C&O 370: Deterministic OR Models

Strong IP Formulations

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Guidelines for Strong Formulations

LP vs IP

- Good linear programming formulations have as few variables and constraints as possible.

  Remember: Running time of LP solvers depends heavily on number of variables and on number of constraints.

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  Remember: Running time of LP solvers depends heavily on number of variables and on number of constraints.
- Different for IP!
  - Computational experiments suggest that the choice in formulation crucially influences solution time and sometimes solvability
  - Feasible region of LP relaxation resembles convex hull of feasible integer points closely

LP Relaxation

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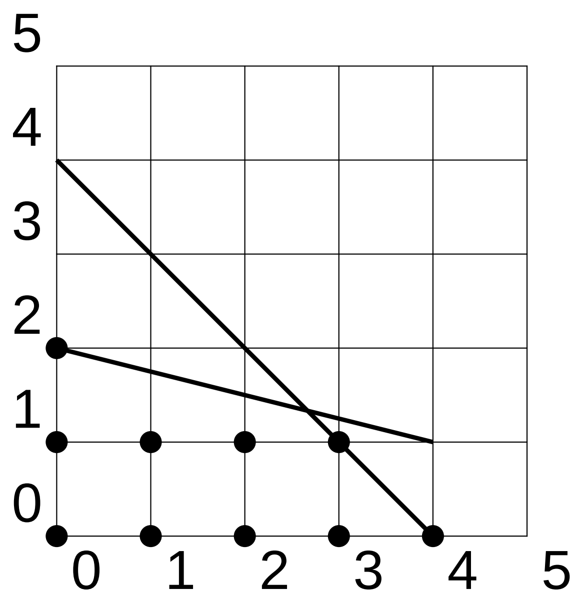
- Example

$$\begin{aligned} \max \quad & 3x_1 + 10x_2 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 8 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{aligned} \qquad \text{(IP)}$$

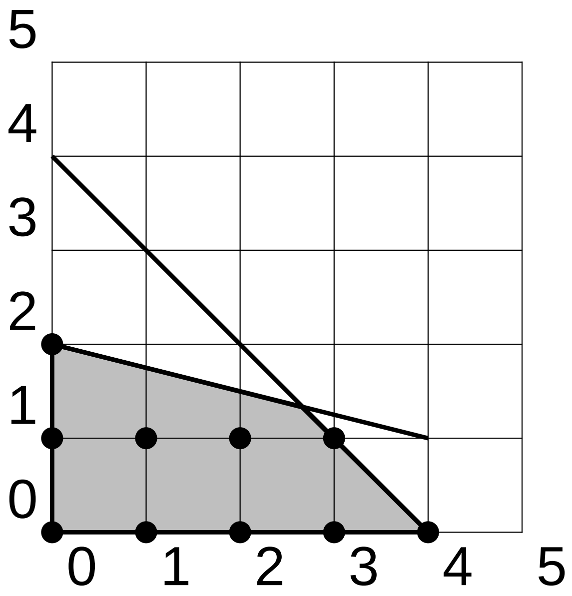
LP Relaxation

- Geometric view:



LP Relaxation

- We obtain the linear programming relaxation of an integer program by dropping the integrality constraints



Convex Hull

- We have seen: optimal solution to LP relaxation is fractional. Can we write a different LP with the same set of feasible integer solutions for which has an integral optimal solution?

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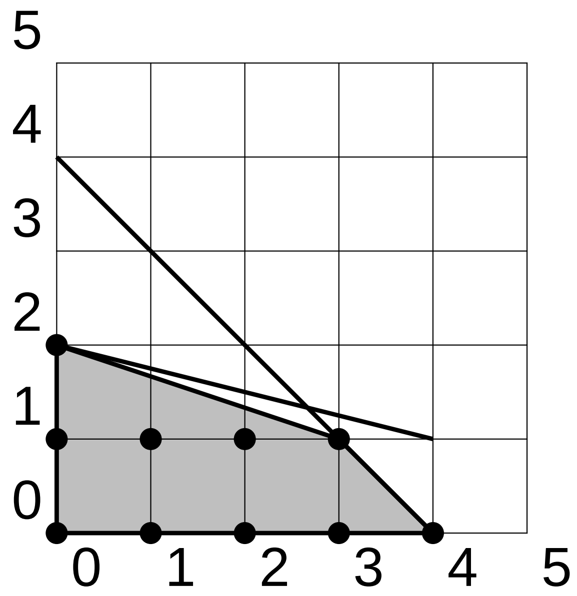
- Yes! Let X be the set of all solutions to original IP. Then define the convex hull of X as

$$
\begin{aligned}
CH(X) := \{x \in \mathbb{R}^n \; : \qquad\qquad & x = \sum_{\bar{x} \in X} \lambda_{\bar{x}} \cdot \bar{x}, \\
& \sum_{\bar{x} \in X} \lambda_{\bar{x}} = 1 \\
& \lambda_{\bar{x}} \geq 0 \quad \forall \bar{x} \in X\}
\end{aligned}
$$

Convex Hull

- The convex hull $CH(X)$ of feasible integer solutions X is the smallest polyhedron containing X:



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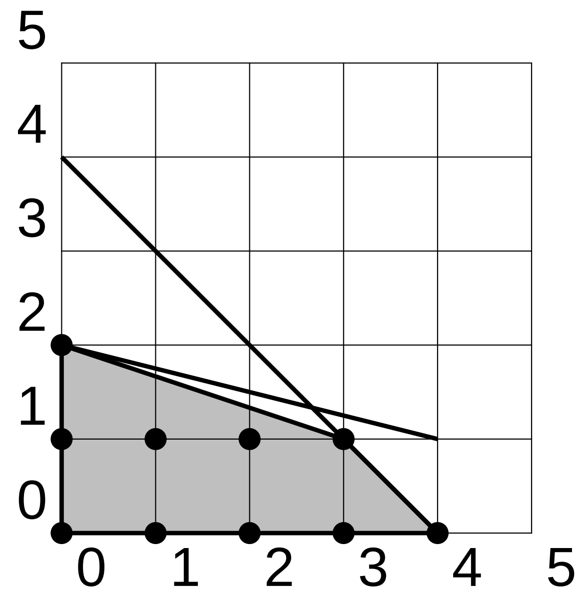

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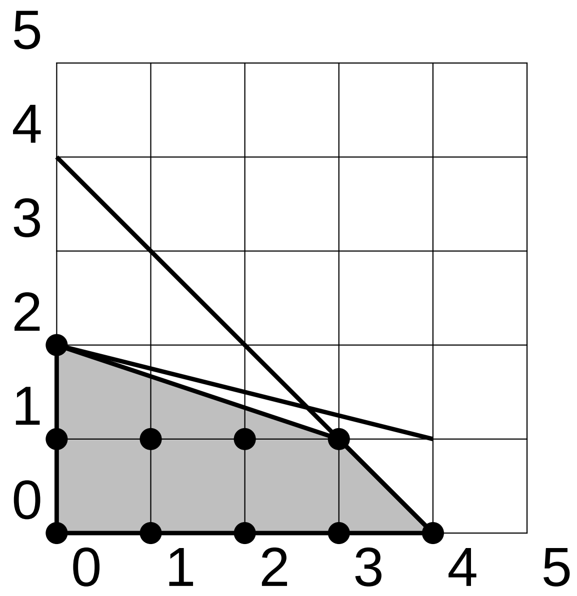

- If P is the feasible region of an LP relaxation then $CH \subseteq P$
- Each vertex of the convex hull corresponds to an integer solution!

Valid Inequalities

Introduction

- In this class, we are interested in integer programs of the following general form:

$$\max\{c^T x \,:\, x \in X\} \tag{IP}$$

and $X = \{x \,:\, Ax \leq b, X \in \mathbb{Z}^n_+\}$.

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- Fact: There is $\widetilde{A}$ and $\widetilde{b}$ such that

$$\mathtt{CH}(X) = \{x : \widetilde{A}x \leq \widetilde{b}\}$$

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- Fact: There is $\widetilde{A}$ and $\widetilde{b}$ such that

$$\mathtt{CH}(X) = \{x \, : \, \widetilde{A}x \leq \widetilde{b}\}$$

- $\widetilde{A}$ may be be huge! We will not be able to generate a description of the convex hull in polynomial time for all problems.

Valid Inequalities

- A more tractable task: Find valid inequalities for $X := \{x : Ax \leq b, x \text{ integer}\}$.

  An inequality

  $$\pi x \leq \pi_0$$

  is valid for X if is satisfied for all $x \in X$.

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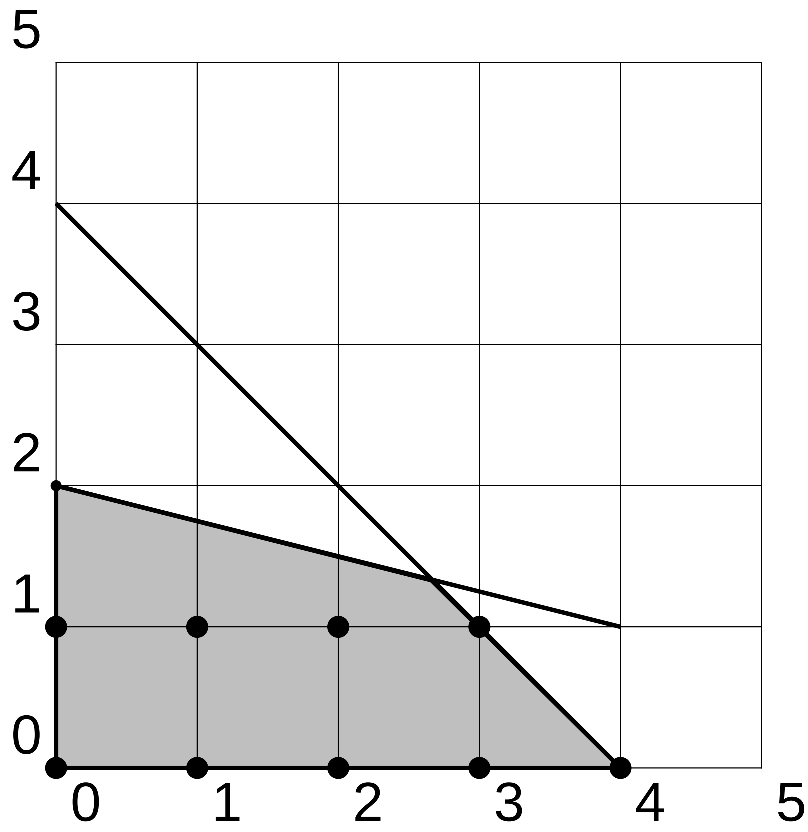
Geometric view of LP relaxation:



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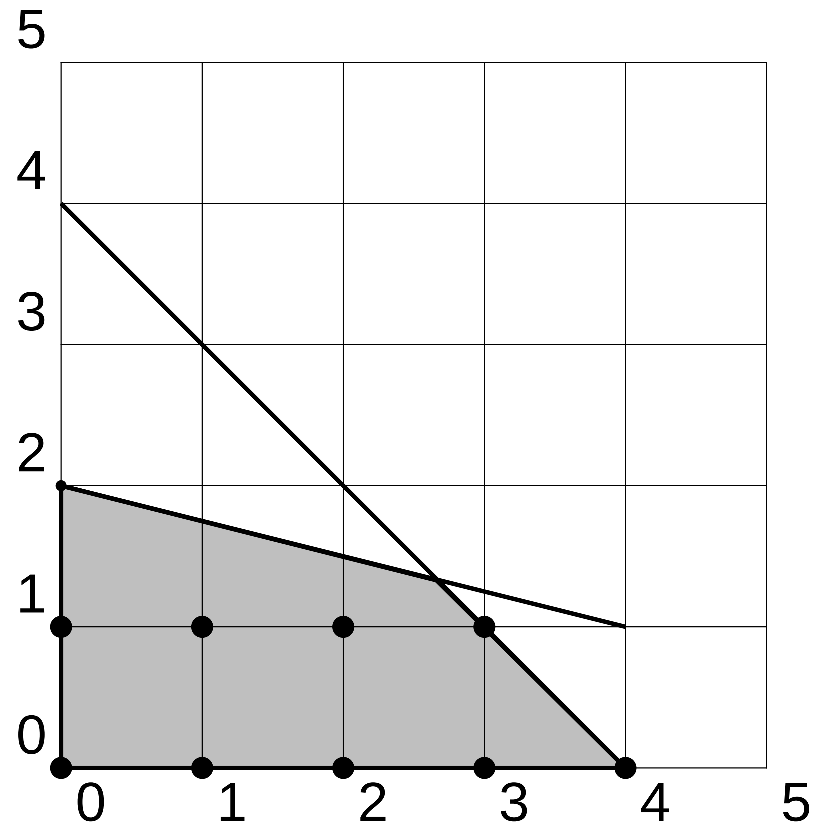
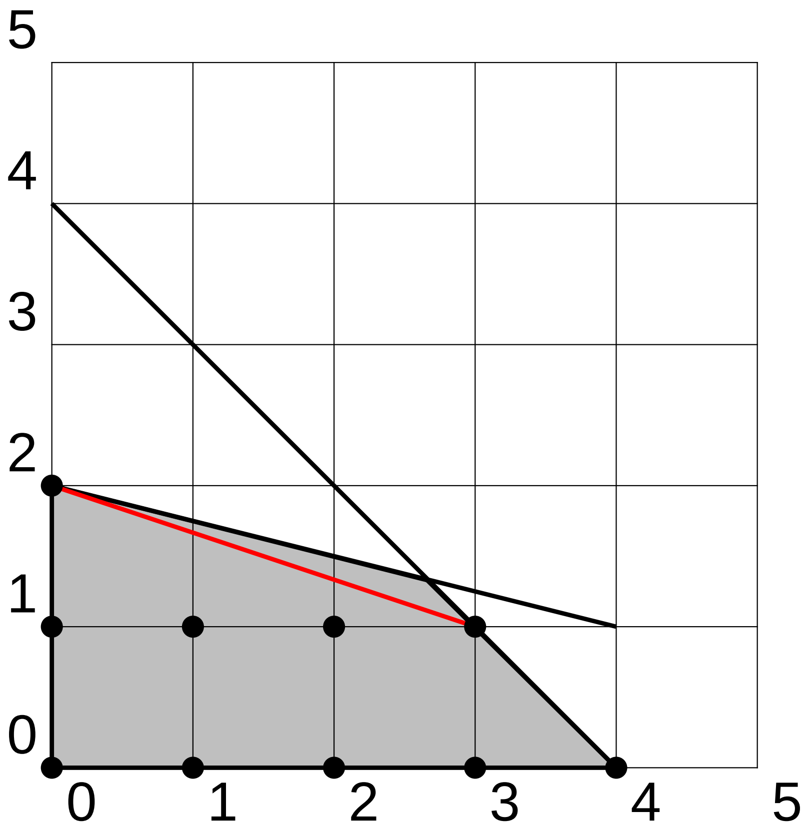


Optimum solution: $x_1 = 8/3, x_2 = 4/3$.

Can you find a good valid inequality for this example?

Introduction

Geometric view:



Inequality $x_1/3 + x_2 \leq 2$ is valid for (IP)!

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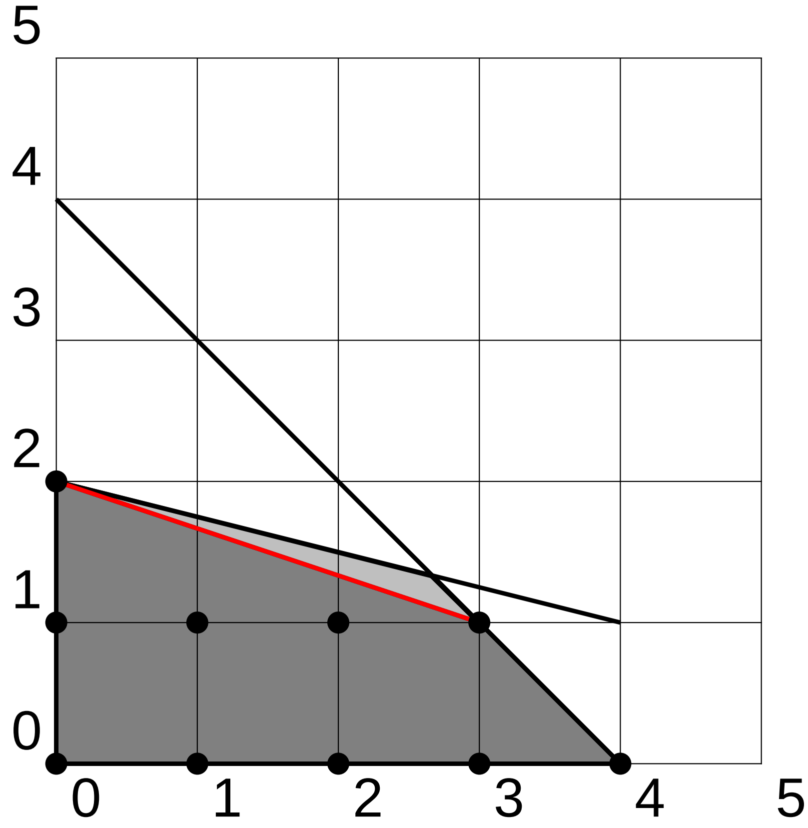

Inequality $x_1/3 + x_2 \leq 2$ is valid for (IP)!
Its addition to existing inequalities yields the convex hull of all feasible integer solutions.

Introduction

- In example, inequality $x_1/3 + x_2 \leq 2$ was useful as its addition to original constraints yielded $\mathtt{CH}(X)$.

  Remember last class: Adding this inequality gives us the optimum integer solution at once! No branch and bound search necessary!

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- What are the useful valid inequalities in general?
- How do we find these inequalities? Are there systematic ways?

Finding Valid Inequalities

- Another set of integer solutions:

$$X := \{x \in \{0,1\}^5 \,:\, 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

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- How about $x_1 = 1$ and $x_2 = 0$?
- This implies $3 + 2x_3 - 3x_4 + x_5 \geq 3 - 3 = 0$.
- Implies: Whenever $x_1 = 1$ then x_2 must have value 1 as well. Valid inequality:

$$x_1 \leq x_2$$

Finding Valid Inequalities

- Another IP:

$$\max(x - 5y) \quad \text{s.t. } (x, y) \in X$$

with $X := \{(x, y) \,:\, x \le 100 \cdot y, 0 \le x \le 5, y \in \{0, 1\}\}$

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- $x \leq 100 \cdot y$ is a big-M constraint where the M is chosen poorly.

  Is there a good valid inequality? Can you find a better M?

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  is valid! Variable x can only be positive if $y = 1$. Whenever $y = 1$, x must have value at most 5.

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- $\mathtt{CH}(X) = \{(x, y) \,:\, x \leq 5y, 0 \leq y \leq 1\}$.

Finding Valid Inequalities

- One more example:

$$X := \{x \in \mathbb{Z}^4_+ \; : \; 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$$

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is valid for X for all $\alpha \geq 0$.

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- Valid inequality for $\alpha = \frac{1}{11}$:

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + \frac{11}{11}x_3 + \frac{6}{11}x_4 \geq \frac{72}{11}$$

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- Left-hand side is integer! Can round up right-hand side:

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

This inequality is valid for original set X.

Chvátal-Gomory Procedure

Valid Inequalities for LP

- Back to IP example from last class:

$$
\begin{aligned}
\max \quad & 3x_1 + 10x_2 && \text{(IP)} \\
\text{s.t.} \quad & x \in P \\
& P = \{(x_1, x_2) \quad : \quad x_1 + 4x_2 \le 8, && (1) \\
& \qquad\qquad\qquad\qquad x_1 + x_2 \le 4, x \ge 0\} \\
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- Notice that the inequality

$$u_1(x_1 + 4x_2) + u_2(x_1 + x_2) \leq 8u_1 + 4u_2$$

is valid for P for any $u_1, u_2 \geq 0$

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  is valid for P for any $u_1, u_2 \geq 0$
- In fact: Any valid inequality for P can be obtained in this way.

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- Let's try this with $u_1 = 2/3, u_2 = 1/3$:

$$\frac{2}{3}(x_1 + 4x_2) + \frac{1}{3}(x_1 + x_2) \leq \frac{16}{3} + \frac{4}{3}$$

Valid Inequalities for LP

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- Let's try this with $u_1 = 2/3, u_2 = 1/3$:

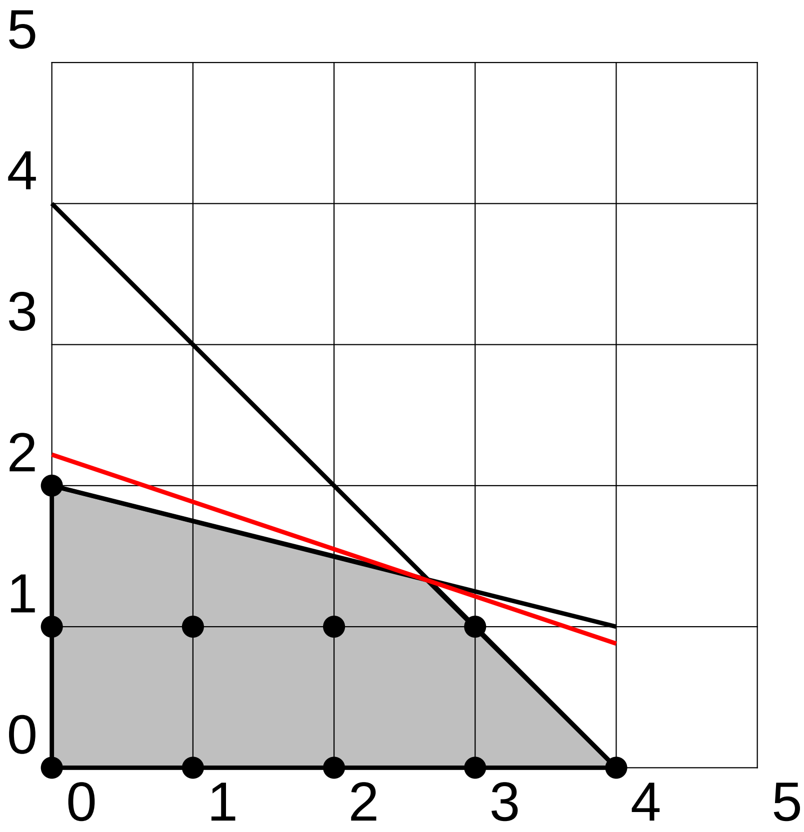
$$\frac{2}{3}(x_1 + 4x_2) + \frac{1}{3}(x_1 + x_2) \leq \frac{16}{3} + \frac{4}{3}$$

- ... and this is equivalent to

$$x_1 + 3x_2 \leq \frac{20}{3}$$

Valid Inequalities for LP

Geometric view:



Red line is the inequality $x_1 + 3x_2 \leq \frac{20}{3}$.
It is clearly satisfied by all points in P.

- Have seen that inequality

$$x_1 + 3x_2 \leq \frac{20}{3} \qquad (1)$$

is valid for P.

Strengthening Inequalities

- Have seen that inequality

$$x_1 + 3x_2 \leq \frac{20}{3} \tag{1}$$

  is valid for P.

- Every feasible solution for the LP relaxation satisfies this inequality.

  We haven't gained anything, have we?

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- Well, if x_1, x_2 are integer, then the left-hand side of (1) is integer.
- For every feasible integer solution in X, the left-hand side of (1) has value at most $\lfloor 20/3 \rfloor = 6$.

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- For every feasible integer solution in X, the left-hand side of (1) has value at most $\lfloor 20/3 \rfloor = 6$.
- Inequality $x_1 + 3x_2 \leq 6$ is valid for $\mathtt{CH}(X)$ but not valid for P.

Strengthening Inequalities

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$$x_1 + 3x_2 \leq \frac{20}{3} \tag{1}$$

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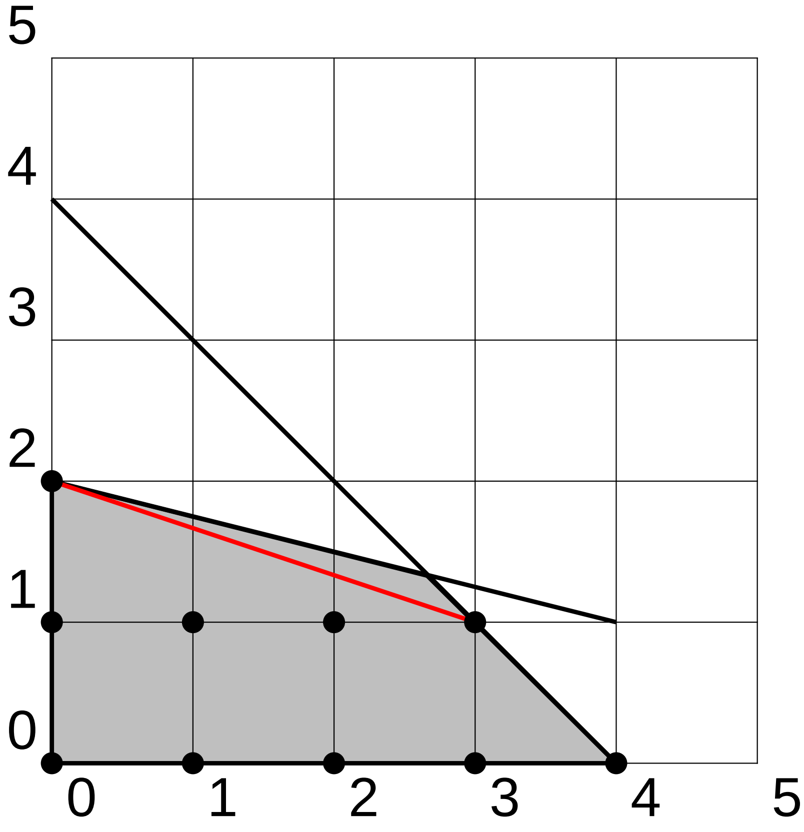
  We haven't gained anything, have we?
- Well, if x_1, x_2 are integer, then the left-hand side of (1) is integer.
- For every feasible integer solution in X, the left-hand side of (1) has value at most $\lfloor 20/3 \rfloor = 6$.
- Inequality $x_1 + 3x_2 \leq 6$ is valid for $\texttt{CH}(X)$ but not valid for P.
- We gained strength over the LP relaxation of (IP).

Geometric view:



Red line is the inequality $x_1 + 3x_2 \leq 6$.
Adding this inequality gives the convex hull $\mathtt{CH}(X)$ of all integer solutions in X.

CG Procedure

- Suppose you have a valid inequality for the polyhedron P given by the relaxation of your integer program:

$$\sum_{j=1}^{n} a_j x_j \leq b$$

How can we strengthen this inequality to lead to a valid inequality for X?

CG Procedure

- The Chvátal-Gomory procedure:
  1. x_i is non-negative for all $i \in \{1, \ldots, n\}$. So the inequality

$$\sum_{j=1}^{n} \lfloor a_j \rfloor x_j \leq b \tag{1}$$

     is valid for P as well.

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     is valid for P as well.
  2. The left-hand side of (1) is integer for $(x_1, \ldots, x_n) \in X$. Therefore,

     $$\sum_{j=1}^{n} \lfloor a_j \rfloor x_j \leq \lfloor b \rfloor$$

     is a valid inequality for X.

Discussion

- Notice that the linear program

$$\begin{aligned} \max \quad & 3x_1 + 10x_2 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 8 \\ & x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

describes the convex hull $\texttt{CH}(X)$ of all feasible integer solutions for the original LP.

Discussion

- Notice that the linear program

$$\begin{aligned} \max \quad & 3x_1 + 10x_2 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 8 \\ & x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

  describes the convex hull $\mathtt{CH}(X)$ of all feasible integer solutions for the original LP.
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  describes the convex hull $\texttt{CH}(X)$ of all feasible integer solutions for the original LP.
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- CG Procedure is a tool to strengthen valid inequalities for the LP relaxation.

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  Hopefully this reduces the size of our branch & bound tree.
- Disadvantages: The size of the LP formulation may grow quite dramatically. We need to solve an LP at each node in the branch & bound tree.
- There is no good answer here. Need to experiment!

Cutting-Plane Algorithms

General Framework

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- Also know that there maybe too many such inequalities to write them all out. What can we do?
- Cutting-Plane algorithms solve the LP relaxation of the given integer program and add strong valid inequalities one by one.

General Framework

- Suppose you want to solve integer program

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & x \in P_0 \\ & x \text{ integer} \end{aligned} \qquad \text{(IP)}$$

for some polyhedron P_0.

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for some polyhedron P_0.
- Solve the LP relaxation

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of (IP). Let x_0 be the solution.

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- Resolve LP relaxation with P_0 replaced by P_1.
- Continue this way until integral solution is found.

Gomory Cuts

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- Consider general IP of the form

$$\max\{cx \;:\; Ax \leq b, x \geq 0 \text{ and integer}\}$$

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- Solve the linear programming relaxation of (IP) via Simplex.
- Gives a final tableau of the form

BV	x_1	$\cdots$	x_j	$\cdots$	x_i	$\cdots$	x_n	Value
z	$\bar{c}_1$		$\bar{c}_j$		$\bar{c}_i$		$\bar{c}_n$	$\bar{z}$
$\vdots$	$\vdots$		$\vdots$		0		$\vdots$	$\vdots$
x_i	$\bar{a}_{i1}$		$\bar{a}_{ij}$		1		$\bar{a}_{in}$	$\bar{b}_i$
$\vdots$	$\vdots$		$\vdots$		0		$\vdots$	$\vdots$

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- The optimal basis is $\mathcal{B} = \{1, \ldots, m\}$ and the non-basis is $\mathcal{N} = \{1, \ldots, n\} \setminus \mathcal{B}$.

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- The optimal basis is $\mathcal{B} = \{1, \ldots, m\}$ and the non-basis is $\mathcal{N} = \{1, \ldots, n\} \setminus \mathcal{B}$.
- Row of x_i corresponds to:

$$x_i + \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j = \bar{b}_i$$

Any feasible solution to (IP) must satisfy this equation!

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- Use Chvátal-Gomory procedure and conclude that any feasible solution to (IP) must also satisfy

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- From (1):

$$x_i = \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j$$

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- ...and

$$x_i = \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j \tag{2}$$

- Combining (1) and (2) leads to a new valid inequality for (IP):

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor \tag{3}$$

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- Notice that current optimum solution x does not satisfy (1) as $x_j = 0$ for all $j \in \mathcal{N}$.
  x therefore does not satisfy (3) either!

- The new valid inequality is called a Gomory Cut:

$$\sum_{j \in \mathcal{N}} (\overline{a}_{ij} - \lfloor \overline{a}_{ij} \rfloor) x_j \geq \overline{b}_i - \lfloor \overline{b}_i \rfloor$$

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- Add this to optimum tableau and use dual simplex to re-optimize!
- Repeat until optimum solution is integral.

An Example

- Back to IP example from last class:

$$
\begin{aligned}
\max \quad & 3x_1 + 10x_2 && \text{(IP)} \\
\text{s.t.} \quad & x \in P \\
& P = \{(x_1, x_2) \quad : \quad x_1 + 4x_2 \leq 8, && (2) \\
& \qquad\qquad\qquad\qquad x_1 + x_2 \leq 4, x \geq 0\} \\
& x_1, x_2 \text{ integer}
\end{aligned}
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$$x_1 + x_2 \leq 4, x \geq 0\}$$

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- Final tableau:

BV	x_1	x_2	s_1	s_2	Value
z	0	0	7/3	2/3	64/3
x_2	0	1	1/3	-1/3	4/3
x_1	1	0	-1/3	4/3	8/3

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- Both variables x_1 and x_2 are fractional. What is the Gomory cut for x_1 row?

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- Both variables x_1 and x_2 are fractional. What is the Gomory cut for x_1 row?
- Gomory cut formula is

$$\sum_{j \in \mathcal{N}} (\overline{a}_{ij} - \lfloor \overline{a}_{ij} \rfloor) x_j \geq \overline{b}_i - \lfloor \overline{b}_i \rfloor$$

and therefore cut is

$$\frac{2}{3} s_1 + \frac{1}{3} s_2 \geq \frac{2}{3}$$

An Example

- Add new slack-variable s_3 and row

$$-\frac{2}{3}s_1 - \frac{1}{3}s_2 + s_3 = -\frac{2}{3}$$

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- Use dual simplex to remove infeasibility.

An Example

BV	x_1	x_2	s_1	s_2	s_3	Value
z	0	0	1	0	2	20
x_2	0	1	1	0	-1	2
x_1	1	0	-3	0	4	0
s_2	0	0	2	1	-3	2

- All variables have integer values. Done!

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- There are often better cuts to add than Gomory cuts
  - Chvàtal-Gomory cuts, specially tailored cuts, . . .
- Cuts are often used in Branch & Bound
  - Add cuts while you go and reduce B&B tree size.