Discrete Optimization II   COMP 567

Homework 3   Due: Friday April 16, 2010, McConnell 232 (Conor's office)

This assignment uses *lrs* to investigate a combinatorial polytope. First read the handout: http://cgm.cs.mcgill.ca/~avis/courses/567/notes/knapsack_notes.html You will also need to refer to the lrs Users Guide: http://cgm.cs.mcgill.ca/~avis/C/lrs.html

Consider the $n(n-1)$variables x_{ij}, $1 \le i \le n, 1 \le j \le n$, $i \ne j$ defined on the edges of a **directed** complete graph on n vertices. This means that for each pair of vertices i and j there are **distinct** edges ij and ji. Consider the polytope defined by the following linear system, where we choose **every** distinct set of 3 indices $1 \le i, j, k \le n$:

$$x_{ij} + x_{jk} + x_{ki} = x_{ji} + x_{ik} + x_{kj} \tag{1}$$

$$x_{ij} + x_{jk} + x_{ki} \le 1 \tag{2}$$

$$x_{ik} \le x_{ij} + x_{jk} \tag{3}$$

In addition, all variables are nonnegative. The indices are a bit tricky, so first try to write down the system for $n = 3$. You should get one equation of type (1), two inequalities of type (2) and 6 inequalities of type (3). In addition there are 6 nonnegativity inequalities, for a total of 15 constraints. The lrs input file for $n = 3$ is shown below. Note that the equation is entered using the linearity option, and the nonnegativity constraints must be explicitly given when using this option. The columns correspond to variables in order x_{12}, x_{13}, x_{23}, x_{21}, x_{31}, x_{32}.

```
H-representation
linearity 1 1
begin
15 7 rational
0  1 -1  1 -1  1 -1
0 -1  1  0  0  0  1
0  1 -1  1  0  0  0
0  0  0  1 -1  1  0
0  0  1 -1  1  0  0
0  0  0  0  1 -1  1
0  1  0  0  0  1 -1
1 -1  0 -1  0 -1  0
1  0 -1  0 -1  0 -1
0  1  0  0  0  0  0
0  0  1  0  0  0  0
0  0  0  1  0  0  0
0  0  0  0  1  0  0
0  0  0  0  0  1  0
0  0  0  0  0  0  1
end
```

In fact some of these constraints are redundant, and you could get a minimum description by running the program *redund.*

Here is what I would like you to do. Repeat these steps for $n = 3, 4, 5$.

(a) Create the lrs input files for the above system. Compute the vertices using lrs.

(b) Extract the integer vertices and discard the fractional ones. What subgraphs of the directed complete graph do the integer vertices correspond to?

(c) Compute the facets of the integer vertices using lrs again. For $n = 4$ only: find two new non-equivalent facets. Note: Because of equation (1) some facets may seem "new" but are not. For example if we add the equation (1)

$$x_{21} + x_{13} + x_{32} = x_{12} + x_{23} + x_{31}$$

to the inequality (2)

$$x_{34} \le x_{32} + x_{24}$$

we get

$$x_{21} + x_{13} + x_{34} \le x_{12} + x_{23} + x_{31} + x_{24}$$

which is not a new facet, just a rewriting of the old one. To get rid of these cases, draw each facet as a directed graph called a support graph. The edges of the support graph correspond to coefficients which are non zero. Discard any support graphs that contain a directed cycle of length at least three (eg. edges $ij,\ jk,\ ki$ form a directed cycle of length 3) when looking for new facets.

(d) For $n = 4$ only: show how to obtain your two new facets by using the Chvatal-Gomory procedure.