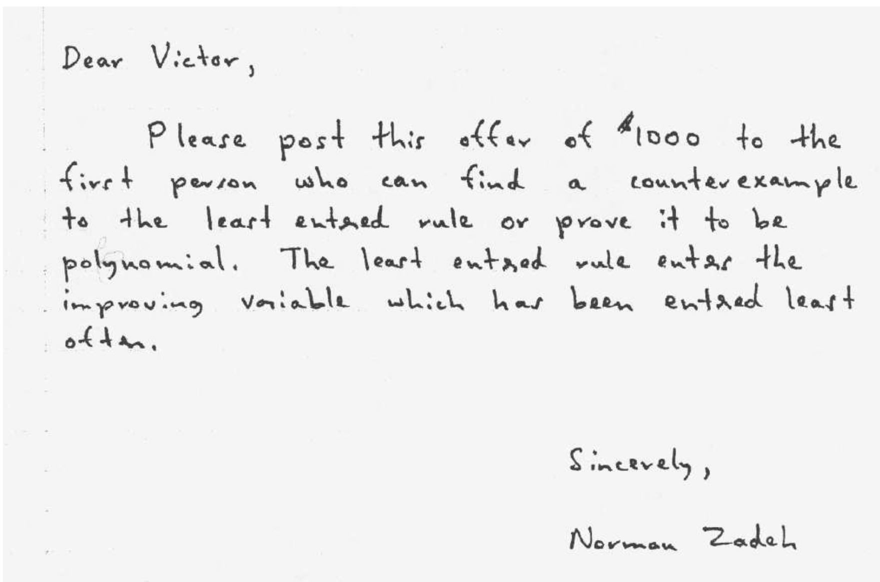
Postscript

The paper reprinted above is a 1980 technical report [9] issued by the (now defunct) Department of Operations Research at Stanford University. Although it was never published in a journal, and went out of print, it contains a promising pivot rule for linear programming that has resisted analysis and entered the folklore of mathematical programming. In fact a little known prize goes with a successful analysis of its performance, as described below in a figure and caption excerpted from Günter Ziegler's paper [10], and included here with his kind permission:



Dear Victor,

Please post this offer of $1000 to the first person who can find a counterexample to the least entred rule or prove it to be polynomial. The least entred rule enters the improving variable which has been entred least often.

Sincerely,

Norman Zadeh

Figure 6. *Z*adeh's offer

The Least Entered rule was proposed by Norman Zadeh around 1980, and he offered $1000 to anyone who can prove or disprove that this rule is polynomial in the worst case; see the text of Figure 6 in Zadeh's handwriting (from a letter to Victor Klee, reproduced with his kind permission). Just to encourage the readers to try their luck on this problem, we want to mention that according to a recent magazine report [4], Norman Zadeh is now a successful businessman for whom it should be no problem to pay for the prize once you have solved the problem. Good luck! [10]

Early references to the pivot rule are contained in Klee and Kleinschmidt [7], Fathi and Tovey [2], and Shamir [5]. In Terlaky and Zhang's [6] 1993 survey of pivot rules for linear programming, the last paragraph reads:

> *To conclude the paper we note that the hardest and long standing open problems in the theory of linear programming are still concerned with pivot methods. These include the d-step conjecture [7] and the question of whether there exists a polynomial time pivot rule or not. For the last problem Zadeh's rule [9] might be a candidate. At least it is still notproved tobe exponential in the worst case. [6]*

As for progress on analyzing Zadeh's rule, the only result to date that I know of was obtained for simple polytopes in 3-dimensions by Kaibel et al. [3]. For such a polytope with n facets, the longest pivot path that the simplex method could take would have at most $2n-5$ pivots. They show that this bound is essentially achieved by many common pivot rules, including Zadeh's rule, that the greatest improvement rule requires at most $1.5n$ pivots, and that the random edge rule does somewhat better with at most $1.4943n$ pivots.

However, reading Zadeh's paper one sees its main thrust was not a new pivot rule. Zadeh makes two other contributions. One was that the bad examples could be achieved with small integer coefficients, and so had nothing to do with the *size* of the input coefficients. The second was in suggesting a general framework to understand all such examples. He points out in the introduction that all then known examples of exponential worst case behaviour of the simplex method occur in *deformed products* of polytopes. This construction was formalized and extended to many more recent examples almost twenty years later by Amenta and Ziegler [1]. Zadeh also notes that the bad examples for the network simplex method given in his 1973 paper [8] were also deformed product constructions. The network example is not included in [1], but a formal statement of its deformed product structure is given in the paper by Ziegler cited earlier, where it is preceded by the remark:

> *It may seem surprising that even these examples are iterated deformed products[10].*

I doubt Norman was surprised!

References

[1] N. Amenta and G. Ziegler. *Deformed Products and Maximal Shadows of Polytopes*, volume 223 of *Contemporary Mathematics*, pages 57–90. American Mathematics Society, Providence, 1999.

[2] Y. Fathi and C. Tovey. Affirmative action algorithms. *Math. Program.*, 34(3):292–301, 1986.

[3] V. Kaibel, R. Mechtel, M. Sharir, and G. M. Ziegler. The simplex algorithm in dimension three. *SIAM J. Comput.*, 34(2):475–497, 2005.

[4] S. Lubove. See no evil. *Forbes*, 168:68–70, 2001.

[5] R. Shamir. The efficiency of the simplex method: a survey. *Manage. Sci.*, 33(3):301–334, 1987.

[6] T. Terlaky and S. Zhang. Pivot rules for linear programming–a survey. *Annals of Operations Research*, 46:203–233, 1993.

[7] V. and P. Kleinschmidt. The d-step conjecture and its relatives. *Math. Oper. Res.*, 12(4):718–755, 1987.

[8] N. Zadeh. A bad example for the simplex method and other minimum cost flow algorithms. *Mathematical Programming*, 5:255–266, 1973.

[9] N. Zadeh. What is the worst case behaviour of the simplex algorithm? Technical Report 27, Department of Operations Research, Stanford University, 1980.

[10] G. M. Ziegler. Typical and extremal linear programs. In M. Grötschel, editor, *The Sharpest Cut: The Impact of Manfred Padberg and His Work*, volume 4 of *MPS-SIAM Series on Optimization*, chapter 14, pages 217–230. SIAM, Philadelphia, PA, 2004.

David Avis, April 15, 2006, Marseille