Integer Programming 308-761B
Class Test March 12, 1999

1. A *kernel* in an directed graph $G = (V, E)$ is a subset K of the vertices so that:
(i) No two vertices in K are joined by an edge, and
(ii) For every vertex i not in K, there is an directed edge $ij \in E$ for some $j \in K$.
In other words the kernel is an independent set of vertices so that each vertex is either in the kernel, or has an edge directed into it.
Formulate the problem of finding a kernel of minimum cardinality (if one exists) as an integer program.

2. Show how to model the following conditions as standard linear constraints in an ILP, and prove the correctness of your approach:

(i) $x \bmod p = 1$, where x is an integer variable, and p is an integer constant.

(ii) $x = \lfloor y/k \rfloor$, where y is an integer variable, and k is an integer constant.

(iii) $xy = 0, 0 \le x \le M, 0 \le y \le M$, where x and y are real variables, and M is a constant.

3. Consider the IP

$$\textit{maximize} \; -2x_1 + x_2$$

$$-3x_1 + x_2 \le 0$$

$$x_1 + x_2 \le 5$$

$$x_1, x_2 \ge 0 \text{ and } \textit{integer}$$

The LP solution to this is $x_1 = 5/4, x_2 = 15/4$.

Use the Chvátal-Gomory procedure to generate cut(s) for this problem until you get an integer solution. You may solve the LP(s) graphically if you wish. (You may use any method you like to choose the cut(s), they do not have to be generated from the optimum LP dictionary.)

4. Is the n by n matrix with 1 on the main diagonal and -1 elsewhere totally unimodular for all n? Give a proof or counterexample.