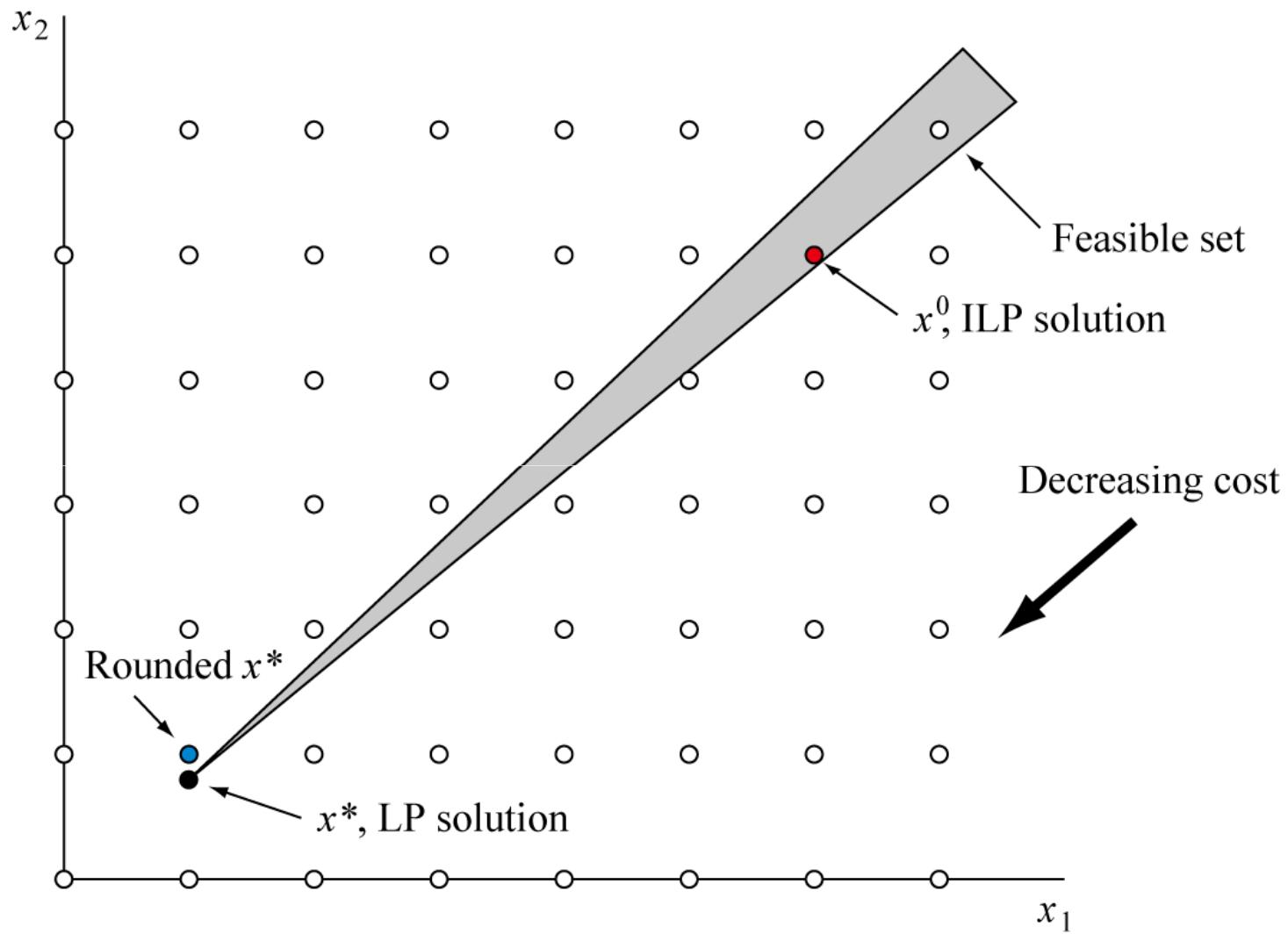


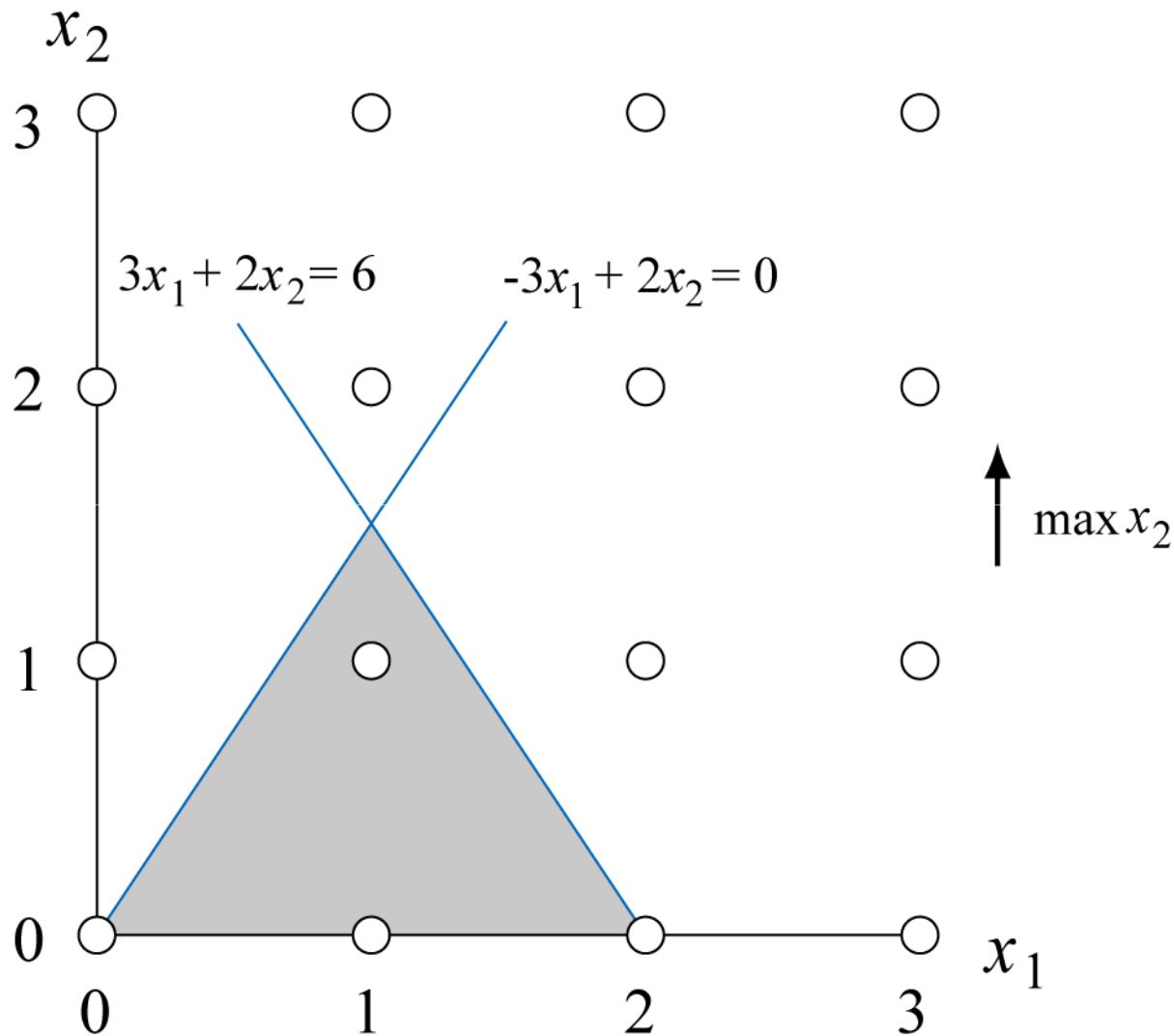
# Gomory's cutting plane algorithm for integer programming

Prepared by Shin-ichi Tanigawa

## Rounding does not give any useful result



We first solve the LP-relaxation



$$\begin{aligned}x_3 &= 6 - 3x_1 - 2x_2 \\x_4 &= 0 + 3x_1 - 2x_2 \\z &= \end{aligned}\quad x_2$$

Optimize using primal simplex method

$$x_3 = 6 - 3x_1 - 2x_2$$

$$x_4 = 0 + 3x_1 - 2x_2$$

$$z = x_2$$

## Optimize using primal simplex method

$$x_3 = 6 - 3x_1 - 2x_2$$

$$x_4 = 0 + 3x_1 - 2x_2$$

$$z = x_2$$



$$x_3 = 6 - 6x_1 + x_4$$

$$x_2 = 0 + \frac{3}{2}x_1 - \frac{1}{2}x_4$$

$$z = \frac{3}{2}x_1 - \frac{1}{2}x_4$$

## Optimize using primal simplex method

$$x_3 = 6 - 3x_1 - 2x_2$$

$$x_4 = 0 + 3x_1 - 2x_2$$

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$$x_3 = 6 - 6x_1 + x_4$$

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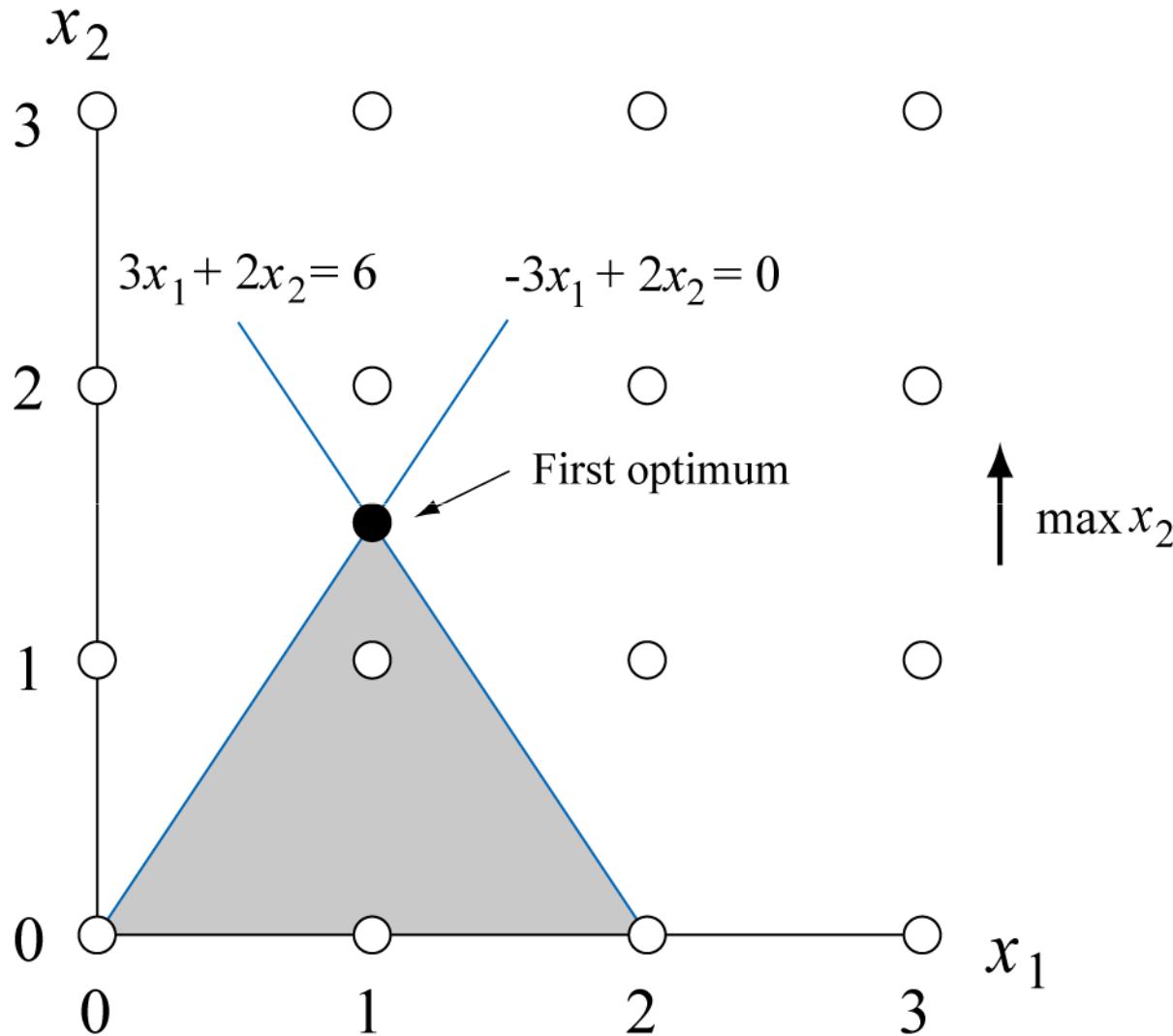


$$x_1 = 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4$$

$$x_2 = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$$

$$z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$$

## The optimal solution is fractional



$$\begin{aligned}x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4\end{aligned}$$

## Generating an objective row cut

$$z + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2} \quad (1)$$

$$\begin{aligned}x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4\end{aligned}$$

↗

## Generating an objective row cut

$$z + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2} \quad (1)$$

$$\begin{aligned}x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4\end{aligned}$$



$$z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 \leq \frac{3}{2} \quad (\text{weakening})$$

## Generating an objective row cut

$$z + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2} \quad (1)$$

$$\begin{aligned} x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \end{aligned}$$



$$z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 \leq \frac{3}{2} \quad (\text{weakening})$$

$$z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 \leq \left\lfloor \frac{3}{2} \right\rfloor \quad (2) \quad (\text{for integers})$$

## Generating an objective row cut

$$x_1 = 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4$$

$$x_2 = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$$

$$z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$$



$$z + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2} \quad (1)$$

$$z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 \leq \frac{3}{2} \quad (\text{weakening})$$

$$z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 \leq \left\lfloor \frac{3}{2} \right\rfloor \quad (2) \quad (\text{for integers})$$

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$

(2) - (1)

Cutting plane is violated by current optimum solution

## Generating an objective row cut

$$\begin{aligned}x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4\end{aligned}$$

$$\begin{aligned}x_3 &= 6 - 3x_1 - 2x_2 \\x_4 &= 0 + 3x_1 - 2x_2\end{aligned}$$

$\rightarrow$

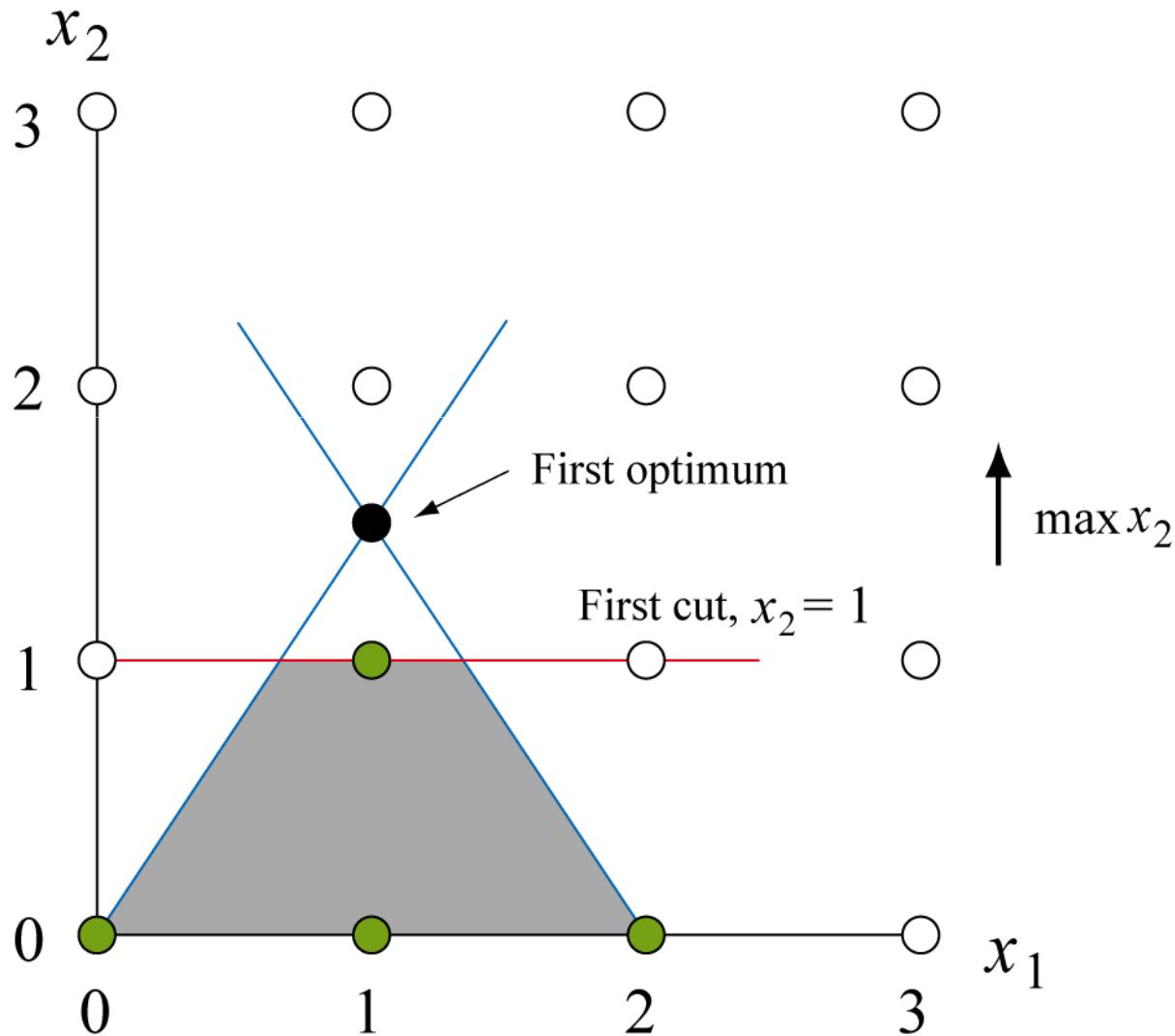
$$\begin{aligned}z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 &\leq \frac{3}{2} \quad (\text{weakening}) \\z + \left\lfloor \frac{1}{4} \right\rfloor x_3 + \left\lfloor \frac{1}{4} \right\rfloor x_4 &\leq \left\lfloor \frac{3}{2} \right\rfloor \quad (2) \quad (\text{for integers}) \\-\frac{1}{4}x_3 - \frac{1}{4}x_4 &\leq -\frac{1}{2} \quad (2) - (1)\end{aligned}$$

$\leftrightarrow$

$$x_2 \leq 1 \quad (\text{substitute for slacks})$$

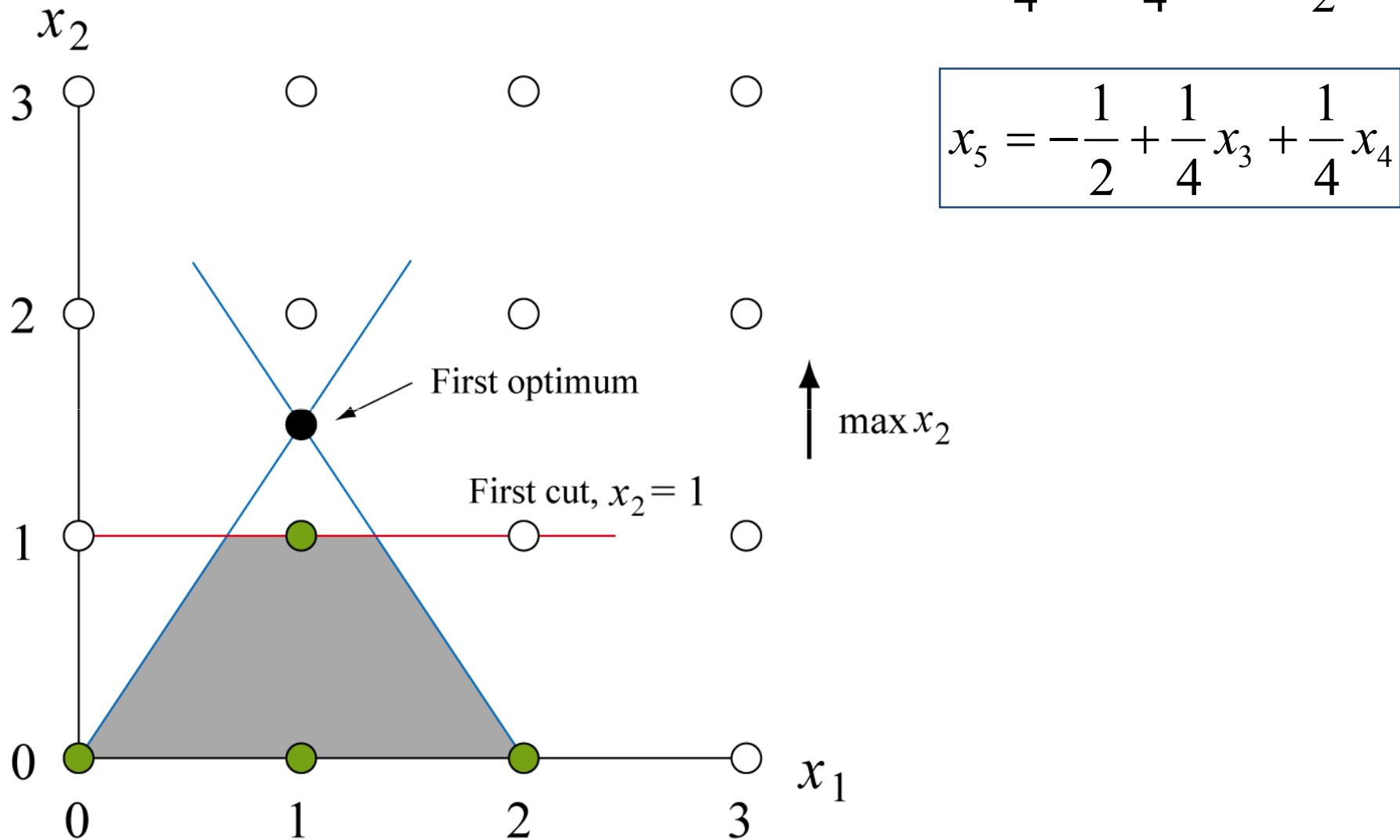
The first cutting plane:

$$x_2 \leq 1 \iff -\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$



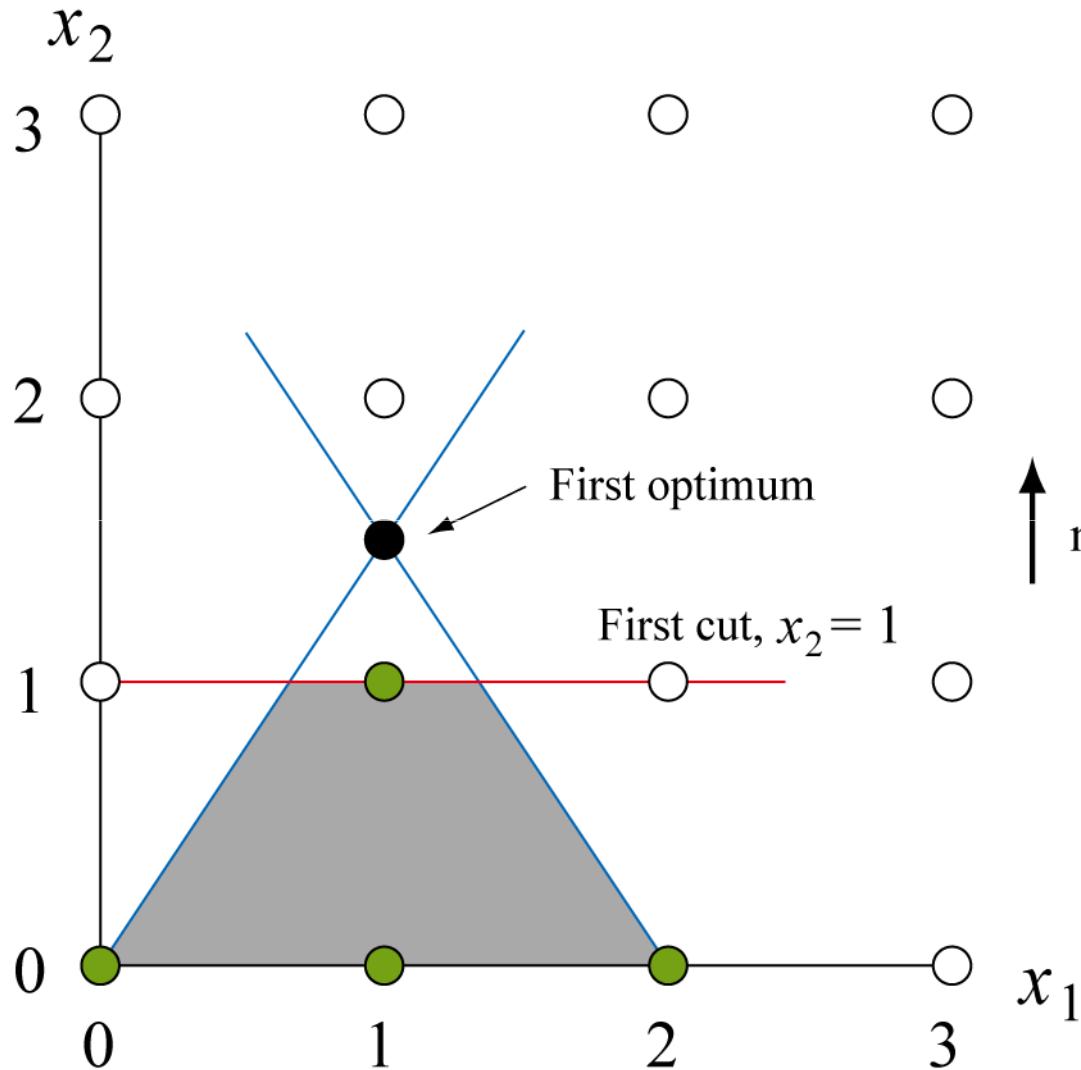
A new slack variable is added:

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$



The new cut is added to the dictionary

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$



$x_1 = 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4$
$x_2 = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$
$\max x_2$
$x_5 = -\frac{1}{2} + \frac{1}{4}x_3 + \frac{1}{4}x_4$
$z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$

## Re-optimize using dual simplex method

$$\begin{aligned}x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\x_5 &= -\frac{1}{2} + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4\end{aligned}$$

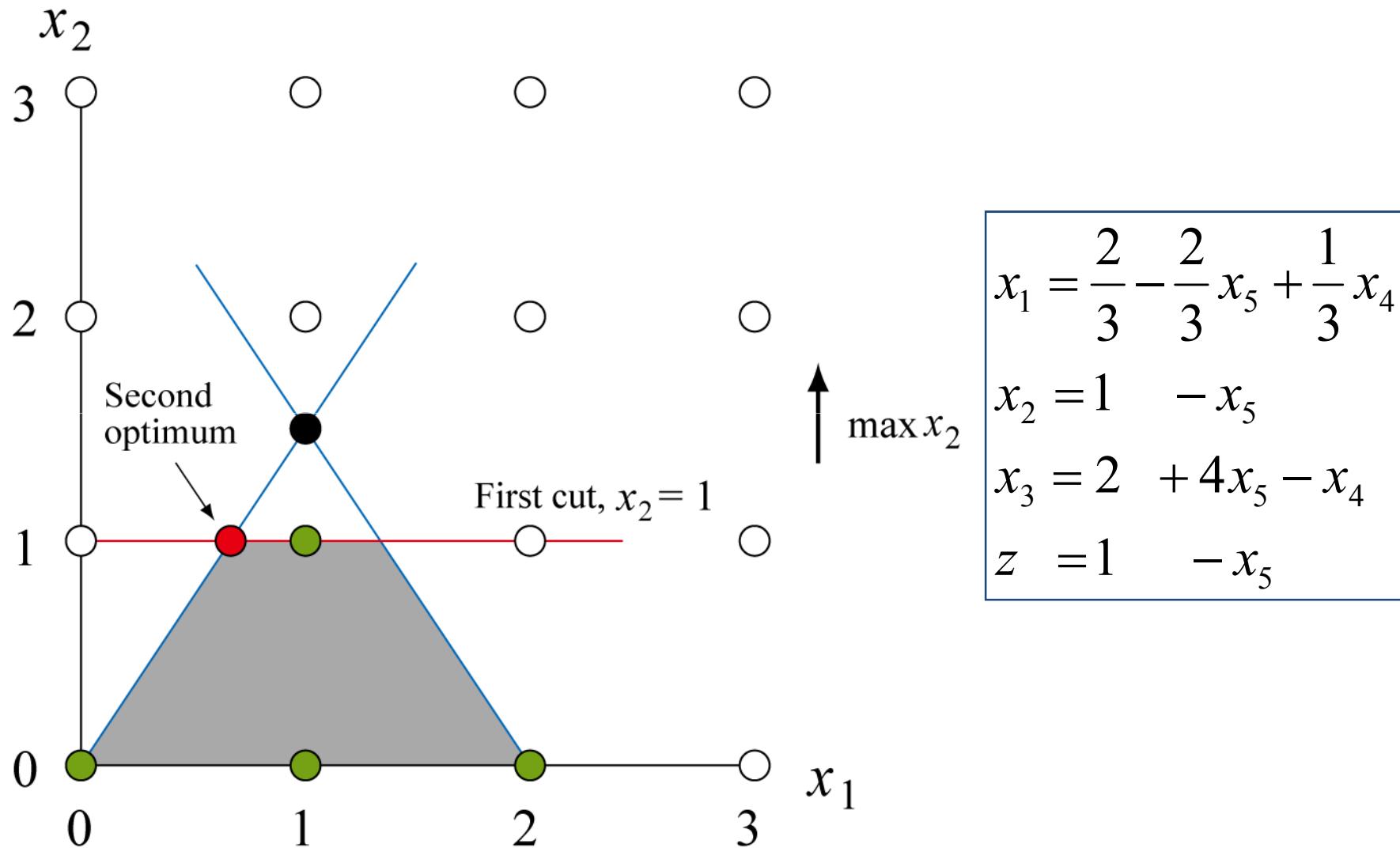
## Re-optimize using dual simplex method

$$\begin{aligned}x_1 &= 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\x_2 &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\x_5 &= -\frac{1}{2} + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\z &= \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4\end{aligned}$$



$$\begin{aligned}x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\x_2 &= 1 - x_5 \\x_3 &= 2 + 4x_5 - x_4 \\z &= 1 - x_5\end{aligned}$$

A new fractional solution has been found



## Generating a constraint row cut

$$x_1 + \frac{2}{3}x_5 - \frac{1}{3}x_4 = \frac{2}{3} \quad (1)$$

$$\begin{aligned} x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\ x_2 &= 1 - x_5 \\ x_3 &= 2 + 4x_5 - x_4 \\ z &= 1 - x_5 \end{aligned}$$

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## Generating a constraint row cut

$$x_1 + \frac{2}{3}x_5 - \frac{1}{3}x_4 = \frac{2}{3} \quad (1)$$

$$\begin{aligned} x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\ x_2 &= 1 - x_5 \\ x_3 &= 2 + 4x_5 - x_4 \\ z &= 1 - x_5 \end{aligned}$$

$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lceil -\frac{1}{3} \right\rceil x_4 \leq \frac{2}{3} \quad (\text{weaken})$$

## Generating a constraint row cut

$$x_1 + \frac{2}{3}x_5 - \frac{1}{3}x_4 = \frac{2}{3} \quad (1)$$

$$\begin{aligned} x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\ x_2 &= 1 - x_5 \\ x_3 &= 2 + 4x_5 - x_4 \\ z &= 1 - x_5 \end{aligned}$$



$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lfloor -\frac{1}{3} \right\rfloor x_4 \leq \frac{2}{3}$$

$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lfloor -\frac{1}{3} \right\rfloor x_4 \leq \left\lfloor \frac{2}{3} \right\rfloor \quad (2)$$

(valid for integers)

## Generating a constraint row cut

$$x_1 + \frac{2}{3}x_5 - \frac{1}{3}x_4 = \frac{2}{3} \quad (1)$$

$$\boxed{x_1 = \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4}$$

$x_2 = 1 - x_5$

$x_3 = 2 + 4x_5 - x_4$

$z = 1 - x_5$



$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lceil -\frac{1}{3} \right\rceil x_4 \leq \frac{2}{3}$$

$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lceil -\frac{1}{3} \right\rceil x_4 \leq \left\lfloor \frac{2}{3} \right\rfloor \quad (2)$$

$$\boxed{-\frac{2}{3}x_5 - \frac{2}{3}x_4 \leq -\frac{2}{3}}$$

$$(2) - (1)$$

## Generating a constraint row cut

$$x_1 + \frac{2}{3}x_5 - \frac{1}{3}x_4 = \frac{2}{3} \quad (1)$$

$$\begin{aligned} x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\ x_2 &= 1 - x_5 \\ x_3 &= 2 + 4x_5 - x_4 \\ z &= 1 - x_5 \end{aligned}$$



$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lceil -\frac{1}{3} \right\rceil x_4 \leq \frac{2}{3}$$

$$x_1 + \left\lfloor \frac{2}{3} \right\rfloor x_5 + \left\lceil -\frac{1}{3} \right\rceil x_4 \leq \left\lfloor \frac{2}{3} \right\rfloor \quad (2)$$

$$-\frac{2}{3}x_5 - \frac{2}{3}x_4 \leq -\frac{2}{3}$$

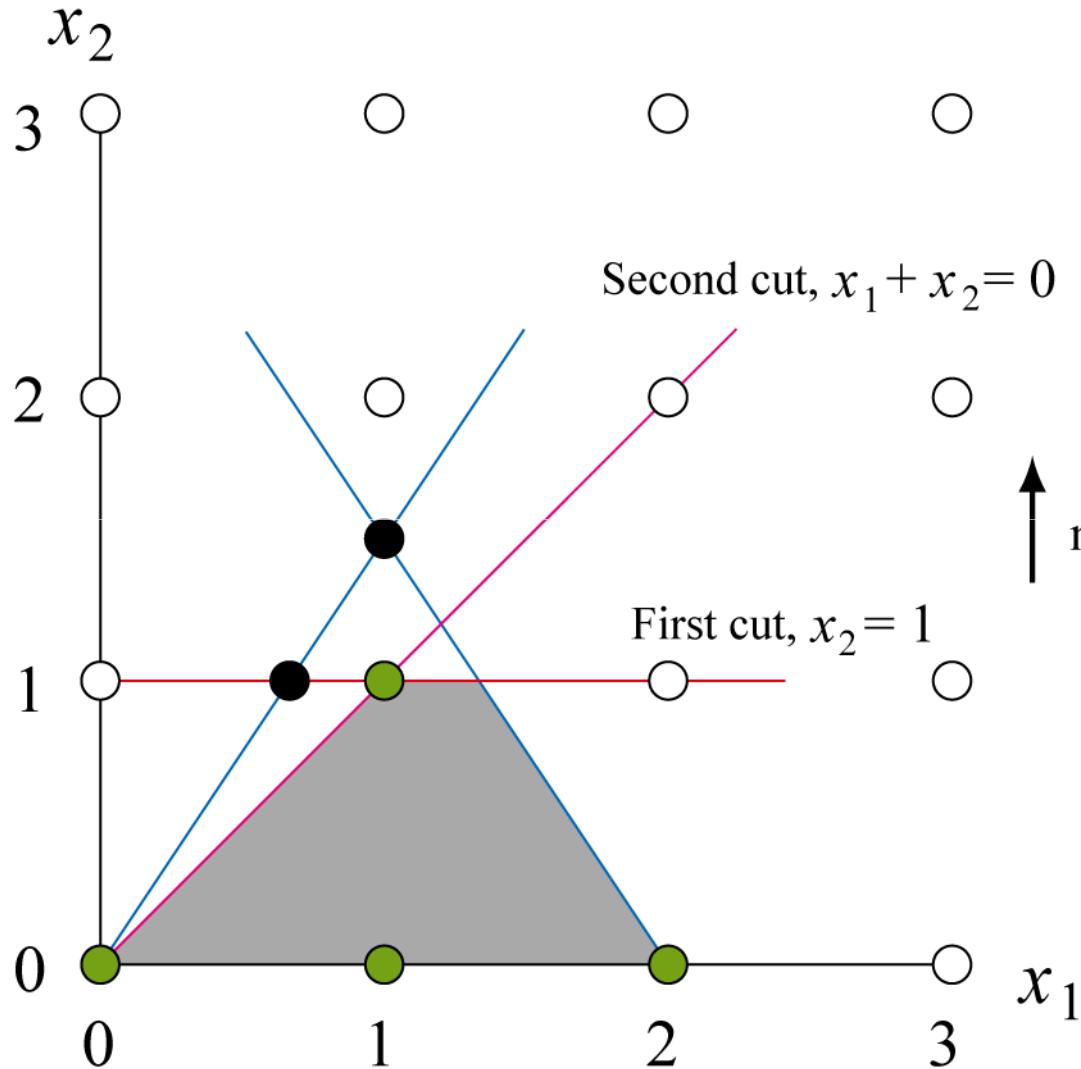
$$(2) - (1)$$

$$\begin{aligned} x_3 &= 6 - 3x_1 - 2x_2 \\ x_4 &= 0 + 3x_1 - 2x_2 \end{aligned}$$



$$x_1 - x_2 \geq 0$$

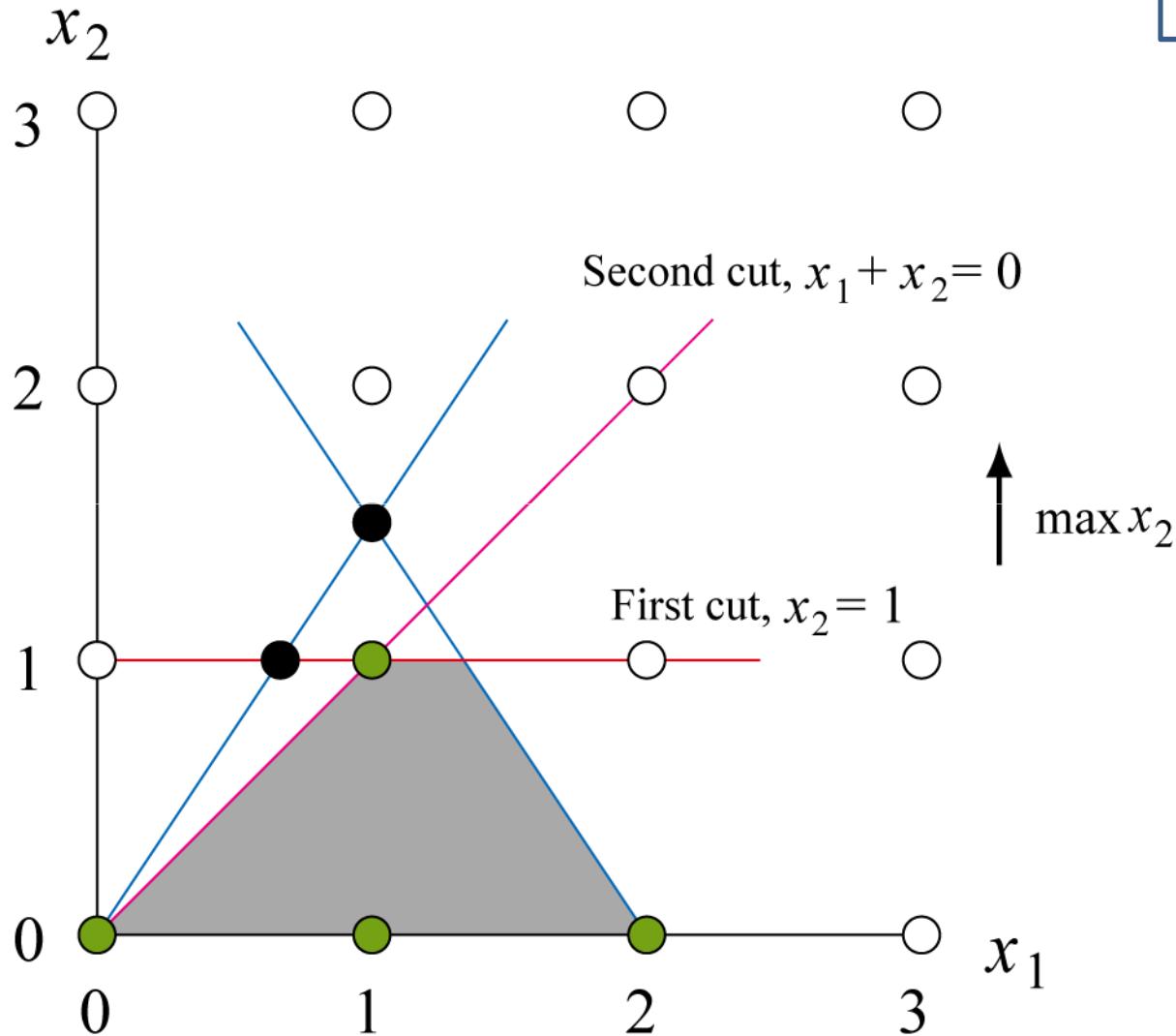
## The second cutting plane



$$x_1 - x_2 \geq 0$$

$$-\frac{2}{3}x_5 - \frac{2}{3}x_4 \leq -\frac{2}{3}$$

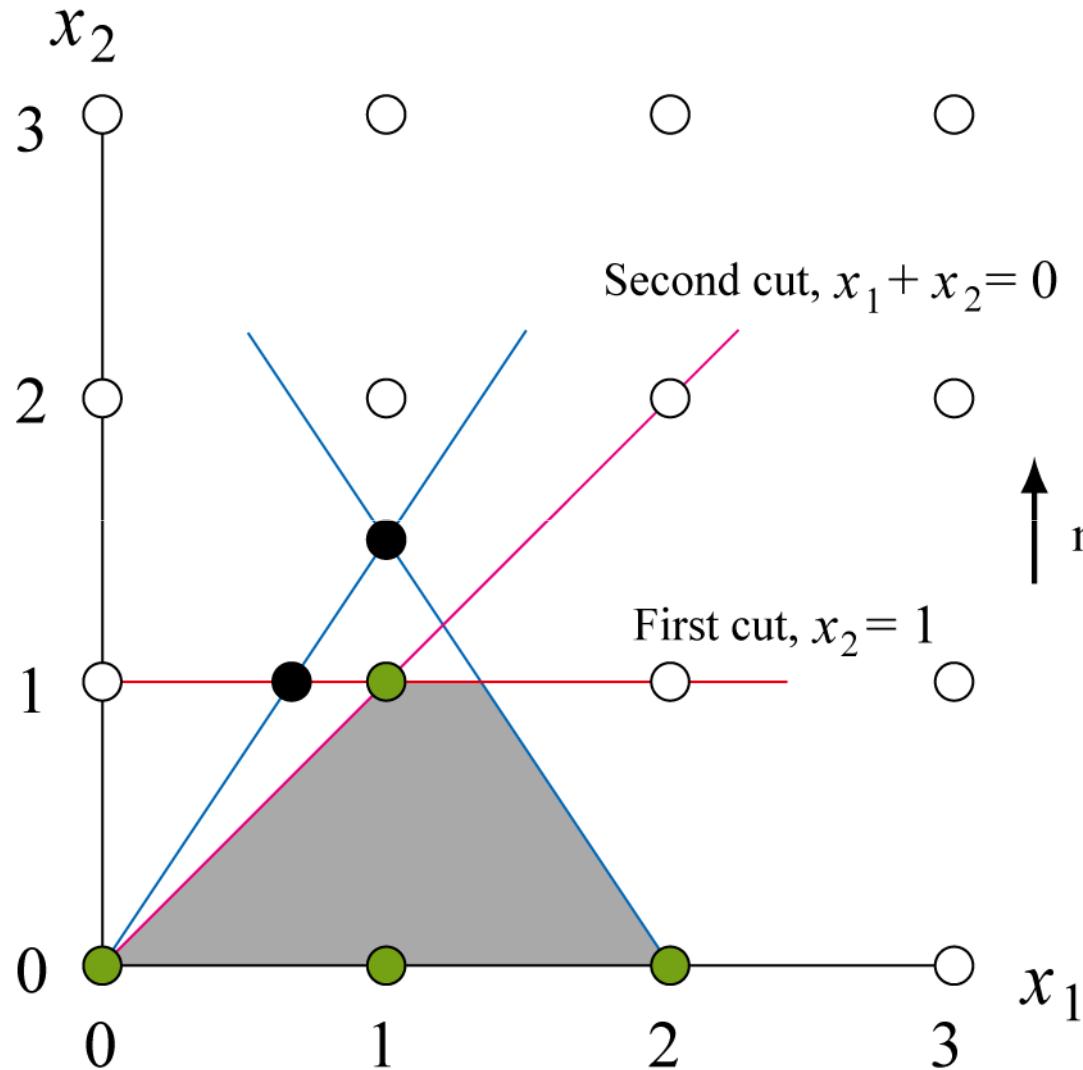
Add a new slack variable:



$$-\frac{2}{3}x_5 - \frac{2}{3}x_4 \leq -\frac{2}{3}$$

$$x_6 = \frac{2}{3} + \frac{2}{3}x_5 + \frac{2}{3}x_4$$

The new cut is inserted into the optimum dictionary



$$-\frac{2}{3}x_5 - \frac{2}{3}x_4 \leq -\frac{2}{3}$$

$$x_1 = \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4$$

$$x_2 = 1 - x_5$$

$$x_3 = 2 + 4x_5 - x_4$$

$$x_6 = \frac{2}{3} + \frac{2}{3}x_5 + \frac{2}{3}x_4$$

$$z = 1 - x_5$$

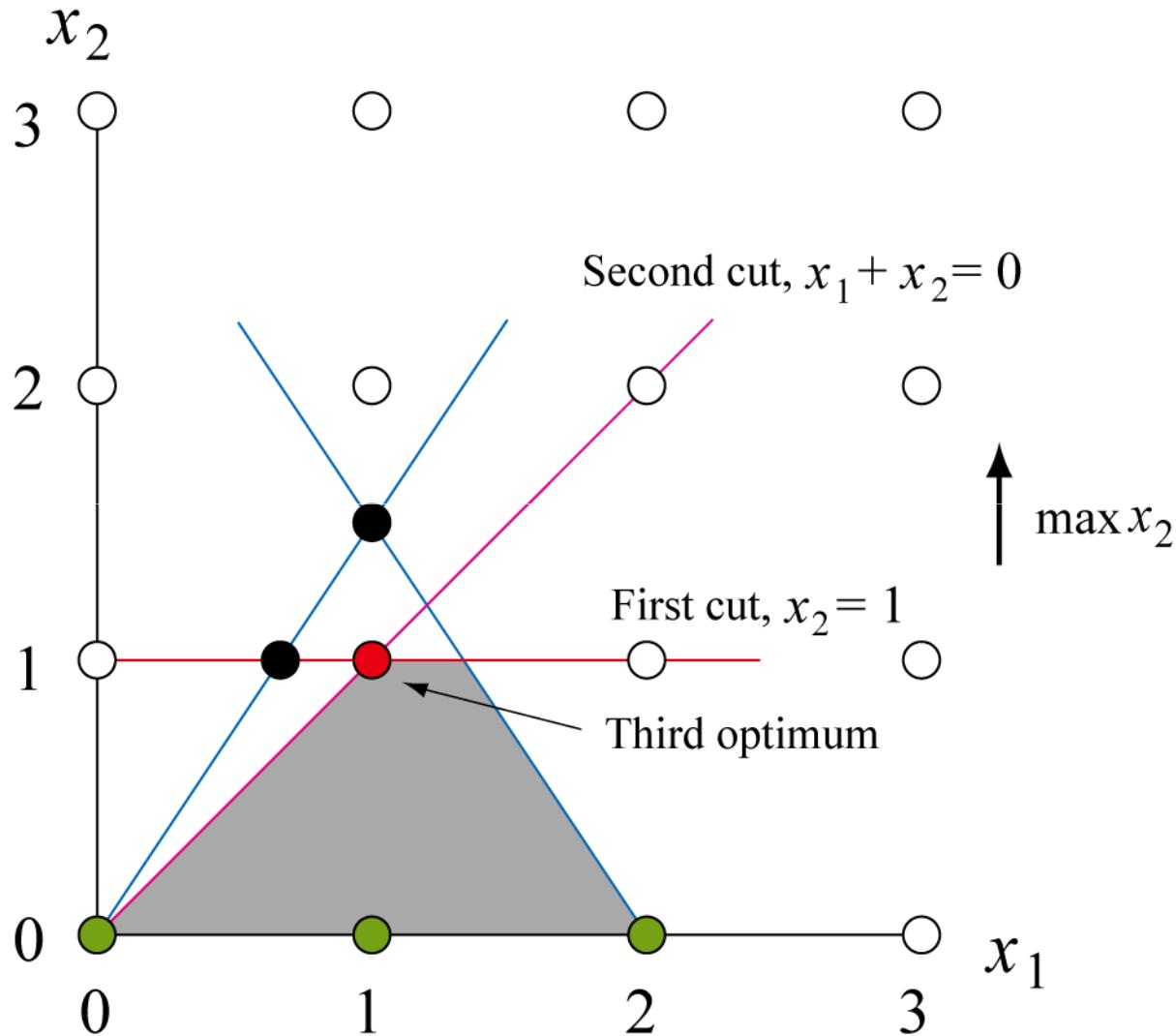
## Re-optimize using dual simplex method

$$\begin{aligned}x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\x_2 &= 1 - x_5 \\x_3 &= 2 + 4x_5 - x_4 \\x_6 &= -\frac{2}{3} + \frac{2}{3}x_5 + \frac{2}{3}x_4 \\z &= 1 - x_5\end{aligned}$$



$$\begin{aligned}x_1 &= 1 - x_5 + \frac{1}{2}x_6 \\x_2 &= 1 - x_5 \\x_3 &= 1 + 5x_5 - \frac{3}{2}x_6 \\x_4 &= 1 - x_5 + \frac{2}{3}x_6 \\z &= 1 - x_5\end{aligned}$$

## The new optimum solution is integral



$$\begin{aligned}x_1 &= 1 - x_5 + \frac{1}{2}x_6 \\x_2 &= 1 - x_5 \\x_3 &= 1 + 5x_5 - \frac{3}{2}x_6 \\x_4 &= 1 - x_5 + \frac{2}{3}x_6 \\z &= 1 - x_5\end{aligned}$$