqslant 2^{2m+4}$ we can find the asymptotic behavior of

$$A = 1 + \mathrm{o}(1), \qquad A = 1 - \mathcal{O}(2^{-m^2}).$$

Suppose that S is a set with the desired property for RPP. Define $T = S \cup \{n+1, n+2\}$. We show that T is a set with the desired property for RSSP. We have

$$\sum_{i \in T} f_i \leqslant 2^{-m^2}(2^{m+1} - 1) + 1 - 2^{-m^2+m+1} = 1 - 2^{-m^2} < 1.$$

On the other hand,

$$\sum_{i\in T} f_i \geqslant 2^{-m^2}(2^{m+1}-2)+1-2^{-m^2+m+1}$$
$$= 1-2^{-m^2+1} > A$$

and hence T has the desired property for RSSP.

Now let $T\subset\{1,\dots,n+4\}$ be such that $A<\sum_{i\in T} f_i<1$. Let $S=T\cap N$. Then since $\sum_{i:\ i>n, i\in T} f_i<1$ and since

$$\sum_{i:\ i>n,i\in T} f_i + \mathrm{o}(1) = \sum_{i:\ i>n,i\in T} f_i + 2^{-m^2}2(2^{m+1}-1)$$
$$\geqslant \sum_{i\in T} f_i > A = 1+\mathrm{o}(1),$$

it follows that $\sum_{i:\ i>n,i\in T} f_i = 1+\mathrm{o}(1)$.

Suppose that $f_{n+4}\in T$. Then

$$\sum_{i\in T} f_i \leqslant 1-2^{-[(m+1)(m-2)]/2}+2^{-m^2}\sum_{i\in S} r_i$$
$$\leqslant 1-2^{-[(m+1)(m-2)]/2}+2^{-m^2}2(2^{m+1}-1)$$
$$= 1-\Omega(2^{-m^2/2})+2^{-m^2+m+2}.$$

On the other hand, $\sum_{i\in T} f_i > A = 1-\mathcal{O}(2^{-m^2})$. So we get that $1-\Omega(2^{-m^2/2})\geqslant 1-\mathcal{O}(2^{-m^2})$, which is a contradiction for big enough m.

Hence $f_{n+4}\notin T$. But then either $f_{n+1}\in T$ or $f_{n+3}\in T$. Without loss of generality, we can assume that $f_{n+1}\in T$. Therefore $f_{n+2}\in T,\ f_{n+3}\notin T$.

From $\sum_{i\in T} f_i<1$, it follows by the definition of f_i that $\sum_{i\in S} r_i \leqslant 2^{m+1}-1$. Also from

$$\sum_{i\in T} f_i = 1-2^{-m^2+m+1}+2^{-m^2}\sum_{i\in S} r_i > A$$

and by the definition of A it follows that $\sum_{i\in S} r_i > 2^{m+1}-2-\mathcal{O}(2^{-2m^2})$ and hence $\sum_{i\in S} r_i \geqslant 2^{m+1}-2$ for big enough m. □

Now we will show that DSP for BIP is NP-complete even when the number of constraints m equals 1.

Let $a,c\in\mathbb{Z}^n_+$ and $b\in\mathbb{Z}_+$ be given. Suppose we are given also a solution $x^*\notin B^n$ to the problem $\max\{cx:\ ax\leqslant b,\ 0\leqslant x\leqslant 1\}$.

Theorem 1. *DSP is NP-complete when the number of constraints equals* 1.

Proof. The reduction is from RSSP. Let f,n be the input for RSSP. Define $e\in\mathbb{Z}^n_+$ to be the vector of all 1's and let $M\in\mathbb{Z}_+$ be the smallest power of 2 such that $Mf_i\in\mathbb{Z}_+$ for all $i=1,\dots,n$. Also define

$$a=M^2f,\qquad c=M^2f,\qquad b=M^2f(e-f),$$
$$x^*=e-f.$$

It is clear that x^* is an optimal solution and therefore this is a valid input for DSP.

Now, it is easy to see that an answer to RSSP is yes if and only if there exists a violated cover inequality for DSP. □

3. Polynomially solvable case of the separation problem

Suppose we have an optimal extreme point solution to Eq. (1). Then it is natural to formulate the separation problem

Problem DSP1

Input: (A,d,c) and an optimal extreme point x^* to Eq. (1).

Output: Is there a row (a,b) of (A,d) such that $\min\{\sum_{i\in N}(1-x_i^*)z_i:\ \sum_{i\in N} a_iz_i\geqslant b+1,\ z\in B^n\}<1$, i.e. is there a violated cover inequality for the given extreme point LP solution x^*?

Theorem 2. *If the number of constraints in* Eq. (1) *is a constant* (*resp. polylogarithmic*), *then DSP1 can be solved in linear* (*resp. polynomial*) *time.*

Proof. Let the number of constraints be a constant α. Since x^* is an extreme point, we know that at most α components of x^* are fractional and all others are 0 or 1. Without loss of generality, $x_1^*=x_2^*=\cdots=x_u^*=0$

and $x^*_{u+1}, x^*_{u+2}, \ldots, x^*_{u+\alpha}$ are fractional and all other components of x^* equal one. The separation problem is

$$\min\left\{\sum_{i=1}^{u} z_i + \sum_{i=1}^{\alpha}(1-x^*_{u+i})z_{u+i}\colon \sum_{i=1}^{u+\alpha} a_i z_i \geqslant b+1-\sum_{i=u+\alpha+1}^{n} a_i, \in B^{u+\alpha}\right\}. \quad (3)$$

We present a linear time algorithm to solve Eq. (3). Fix a subset $S \subseteq \{u+1, u+2, \ldots, u+\alpha\}$ and assign $z_j = 1$ if $j \in S$ and 0 otherwise. Now solve

$$\sum_{j\in S}(1-x^*_j) + \min\left\{\sum_{i=1}^{u} z_i\colon \sum_{i\in N} a_i z_i \geqslant b+1 - \sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j, z \in B^u\right\}. \quad (4)$$

Next we present a subroutine that solves problem (4) in linear time.

Find the median of $a_1, \ldots, a_{u+\alpha}$. Suppose a_k is the median. Define $L_k = \{i\colon a_i < a_k\}$ and $U_k = \{i\colon a_i \geqslant a_k\}$. Compute $l = \sum_{i\in U_k} a_i$. If $l < b+1-\sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j$, then set $z_i = 1$ for all $i \in U_k$ and recursively solve the problem

$$\sum_{j\in S}(1-x^*_j) + \min\left\{\sum_{i\in L_k} z_i\colon \sum_{i\in L_k} a_i z_i \geqslant b+1 - \sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j - \sum_{i\in U_k} a_i,\ z_i \in B^{|L_k|}\right\}.$$

If $l \geqslant b+1-\sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j$, then set $z_i = 0$ for all $i \in L_k$ and recursively solve the problem

$$\sum_{j\in S}(1-x^*_j) + \min\left\{\sum_{i\in U_k} z_i\colon \sum_{i\in U_k} a_i z_i \geqslant b+1 - \sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j,\ z_i \in B^{|U_k|}\right\}.$$

The complexity of this algorithm is $\mathcal{O}(n)$ since we can find the median with a linear time algorithm, see e.g. [1], and we precompute the sums $\sum_{j\in S}(1-x^*_j)$, $\sum_{i=u+\alpha+1}^{n} a_i$, and $\sum_{j\in S} a_j$.

Now iterate the above step for all the subsets S of $\{u+1, u+2, \ldots, u+\alpha\}$ and for all rows and take a solution with the smallest value. The overall complexity of the algorithm is $\alpha 2^\alpha \mathcal{O}(n)$. Since α is a constant, this yields a linear time algorithm.

The same procedure yields a polynomial time separation algorithm if the number of constraints is polylogarithmic in the input size. □

Since the linear time separation algorithm has a large constant in its bound, we present an alternative way to solve Eq. (4) with complexity $\mathcal{O}(n\log n)$. First observe that the problem can be solved by a greedy algorithm assuming that $a_1, a_2, \ldots, a_n$ are in nonincreasing order. So first, we sort the a_i in nonincreasing order. Assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. For each j, compute $S_j = \sum_{i=1}^{j} a_i$. Now with binary search find a $1 \leqslant k \leqslant u+\alpha$ such that $S_k < b+1-\sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j$ and $S_{k+1} \geqslant b+1-\sum_{u+\alpha+1}^{n} a_i \sum_{j\in S} a_j$. Then the solution to Eq. (4) is $z_1 = z_2 = \cdots = z_k = 1$, $z_{k+2} = \cdots = z_u = 0$, and $z_{k+1} = S_{k+1} - (b+1-\sum_{i=u+\alpha+1}^{n} a_i - \sum_{j\in S} a_j)$.

The preprocessing step has complexity $\mathcal{O}(n\log n)$, while the binary search has complexity $\mathcal{O}(\log n)$. So this procedure yields an algorithm with time complexity $\mathcal{O}(n\log n) + 2^\alpha \mathcal{O}(\log n)$.

Next we prove that the problem becomes difficult if the number of constraints grows linearly.

Theorem 3. *DSP1 is NP-complete.*

Proof. We do a reduction from RSSP, which is NP-complete by Proposition 2. Let f, n be the data for RSSP.

We construct an instance of BIP with $2n$ columns and $2n+1$ rows. Let K be a node–node adjacency matrix of a cycle on n nodes. Define the system of inequalities as

$$fx \leqslant f(e-f),$$
$$x_i + 2y_1 + 2y_2 + 2y_3 \leqslant 4 - f_i, \quad i \in N,$$
$$Ky \leqslant e.$$

Let $fx + ey$ be the objective function. Also let $x^* = (e-f, \frac{1}{2}e)$. We can multiply all the rows and the objective function coefficients by a power of 2 to get an instance of BIP with integral data.

First we have to prove that this is a valid input for DSP1. It is easy to see that x^* is feasible. By summing

$Ky \leqslant e$ over all rows it follows that x^* is an optimal solution to the LP relaxation.

Next we show that x^* is an extreme point. If n is odd, then K is nonsingular. Hence setting the last $2n$ inequalities of the LP as equalities yields a nonsingular system of $2n$ equations and $2n$ unknowns with the unique solution x^*. If n is even, then the first $2n$ rows of the LP give a nonsingular system of equations having $2n$ unknowns and the unique solution x^*. The nonsingularity can be seen by sequentially subtracting the rows $2, 3, 4 \ldots 2n$ from the first row.

If we have a yes answer to RSSP, then it yields a violated cover inequality for the top row of the system.

Suppose now that we have a yes answer to DSP1. The violated cover inequality cannot come from $Ky \leqslant e$ since these rows are already covers. All the cover inequalities for $x_i + 2y_1 + 2y_2 + 2y_3 \leqslant 4 - f_i$ are

$$y_u + y_v \leqslant 1, \quad u, v \in \{1,2,3\},\ u \neq v,$$

$$x_i + y_u + y_v \leqslant 2, \quad u, v \in \{1,2,3\},\ u \neq v,$$

$$y_1 + y_2 + y_3 \leqslant 2,$$

$$x_i + y_1 + y_2 + y_3 \leqslant 3.$$

However, none of the above inequalities is violated by x^*.

Hence, a violated cover inequality has to come from the row $fx \leqslant e(f - e)$. But then this inequality produces a yes answer for RSSP. □

Acknowledgements

We are grateful to an anonymous referee whose comments led to an improved version of the paper.

References

[1] T.H. Cormen, C.E. Leiserson, R.L. Rivest, Introduction to Algorithms, McGraw-Hill, New York, 1989.

[2] H. Crowder, E.L. Johnson, M.W. Padberg, Solving large scale 0–1 linear programming problems, Oper. Res. 31 (1983) 803–834.

[3] C.E. Ferreira, On combinatorial optimization problems arising in computer systems design, Ph.D. Thesis, Technische Universität Berlin, 1994.

[4] C.E. Ferreira, A. Martin, R. Weismantel, Solving multiple knapsack problems by cutting planes, 1994, preprint.

[5] K. Hoffman, M.W. Padberg, LP-based combinatorial problem solving, Ann. Oper. Res. 4 (1985) 145–194.

[6] G.L. Nemhauser, L.A. Wolsey, Integer and Combinatorial Optimization, Wiley, New York, 1988, pp. 386–392.

[7] M.W. Padberg, G. Rinaldi, Optimization of a 532-City traveling salesman problem by branch and cut, Oper. Res. Lett. 6 (1987) 1–8.

[8] M.W. Padberg, G. Rinaldi, A branch-and-cut algorithm for the resolution of large scale symmetric traveling salesman problems, SIAM Rev. 33 (1991) 60–100.

[9] C.H. Papadimitriou, Computational Complexity, Addison-Wesley, Reading, MA, 1994.