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EASILY COMPUTABLE FACETS OF THE KNAPSACK POLYTOPE*

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It is known that facets and valid inequalities for the knapsack polytope P can be obtained by lifting a simple inequality derived from each minimal cover. We study the computational complexity of such lifting. In particular, we show that the task of computing a lifted facet can be accomplished in $O(ns)$ where $s \leqslant n$ is the cardinality of the minimal cover. Also, for a lifted inequality with integer coefficients, we show that the dual tasks of recognizing whether the inequality is valid for P or is a facet of P can be done within the same time bound.

1. The convex hull of solutions of combinatorial problems has been studied extensively over the past few decades. Numerous results are now available on the facial structure of problems such as the traveling salesman problem, the knapsack and multi-knapsack problems, the set covering, packing and partitioning problems, plant location problems, scheduling problems, etc. For a recent survey on these results and their applications, the reader is referred to [G], [GP] and [Pu].

In spite of the wealth of studies on facets, there are few results concerning the computational complexity issues involved. A notable exception is the work of [PY], [PW] on the complexity of recognizing the facets of the traveling salesman polytope.

In this note we study the computational complexity of computing and recognizing facets and valid inequalities of the binary knapsack polytope. This polytope is very useful, since for any 0–1 integer programming problem, each constraint individually, or each individual aggregation of several constraints, can be regarded as a knapsack problem. Thus, facets and valid inequalities for the knapsack polytope can be used for the general integer problem. This approach is utilized effectively, for example, in [CJP].

We are interested in a family of facets obtained from minimal covers. The existence of such facets has been known for over 15 years [B], [P1], [W], and their properties have been investigated in great detail, e.g., [B], [BZ1], [HJP], [Pe]. We show that, for this family, the tasks of computing a facet, or of recognizing whether a given inequality with integer coefficients is a facet or valid inequality, can be done simply and efficiently using an algorithm whose running time is bounded by $O(n^2)$.

2. Preliminaries. Consider the inequality

$$\sum_{j \in N} a_j x_j \leqslant a_0 \tag{1}$$

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where $0 < a_j \leqslant a_0$ are positive integers and $x_j = 0$ or 1, $j \in N = \{1, \dots, n\}$. The *knapsack polytope* P is the convex hull of 0– points satisfying (1).

A set $S \subseteq N$ is called *a minimal cover* for P if $\sum_{j \in S} a_j > a_0$, and S is minimal with respect to this property. We denote by s the cardinality of S. For any subset $V \subset N$, let $P_V = \text{conv}\{x \in \{0,1\}^V | \sum_{j \in V} a_j x_j \leqslant a_0\}$. It is known [B], [P1], [W] that if S is a minimal cover, then the inequality

(2) $$\sum_{j \in S} x_j \leqslant s - 1$$

is a facet of P_S. It is also known [NT], [P1], [P2] that facets and valid inequalities of lower dimensional polytopes can be "lifted" into n-space so as to yield facets or valid inequalities of P. A procedure for computing such lifted facets was given by Padberg [P1], [P2] (see also [Po]). When lifting is applied to (2), one gets inequalities of the form

(3) $$\sum_{j \in S} x_j + \sum_{j \in N-S} \beta_j x_j \leqslant s - 1.$$

We call any inequality of the form (3) (not necessarily valid or a facet) a *lifting* from the minimal cover S.

Padberg's lifting procedure is sequential, in that the lifting coefficients β_j are computed one by one in a given sequence. The computation of each coefficient requires that a certain binary knapsack problem of size between s and n be solved to optimality. The coefficients obtained in this way depend on the sequence in which they are calculated and, in general, there may be an exponential number of sequences yielding distinct facets of P. Moreover, there may exist facets of P which are liftings from S, but which cannot be obtained by Padberg's algorithm for any sequence of $N - S$ [BZ1], [Z]. A general characterization of all the liftings of a lower dimensional facet or valid inequality is gen in [Z] and specialized to liftings from S in [BZ1]. We note that not all the facets of P are liftings from some minimal cover S. A generalization of this form, which accounts for all the facets of P, is given in [BZ2].

In this note we study the computational complexity of the following three tasks:

$P1$. Given a sequence π of $N - S$, compute the lifted facet associated with this sequence.

$P2$. Given a lifting (3) from S, is it a facet for P?

$P3$. Given a lifting (3) from S, is it valid for P?

As noted earlier, $P1$ requires a solution of a sequence of $n - s$ binary knapsack problems. $P2$ and $P3$ seem more difficult, since they potentially require enumerating all sequences of $N - S$. Nevertheless, we have:

$A1$. The complexity of $P1$ is $O(ns)$.

$A2$. If the coefficients β_j: $j \in N - S$ are integers, the complexity of $P2$ and $P3$ is $O(ns)$.

We devote the remainder of this note to the proof of $A1$ and $A2$.

3. Properties of sequentially lifted facets. In this section we sumarize some known results concerning sequential liftings from S. We begin by describing Padberg's sequential procedure, specialized to such liftings. Let π be a sequence of $N - S$, i.e., a one-to-one mapping from $\{1, \dots, n - s\}$ to $N - S$ and let $T(i) = (\pi_1, \dots, \pi_i\}$, $i = 1, \dots, n - s$.

Proposition 1 [P1], [P2]. *For each* $i = 1, \ldots, n - s$, *consider the knapsack problem* K_{π_i} *defined recursively as follows*

$$z_{\pi_i} = \max \sum_{j \in S} x_j + \sum_{j \in T(i-1)} \beta_j x_j$$

subject to:

$$\sum_{j \in S} a_j x_j + \sum_{j \in T(i-1)} a_j x_j \leqslant a_0 - a_{\pi_i},$$

$$x_j = 0, 1, \quad j \in S' \cup T(i-1),$$

and let

$$\beta_{\pi_i} = s - 1 - z_{\pi_i}.$$

Then for $i = 1, \ldots, n - s$, *each inequality*,

$$\sum_{j \in S} x_j + \sum_{j \in T(i)} \beta_j x_j \leqslant s - 1$$

is a facet of $P_{S \cup T(i)}$. *In particular* (3) *is a facet of* P.

The following properties of the lifting coefficients β_j, $j \in N - S$, are useful. Propositions 2–6 are adopted from [BZ1]. See also [HJP], [Pe].

Let l_t, $t = 0, \ldots, s$ be the sum of the t smallest a_j, $j \in S$, and let b_t be the sum of the t largest. For any number $0 \leqslant a \leqslant a_0$ let $\gamma(a)$ be the smallest integer t such that $l_{s-1-t} \leqslant a_0 - a$ and let $\alpha(a)$ be the largest integer t such that $b_t \leqslant a$. We let $\gamma_i = \gamma(a_i)$ and similarly for α_i. The coefficients α_j, γ_j, $j \in N - S$ play a crucial role with respect to liftings of S:

Proposition 2. *If* (3) *is valid*, $\beta_i \leqslant \gamma_i$, $i \in N - S$.

Proposition 3. *If* (3) *is a facet*, $\beta_i \geqslant \alpha_i$, $i \in N - S$. (*In fact, a much weaker condition on* (3) *is sufficient.*)

Proposition 4. *For every* $0 \leqslant a \leqslant a_0$, $\alpha(a) \leqslant \gamma(a) \leqslant \alpha(a) + 1$.

In view of Propositions 2–4, let $I = \{i \in N - S\colon \alpha_i = \gamma_i\}$ and let $J = \{i \in N - S\colon \gamma_i = \alpha_i + 1\}$. The variables $i \in I$ can be very easily handled with respect to the tasks $P1$–$P3$. Specifically, consider the inequality (4)

$$\sum_{j \in S} x_j + \sum_{j \in J} \beta_j x_j \leqslant s - 1. \tag{4}$$

Proposition 5. (a) (3) *is valid for* P *iff* $\beta_j \leqslant \alpha_j$, $j \in I$ *and* (4) *is valid for* $P_{J \cup S}$. (b) (3) *is a facet of* P *iff* $\beta_j = \alpha_j$, $j \in I$ *and* (4) *is a facet of* $P_{J \cup S}$.

We conclude this section by a characterization of those lifted facets (3) which can be obtained by sequential lifting.

Proposition 6. *A lifted inequality* (3) *which is a facet of* P *can be obtained by sequential lifting for some sequence* π *of* $N - S$ *iff all the coefficients* β_j, $j \in N - S$ *are integer*.

4. The algorithms. In this section we give the algorithms which support A1 and A2. In view of Proposition 5, we restrict our attention to the set J. We will thus consider liftings from S of the form (4). We assume below that the partial sums b_t, l_t,

$t = 1, \ldots, s-1$ are available, and that the set J is identified. This preprocessing phase can be easily done in $O(n \log s)$ effort. The computational requirements reported in the remainder of this section are in addition to this amount.

4.1. *The task P1: Computing a facet.* We first consider the task of computing a sequentially lifted facet (4). Using a standard dynamic programming technique, consider, for each $i = 1, \ldots, |J|$, the set of dual knapsack problems $D_{\pi_i}(z)$ for $z = 0, \ldots, s-1$:

$$A_{\pi_i}(z) = \min \sum_{j \in S} a_j + \sum_{j \in T(i-1)} a_j x_j$$

subject to:

$$\sum_{j \in S} x_j + \sum_{j \in T(i-1)} \beta_j x_j \geqslant z,$$

$$x_j = 0 \quad \text{or} \quad 1, \qquad j \in S \cup T(i-1).$$

Clearly, the problem K_{π_i} of Padberg's procedure is related to the set of problems $D_{\pi_i}(z)$, $z = 0, \ldots, s-1$ via the relation: $Z_{\pi_i} = \max\{z\colon A_{\pi_i}(z) \leqslant a_0 - a_{\pi_i}\}$. This suggests the following algorithm:

Algorithm Lift.

Input: a sequence π of the set J. The partial sums l_t, $t = 1, \ldots, s-1$.
Output: The lifted facet (4) which corresponds to π.
Begin Lift
(1) Let $A_{\pi_1}(0) = 0$, $A_{\pi_1}(z) = l_z$, $z = 1, \ldots, s-1$.
For $j = 1, \ldots, |J|$:
(2) $z_{\pi_j} = \max\{z\colon A_{\pi_j}(z) \leqslant a_0 - a_{\pi_j}\}$.
(3) $\beta_{\pi_j} = s - 1 - z_{\pi_j}$.
(4) For $z = 0$ to $s-1$.
If $z < \beta_{\pi_j}$, $A_{\pi_{j+1}}(z) = A_{\pi_j}(z)$.
Else $A_{\pi_{j+1}}(z) = \min\{A_{\pi_j}(z), A_{\pi_j}(z - \beta_{\pi_j}) + a_{\pi_j}\}$.
End Lift

Lift is a typical dynamic programming algorithm of complexity $O(n \cdot s)$. The only nonstandard feature here is that the coefficients β_{π_j}, used for the update of $A_{\pi_j}(\cdot)$ into $A_{\pi_{j+1}}(\cdot)$, are not given in advance but are computed as one goes along. However, β_{π_j} is computed in step (3), *before* it is used in step (4). This makes for an interesting property of Lift, namely, the effort to lift all the way from (2) to (4) is the same as the work needed to compute just the last coefficient of (4), given that the other coefficients are known.

4.2. *The tasks P2 and P3: Recognizing lifted facets and valid inequalities.* We now examine how algorithm Lift can be used to perform the tasks $P2$ and $P3$. Consider a lifting (4) with *integer* coefficients. By Proposition 6, (4) is a facet of $P_{S \cup J}$ iff there exists a sequence π of J which yields (4) via Algorithm Lift. The difficulty is to identify the sequence π or to prove that none exists. To that end, let $J_1 = \{j \in J\colon \beta_j = \alpha_j + 1\}$, $J_2 = J \setminus J_1$ and let a *reversal* in π be any index $1 \leqslant i < n - s$ such that $\pi_{i+1} \in J_1$, $\pi_i \in J_2$.

LEMMA 2. *Let π be any sequence of $N - S$ which contains no reversals. Then* (4) *is a facet of $P_{S \cup J}$ iff it can be obtained from* (2) *by sequential lifting according to π.*

PROOF. Assume that (4) is a facet of $P_{S \cup J}$ and let π be an arbitrary sequence of J which yields it by sequential lifting. Then by Propositions 2–4, $\beta_j = \alpha_j$, $j \in J_2$. Consider any reversal i in π and flip π_i and π_{i+1} to obtain a new sequence π'. Note that by calculating π_{i+1} earlier in the sequence (in the ith rather than the $i + 1$th position) we cannot decrease its coefficient $\beta_{\pi_{i+1}}$ (since the feasible region of $K_{\pi_{i+1}}$ is smaller, $z_{\pi_{i+1}}$ cannot increase). Similarly, by delaying the calculation of π_i to the $i + 1$st position, we cannot increase the value β_{π_i}. However, $\beta_{\pi_{i+1}}$ is already at its upper bound, $\gamma_{\pi_{i+1}}$ and β_{π_i} is at its lowest bound, α_{π_i}. Thus, π' yields the same facet (4) but has fewer reversals than π. Thus, there exists a sequence π of J which yields (4), and has no reversals. We have to show that *every* sequence with the latter property yields (4) as a facet. But this is easy since otherwise one can produce two facets for $P_{S \cup J}$ or two facets for $P_{S \cup J_1}$, one of which dominates the other, which is impossible. ∎

In light of Lemma 2 the complexity of $P2$ and $P3$ is $O(ns)$.

5. Summary. We have shown that computing or recognizing a facet or valid inequality (3) can be done in $O(n^2)$ provided: (a) the minimal cover S is specified, and (b) the inequality involves integer coefficients. The complexity of these tasks, when these conditions are relaxed, is resolved in a subsequent paper [HZ].

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