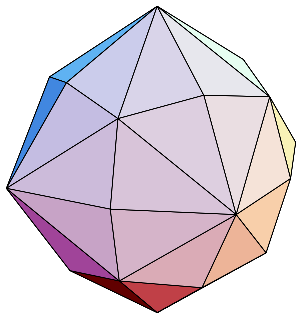
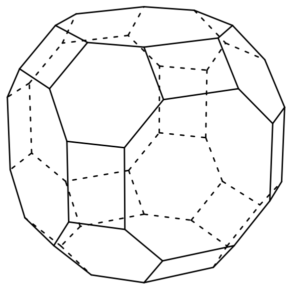
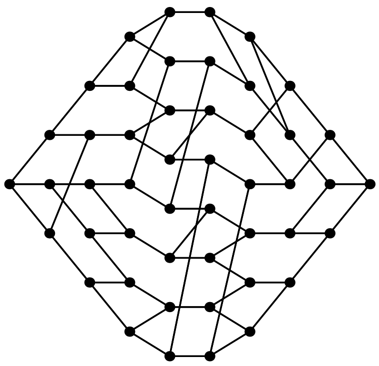
Lecture Notes: Combinatorics of Mathematical Programming and Polyhedral Geometry

Komei Fukuda
fukuda@dma.epfl.ch or fukuda@gssm.otsuka.tsukuba.ac.jp
EPFL, Lausanne, Switzerland, and
University of Tsukuba, Tokyo, Japan







Contents

1 LP, QP and LCP

1.1 LP Duality Theorem

Let A be a real $m \times n$ matrix, let b and c be real m-vector and n-vector, respectively. The following optimization problems LP and LD are known to be a dual pair of linear programs.

$$
\begin{array}{llrl}
\text{(LP)} & \max & f(x) = c^T & x \\
& \text{subject to} & A & x \le b \\
& & & x \ge 0 \\
\\
\text{(LD)} & \min & g(y) = b^T & y \\
& \text{subject to} & A^T & y \ge c \\
& & & y \ge 0
\end{array}
\tag{1.1}
$$

Proposition 1.1 (LP Weak Duality) : *For any feasible solution $\overline{x}$ to LP and any feasible solution $\overline{y}$ to DP,*

$$c^T\overline{x} \quad \le \quad b^T\overline{y},$$

where the equality holds if and only if

$$
\begin{array}{ll}
(a) & i \in \{1, 2, \dots, m\} \text{ and } (b - A\overline{x})_i > 0 \Rightarrow \overline{y}_i = 0, \text{ and} \\
(b) & j \in \{1, 2, \dots, n\} \text{ and } (A^T\overline{y} - c)_j > 0 \Rightarrow \overline{x}_j = 0.
\end{array}
$$

Theorem 1.2 (LP Duality Theorem) : *The following statements hold:*

(a) *If both LP and LD are feasible, then each has an optimal solution, and the optimal values are equal.*

(b) *If LP (LD) is infeasible, then its dual problem LD (LP) is either infeasible or unbounded.*

1.2 QP Duality Theorem

Let A be a real $m \times n$ matrix, let G and H be symmetric positive definite matrice of order n and m, and let b and c be real m-vector and n-vector, respectively. The following optimization problems QP and QD are known to be a dual pair of convex quadratic programs.

$$
\begin{array}{llrlrl}
\text{(QP)} & \max & f(x,y) = c^Tx - \frac{1}{2}x^TG & x- & \frac{1}{2}y^TH & y \\
& \text{subject to} & A & x- & H & y \le b \\
& & & x \ge & 0 & \\
\\
\text{(QD)} & \min & g(x,y) = b^Ty + \frac{1}{2}x^TG & x+ & \frac{1}{2}y^TH & y \\
& \text{subject to} & G & x+ & A^T & y \ge c \\
& & & & & y \ge 0
\end{array}
\tag{1.2}
$$

Proposition 1.3 (QP Weak Duality) : *For any feasible solution (x', y') to LP and any feasible solution (x'', y'') to DP,*

$$f(x', y') \quad \leq \quad g(x'', y''),$$

where the equality holds if and only if

$$\begin{array}{ll} (a) & i \in \{1, 2, \ldots, m\} \text{ and } (b - Ax' + Hy')_i > 0 \Rightarrow y''_i = 0, \\ (b) & j \in \{1, 2, \ldots, n\} \text{ and } (Gx'' + A^T y'' - c)_j > 0 \Rightarrow x'_j = 0, \text{ and} \\ (c) & (x' - x'')^T G(x' - x'') = (y' - y'')^T H(y' - y'') = 0. \end{array}$$

Theorem 1.4 (QP Duality Theorem, Cottle [Co63]) : *The following statements hold:*

(a) *If both QP and QD are feasible, then each has an optimal solution, and the optimal values are equal; moreover the optimal solutions can be taken to be the same.*

(b) *If QP (QD) is infeasible, then its dual problem QD (QP) is either infeasible or unbounded.*

1.3 LCP Fundamental Theorems

Let A be a real $n \times n$ matrix, let b a real n-vector. Then the linear complementarity problem is to find vectors w and z in R^n satisfying

$$\begin{array}{ll} (LCP) & w = Az + b, z \geq 0, w \geq 0 \\ & z^T w = 0. \end{array} \tag{1.3}$$

The last condition $z^T w = 0$ is called the *complementarity condition.*

The theory of linear complementarity, introduced by Cottle [Co68], Cottle and Dantzig [CD68] and Lemke [L65], has been extensively studied for the last two decades. One main importance of this theory lies in the fact that it provides a unified framework for the two important optimization problems with wide applications, linear programming and convex quadratic programming.

LCP Formulation of LP

By using the LP duality theorems, Proposition 1.1 and Theorem 1.1, the linear programming problem (1.1) can be reduced to the LCP:

$$\begin{array}{ll} (CLP) & \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} b \\ -c \end{bmatrix}, \\ & \begin{bmatrix} y \\ x \end{bmatrix} \geq 0, \begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \\ & \begin{bmatrix} y \\ x \end{bmatrix}^T \cdot \begin{bmatrix} u \\ v \end{bmatrix} = 0. \end{array} \tag{1.4}$$

LCP Formulation of QP

Similary by using the QP duality theorems, Proposition 1.3 and Theorem 1.4, the convex quadratic programming problem (1.2) can be reduced to the LCP:

$$
\begin{aligned}
(CLP) \quad \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} H & -A \\ A^T & G \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} b \\ -c \end{bmatrix}, \\
\begin{bmatrix} y \\ x \end{bmatrix} &\geq 0, \begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \\
\begin{bmatrix} y \\ x \end{bmatrix}^T &\cdot \begin{bmatrix} u \\ v \end{bmatrix} = 0.
\end{aligned} \tag{1.5}
$$

It is known that solving a general LCP (1.3) is hard: it is *NP*-complete when A and b are rational, see [Chu89]. In fact it has been shown in [KMNY91] (Section 3.4) that the LCP remains *NP*-complete even if the matrix A is restricted to be P_0. (A square matrix is called a P_0*-matrix* if every principal minors are nonnegative.) The main difficulty can be explained by the fact that nobody knows any easy way to recognize when the problem has no solution. For an LCP to be solved efficiently, it is necessary that a polynomially verifiable condition exists which is true if and only if the LCP has no solution. Such a condition may be called a good characterization of the nonexistence of solutions. There are several different classes of matrices A for which such a characterization is known. The P-matrices and the positive semidefinite (PSD-) matrices constitute such classes. The LCPs arising from linear programming problems or convex quadratic programming problems have PSD-matrices. The class of sufficient matrices, which has been recently introduced by Cottle, Pang and Venkateswaran [CPV89], contains properly the two classes and admits also a good characterization of the nonexistence of solutions.

Theorem 1.5 (Samelson-Thrall-Wesler [STW58], Cottle [Co89], Gale-Nikaido [GN65]) : *The following statements are equivalent.*

(a) *LCP has exactly one solution for every b (i.e. A is P-matrix);*

(b) *There exists no z with $z_i \cdot (Az)_i \leq 0 \; \forall i = 1, 2, \ldots, n$;*

(c) *A has positive principal minors.*

Theorem 1.6 (Cottle-Dantzig [CD68], Lemke [L65]) : *If A is positive semidefinite (abbreviated by PSD), then the following statements are equivalent.*

(a) *LCP has a solution;*

(b) *LCP without the complementarity condition has a solution.*

Remark 1.7. *By Farkas' Lemma, the condition (b) above is equivalent to:*

(c) *there exists no y such that $y \geq 0$, $yA \leq 0$ and $yb < 0$.*

A square matrix A is called *column sufficient* if

$$z_i \cdot (Az)_i \leq 0 \ \forall i = 1, 2, \ldots, n \quad \Rightarrow \quad z_i \cdot (Az)_i = 0 \ \forall i = 1, 2, \ldots, n, \tag{1.6}$$

and called *row sufficient* if A^T is column sufficient. If A is both row and column sufficient, it is called *sufficient*.

Remark 1.8. **(a)** *If A is P-matrix, then it is sufficient.*

(b) *If A is PSD, then it is sufficient.*

Theorem 1.9 (Cottle, Pang and Venkateswaran [CPV89]) : *If A is sufficient, then the following statements are equivalent.*

(a) *LCP has a solution;*

(b) *LCP without the complementarity condition has a solution.*

In Section 2, following [FTe92], we shall develop the combinatorial theory of LCP, and obtain a generalization of Theorem 1.9.

2 Combinatorial Theory of LCP

2.1 LCP Duality Theorem

In order to write the LCP (1.3) in a form that will be convenient for extending it to the setting of oriented matroids, we set

$$V(A, b) = \{x \in R^{2n+1} \ | \ [-b \ -A \ I] \, x = 0\}. \tag{2.1}$$

We denote the components of a vector $x \in R^{2n+1}$ by $x = (x_0, x_1, x_2, \ldots, x_{2n})$ instead of the usual $x = (x_1, x_2, \ldots, x_{2n+1})$, while we denote a vector $x \in R^{2n}$ by $x = (x_1, \ldots, x_{2n})$. Then, one can rewrite the LCP (1.3) as

$$\text{find } x \in V(A, b) \text{ satisfying} \tag{2.2}$$

$$x \geq 0, \ x_0 = 1 \tag{2.3}$$

$$x_i \cdot x_{n+i} = 0 \text{ for all } i = 1, \ldots, n. \tag{2.4}$$

A vector x in R^{2n} or R^{2n+1} is called *complementary* if the complementarity condition (2.4) is satisfied. A solution to the LCP is a nonnegative complementary vector x in $V(A, b)$ with $x_0 = 1$.

If the LCP has a solution, then it is easy to see that there is a basic solution, i.e., one having a minimal support among all nonzero vectors in $V(A, b)$, and hence the rational LCP is in NP (see [Sc86], Corollary 3.2b). In general, there is no known good characterization of the nonexistence of a solution to the LCP. However we can write conditions on the linear subspace under which such characterization exists.

A vector x in R^{2n} or in R^{2n+1} is called *strictly sign-preserving* (s.s.p.) if

$$x_i \cdot x_{n+i} \geq 0 \text{ for all } i = 1, \ldots, n \text{ and} \tag{2.5}$$

$$x_j \cdot x_{n+j} > 0 \text{ for some } j, \tag{2.6}$$

and called *strictly sign-reversing* (s.s.r.) if

$$x_i \cdot x_{n+i} \leq 0 \text{ for all } i = 1, \ldots, n \text{ and} \tag{2.7}$$

$$x_j \cdot x_{n+j} < 0 \text{ for some } j. \tag{2.8}$$

Now we have the following general theorem, which we call the duality theorem of linear complementarity.

Theorem 2.1 (LCP Duality Theorem) : *Let V be a linear subspace of R^{2n+1} satisfying the following condition:*

$$\begin{array}{rl} \textit{either} & V \textit{ contains no s.s.r. vector } x \textit{ with } x_0 = 0 \\ \textit{or} & V \textit{ contains no s.s.p. vector } x \textit{ with } x_0 \neq 0, \end{array} \tag{2.9}$$

and suppose the orthogonal complement $V^\perp$ also satisfies the same condition, i.e.,

$$\begin{array}{rl} \textit{either} & V^\perp \textit{ contains no s.s.r. vector } y \textit{ with } y_0 = 0 \\ \textit{or} & V^\perp \textit{ contains no s.s.p. vector } y \textit{ with } y_0 \neq 0. \end{array} \tag{2.10}$$

Then exactly one of the following statements holds:

(a) *there exists a nonnegative complementary vector x in V with $x_0 = 1$;*

(b) *there exists a nonnegative complementary vector y in $V^\perp$ with $y_0 = 1$.*

One can easily see that this theorem is self-dual, i.e., replacing V by $V^\perp$ yields the same theorem. This self-duality makes the statement look simple, and quite importantly it makes a short elementary proof possible. Also, this theorem suggests us to consider the *dual* problem of the LCP (2.2)∼(2.4):

$$\text{find } y \in V(A,b)^\perp \text{ satisfying} \tag{2.11}$$

$$y \geq 0, \; y_0 = 1 \tag{2.12}$$

$$y_i \cdot y_{n+i} = 0 \text{ for all } i = 1, \ldots, n, \tag{2.13}$$

which is solvable if and only if the *primal* problem is not, under the assumptions of theorem.

Remark that if V in the duality theorem has an explicit representation as $V(A,b)$, then one can write the statements (a) and (b) more explicitly as

(a') there exist vectors w, z in R^n satisfying $w = Az + b, z \geq 0, w \geq 0,$ and $z^T w = 0$;

(b') there exists a vector u in R^n satisfying $b^T u = -1, u \geq 0, A^T u \leq 0,$ and $u^T A^T u = 0$.

The assumptions (2.9) and (2.10) for $V(A, b)$ are satisfied by several well-known classes of square matrices A, for any vector b. For example, the class of P-matrices and the class of PSD-matrices satisfy the conditions because, for any b,

$$\begin{aligned} &V(A, b) \text{ contains no s.s.r. vector } x \text{ with } x_0 = 0 \text{ and} \\ &V(A, b)^\perp \text{ contains no s.s.p. vector } w \text{ with } w_0 \neq 0. \end{aligned}$$

In fact, the sufficient matrices are exactly those matrices with these properties. More explicitly, an $n \times n$ matrix A is column sufficient if

$$w = Az \text{ implies } (w, z) \text{ is not s.s.r.,} \tag{2.14}$$

row sufficient if

$$v = -A^T u \text{ implies } (u, v) \text{ is not s.s.p.,} \tag{2.15}$$

and *sufficient* if it satifies both. Theorem 2.1 implies Theorem 1.9: for any sufficient matrix A and any vector b, exactly one of statements (a') or (b') holds.

As an example of linear subspaces V satisfying the conditions (2.9) and (2.10), we can set $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and $V = V(A, b)$. In this example, the primal problem has a complementary solution $x = (x_0, x_1, x_2, x_3, x_4) = (1, 0, 1, 0, 0)$. Remark that if we set $b = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$ and use the same A, the conditions of the theorem are not fulfilled because both the s.s.r vector $(0, 0, 1, 1, -2)$ and the s.s.p. vector $(2, 0, -1, 3, -2)$ are in $V^\perp$. In fact, in this case, neither of V nor $V^\perp$ have any nonnegative complementary solutions, although V contains the nonnegative vector $(1, 0, 1, 0, \frac{1}{2})$. The matrix A is column sufficient (and not row sufficient), and this example explains the difficulty of solving the LCP with a column sufficient matrix in general.

It should be noted that Theorem 2.1 can be considered as a generalization of the LP duality theorem (Theorem 1.2) and Cottle's QP duality theorem (Theorem 1.4 since they correspond to certain subclasses of PSD-matrices.

In the following sections, we will see how the duality theorem, Theorem 2.1, can be generalized and proved in purely combinatorial fashion using the notion of oriented matroids.

2.2 Oriented Matroids

Let E be a finite set. A *sign vector* X *on* E is a vector in $\{+, 0, -\}^E$. The opposite $-X$ of a sign vector X is defined in the obvious way: $-(-0-++) = (+0+--)$. Let X and Y be sign vectors on E. An element $j \in E$ is said to *separate* X and Y if X_j and Y_j are nonzero and opposite in signs. The set of elements separating X and Y is denoted by $D(X,Y)$. The *composition* of X and Y, denoted by $X \circ Y$, is the sign vector on E defined by

$$(X \circ Y)_j = \begin{cases} X_j & \text{if } X_j \neq 0 \\ Y_j & \text{otherwise} \end{cases} \qquad (j \in E).$$

The *support* $\underline{X}$ of X is the set $\{j \in E : X_j \neq 0\}$.

An *oriented matroid on* E is a pair $M = (E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a set of sign vectors on E satisfying the following axioms:

(OM1) $\mathbf{0} \in \mathcal{F}$; (where $\mathbf{0}$ denotes the zero vector)

(OM2) $X \in \mathcal{F} \Longrightarrow -X \in \mathcal{F}$;

(OM3) $X, X' \in \mathcal{F} \Longrightarrow X \circ X' \in \mathcal{F}$;

(OM4) $X, X' \in \mathcal{F}$ and $f \in D(X, X') \Longrightarrow$ there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z_j = (X \circ X')_j$ for all $j \notin D(X, X')$.

We call each member of $\mathcal{F}$ a *vector* of the oriented matroid. Each vector with minimal nonempty support is called a *circuit*. One can define oriented matroids in terms of circuits, see [B77, BL78].

Note that the notion of oriented matroids was introduced independently by Bland-Las Vergnas [BL78] and Folkman-Lawrence [FL78]. The definition of oriented matroids above is due to Edmonds, Fukuda and Mandel [Fu82, Ma82].

For a vector $x \in R^E$, we denote the sign incidence vector of x as $\delta(x)$, i.e., $\delta(x)$ is the sign vector on E with $\delta(x)_j = sign(x_j)$ for each $j \in E$. One can easily check that for any linear subspace V of R^E, the set $\mathcal{F} = \delta(V)$ of all sign vectors coming from V satisfies the axioms (OM1)$\sim$(OM4). Such an oriented matroid is called *representable* or *linear*. There are several interesting constructions of non-representable oriented matroids, see [Fu82, Ma82, BS89, BLSWZ93].

We shall give below some basic properties of oriented matoids without proofs. For those unfamiliar with oriented matroids, good references are [BLSWZ93, BL78, FL78, Fu82, Ma82].

It will be convenient to define the *product* $\alpha \cdot \beta$ (or simply $\alpha\,\beta$) of two signs $\alpha, \beta \in \{+, 0, -\}$ in the natural way: $+$ if both are nonzero and of same sign, $-$ if both are nonzero and of opposite sign, and 0 otherwise. Two sign vectors X and Y are called *orthogonal* (denoted by $X * Y$) if the existence of an element $f \in E$ with $X_f \cdot Y_f \neq 0$ implies the existence of an element $g \in E$ with $X_g \cdot Y_g = -X_f \cdot Y_f$. The *dual* $M^* = (E, \mathcal{F}^*)$ of an oriented matroid M is defined by

$$\mathcal{F}^* = \{Y \in \{+, -, 0\}^E \mid X * Y \text{ for all } X \in \mathcal{F}\}.$$

The dual M^* is also an oriented matroid, and

$$\mathcal{F}^{**} = \mathcal{F} \tag{2.16}$$

Furthermore, if $F = \delta(V)$ for a linear subspace V of R^E then $F^* = \delta(V^\perp)$. A vector, circuit, ... of M^* may be called *covector*, *cocircuit* ... of M.

There are two basic operations on oriented matroids which will be used frequently in this paper. For a subset S of E, let

$$\mathcal{F}\backslash S = \{X\backslash S \mid X \in \mathcal{F} \text{ and } X_j = 0 \text{ for all } j \in S\} \tag{2.17}$$

$$\mathcal{F}/S = \{X\backslash S \mid X \in \mathcal{F}\}, \tag{2.18}$$

where $X\backslash S$ denotes the subvector $(X_j : j \in E\backslash S)$ of X. It is easy to see from the definition, both $(E, \mathcal{F}\backslash S)$ and $(E, \mathcal{F}/S)$ are oriented matroids, which are denoted by $M\backslash S$ and M/S, respectively. When S is a singleton set $\{e\}$, $M\backslash\{e\}$ and $M/\{e\}$ will be denoted by $M\backslash e$ and M/e, respectively. The following relations hold for any oriented matroid $M = (E, \mathcal{F})$:

$$(\mathcal{F}\backslash S)^* = \mathcal{F}^*/S \tag{2.19}$$

$$(\mathcal{F}/S)^* = \mathcal{F}^*\backslash S \tag{2.20}$$

2.3 Oriented Matroid Complementarity Problem

For a nonnegative integer n, let E_{2n} denote a set of $2n$ elements partitioned into prescribed n pairs; for each $e \in E_{2n}$ there is a unique element $\overline{e}$ in E_{2n}, called the complement of e such that $\overline{e} \neq e$ and $\overline{\overline{e}} = e$. For a subset S of E_{2n} let $\overline{S} = \{\overline{e} \mid e \in S\}$. A subset S is called *complementary* if $S \cap \overline{S} = \emptyset$. We use $\hat{E}_{2n}$ to denote $E_{2n} \cup \{g\}$ for a given E_{2n} and a distinguished element g which is assumed to be not in E_{2n}.

One canonical choice of E_{2n} and $\hat{E}_{2n}$ is $\{1, 2, \ldots, 2n\}$ and $\{0, 1, 2, \ldots, 2n\}$ with $g = 0$, respectively, where $\overline{i} = i + n$ if $1 \leq i \leq n$, and $\overline{i} = i - n$ if $n < i \leq 2n$.

For a given oriented matroid $M = (\hat{E}_{2n}, \mathcal{F})$ with $n \geq 0$, the *complementarity problem* (OMCP) is to

$$\text{find } X \in \mathcal{F} \text{ satisfying} \tag{2.21}$$

$$X \geq 0,\ X_g = + \tag{2.22}$$

$$X_e \cdot X_{\overline{e}} = 0 \text{ for all } e \in E_{2n} \tag{2.23}$$

Here, we use the inequality $\geq$ in the obvious way and we shall do so in the sequel. The conditions (2.21), (2.22) and (2.23) are straightforward abstractions of (2.2), (2.3) and (2.4), respectively.

Again, we simply make the "automatic" abstraction of definitions in the LCP case. A sign vector X on E_{2n} or on $\hat{E}_{2n}$ is called *complementary* if the condition (2.23) is satisfied, called *strictly sign-preserving* (s.s.p.) if

$$X_e \cdot X_{\overline{e}} \geq 0 \text{ for all } e \in E_{2n} \text{ and} \tag{2.24}$$

$$X_f \cdot X_{\overline{f}} > 0 \text{ for some } f, \tag{2.25}$$

and called *strictly sign-reversing* (s.s.r.) if

$$X_e \cdot X_{\overline{e}} \leq 0 \text{ for all } e \in E_{2n} \text{ and} \tag{2.26}$$

$$X_f \cdot X_{\overline{f}} < 0 \text{ for some } f. \tag{2.27}$$

The following theorem is a combinatorial generalization of the duality theorem, Theorem 2.1.

Theorem 2.2 (OMCP Duality Theorem) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the following conditions:*

$$\begin{array}{rl} \text{either} & \mathcal{F} \text{ contains no s.s.r. vector } X \text{ with } X_g = 0 \\ \text{or} & \mathcal{F} \text{ contains no s.s.p. vector } X \text{ with } X_g \neq 0 \text{ and,} \end{array} \tag{2.28}$$

$$\begin{array}{rl} \text{either} & \mathcal{F}^* \text{ contains no s.s.r. vector } Y \text{ with } Y_g = 0 \\ \text{or} & \mathcal{F}^* \text{ contains no s.s.p. vector } Y \text{ with } Y_g \neq 0. \end{array} \tag{2.29}$$

Then exactly one of the following statements holds:

(a) *there exists a nonnegative complementary vector X in $\mathcal{F}$ with $X_g = +$;*

(b) *there exists a nonnegative complementary vector Y in $\mathcal{F}^*$ with $Y_g = +$.*

We call an oriented matroid $M' = (E_{2n}, \mathcal{F}')$ *sufficient* if $\mathcal{F}'$ contains no s.s.r. vector and its dual $(\mathcal{F}')^*$ contains no s.s.p. vector. If $M = (\hat{E}_{2n}, \mathcal{F})$ is an single-element extension of a sufficient oriented matroid $M' = (E_{2n}, \mathcal{F}')$ by g, i.e., $M \backslash g = M'$, then clearly M satisfies the conditions (2.28) and (2.29) of Theorem 2.2. Therefore we have the following corollary of the duality theorem.

Corollary 2.3. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be any single-element extension of a sufficient oriented matroid M' by g. Then exactly one of the following statements holds:*

(a) *there exists a nonnegative complementary vector X in $\mathcal{F}$ with $X_g = +$;*

(b) *there exists a nonnegative complementary vector Y in $\mathcal{F}^*$ with $Y_g = +$.*

In this section, we prove the duality theorem in a most elementary way. It will be interesting to compare this proof with an algorithmic proof for (a slightly different version of) this theorem, since they are closely related.

For the elementary proof, what we need is the notion of minors of OMCP, which was introduced by Todd [To84]. Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid with $n \geq 0$. For each complementary subset S of E_{2n}, let $M(S) = (\hat{E}_{2n} \backslash (S \cup \overline{S}), \mathcal{F}(S))$ denote the oriented matroid $M \backslash S / \overline{S}$, which is called a *complementary minor* of M. For $e \in E$, we denote $M(\{e\})$ by $M(e)$. Of course, the set $\hat{E}_{2n} \backslash (S \cup \overline{S})$ can be considered as a set of form $\hat{E}_{2(n-s)}$, for any complementary subset S with $\mid S \mid = s$. From the definitions, we have the following relations:

$$X \in \mathcal{F} \text{ and } X_e = 0 \text{ for all } e \in S \Rightarrow X \backslash (S \cup \overline{S}) \in \mathcal{F}(S); \tag{2.30}$$

$$X' \in \mathcal{F}(S) \Rightarrow \exists X \in \mathcal{F} \text{ such that } X \backslash (S \cup \overline{S}) = X' \text{ and } X_e = 0 \;\forall\, e \in S; \tag{2.31}$$

$$Y \in \mathcal{F}^* \text{ and } Y_{\overline{e}} = 0 \text{ for all } e \in S \Rightarrow Y \backslash (S \cup \overline{S}) \in \mathcal{F}(S)^*; \tag{2.32}$$

$$Y' \in \mathcal{F}(S)^* \Rightarrow \exists Y \in \mathcal{F}^* \text{ such that } Y \backslash (S \cup \overline{S}) = Y' \text{ and } Y_{\overline{e}} = 0 \;\forall\, e \in S. \tag{2.33}$$

These immediately imply the following:

Lemma 2.4. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (2.28) and (2.29) of Theorem 2.2. Then the complementary minor $M(S)$ of M satisfies the same conditions for any complementary subset S of E_{2n}.*

Lemma 2.5. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (2.28) and (2.29) of Theorem 2.2. Then for each $e \in E_{2n}$, at most one of the following four statements holds:*

(a1) *$\exists$ a complementary vector X^1 in $\mathcal{F}$ with $X^1 \backslash \{e, \bar{e}\} \geq 0, X^1_{\bar{e}} = -$ and $X^1_g = +$;*

(a2) *$\exists$ a complementary vector X^2 in $\mathcal{F}$ with $X^2 \backslash \{e, \bar{e}\} \geq 0, X^2_e = -$ and $X^2_g = +$;*

(b1) *$\exists$ a complementary vector Y^1 in $\mathcal{F}^*$ with $Y^1 \backslash \{e, \bar{e}\} \geq 0, Y^1_e = -$ and $Y^1_g = +$;*

(b2) *$\exists$ a complementary vector Y^2 in $\mathcal{F}^*$ with $Y^2 \backslash \{e, \bar{e}\} \geq 0, Y^2_{\bar{e}} = -$ and $Y^2_g = +$.*

Proof. We prove by contradiction. Suppose the conclusion is false, i.e., two of the four statements hold. We have six cases to consider but because of the duality (2.16), (2.19) and (2.20), we have essentially three cases:

Case 1. (a1) and (a2) hold. (This covers the case (b1) and (b2), also.)
Let $Z' = X^1 \circ X^2$. By the axiom (OM3), $Z' \in \mathcal{F}$. It is easy to see from (a1) and (a2) that Z' is s.s.p. and $Z'_g \neq 0$. On the other hand, by using the axiom (OM4) with $X = X^1$, $X' = -X^2$ and $f = g$, we have $Z'' \in \mathcal{F}$ such that $Z''_g = 0$ and $Z''_j = (X^1 \circ -X^2)_j$ for all $j \notin D(X^1, -X^2)$. Again using (a1) and (a2), we have $Z''_j \cdot Z''_{\bar{j}} \leq 0$ for all $j \in E_{2n}$ and $Z''_e \cdot Z''_{\bar{e}} < 0$. This means that Z'' is s.s.r. and $Z''_g = 0$. The existence of Z' and Z'' violates the condition (2.28). Thus (a1) and (a2) cannot hold simultaneously.

Case 2. (a1) and (b1) hold. (This covers the case (a2) and (b2), also.)
Clearly, the vectors X^1 and Y^1 are not orthogonal, a contradiction. Thus (a1) and (b1) cannot hold simultaneously.

Case 3. (a1) and (b2) hold. (This covers the case (a2) and (b1), also.)
Clearly, the vectors X^1 and Y^2 are not orthogonal, a contradiction. Thus (a1) and (b2) cannot hold simultaneously.

This completes the proof. ∎

Now we are ready to prove the duality theorem.

Proof of Theorem 2.2. We use induction on n. If $n = 0$ then the theorem is obvious. Let $n > 0$, and suppose that the theorem is true for all smaller values of n. Assume the assumptions (2.28) and (2.29) are satisfied. Let e be an element of E_{2n}. By Lemma 2.4, the assumptions of the theorem are satisfied by the minors $M(e)$ and $M(\bar{e})$. By inductive hypothesis, exactly one of

(a1') there exists a nonnegative complementary vector $\hat{X}^1$ in $\mathcal{F}(e)$ with $\hat{X}^1_g = +$;

(b1') there exists a nonnegative complementary vector $\hat{Y}^1$ in $\mathcal{F}(e)^*$ with $\hat{Y}^1_g = +$

holds, and exactly one of

(a2') there exists a nonnegative complementary vector $\hat{X}^2$ in $\mathcal{F}(\overline{e})$ with $\hat{X}^2_g = +$;

(b2') there exists a nonnegative complementary vector $\hat{Y}^2$ in $\mathcal{F}(\overline{e})^*$ with $\hat{Y}^2_g = +$

holds. Now assume that the conclusion of the theorem is false, i.e., neither (a) nor (b) of the theorem holds. Then, using the relations (2.31) and (2.33), each of the statements (a1'), (a2'), (b1') and (b2') implies

(a1) $\exists$ a complementary vector X^1 in $\mathcal{F}$ with $X^1 \backslash \{e, \overline{e}\} \geq 0, X^1_{\overline{e}} = -$ and $X^1_g = +$;

(a2) $\exists$ a complementary vector X^2 in $\mathcal{F}$ with $X^2 \backslash \{e, \overline{e}\} \geq 0, X^2_e = -$ and $X^2_g = +$;

(b1) $\exists$ a complementary vector Y^1 in $\mathcal{F}^*$ with $Y^1 \backslash \{e, \overline{e}\} \geq 0, Y^1_e = -$ and $Y^1_g = +$;

(b2) $\exists$ a complementary vector Y^2 in $\mathcal{F}^*$ with $Y^2 \backslash \{e, \overline{e}\} \geq 0, Y^2_{\overline{e}} = -$ and $Y^2_g = +$,

respectively. The induction hypothesis forces that two of the statements (a1), (a2), (b1) and (b2) must hold. But this is impossible from Lemma 2.5. Thus the one of two statements (a) and (b) must hold. It is clear that (a) and (b) cannot hold simultaneously, and the theorem follows. ∎

2.4 Basis, Tableau and the Duality Theorem

In the previous section, we have proved the duality theorem by an elementary inductive argument. The proof was very simple and, in fact, it gives some general idea of how the OMCP can be solved; once you solve two subproblems $M(e)$ and $M(\overline{e})$, then one of the two solutions will yield a solution to the original OMCP.

In this section, we will see how this simple idea can be extended to a finite algorithm. For this, we need the notion of basis, as this is essential in pivoting algorithms for the linear complementarity problem.

Let $M = (E, \mathcal{F})$ be an oriented matroid. A *basis* is a maximal subset B of E such that there is no nonzero vector X in $\mathcal{F}$ with $\underline{X} \subseteq B$. The following properties of basis are well-known:

Property 2.6. *Let $M = (E, \mathcal{F})$ be an oriented matroid.*

(1) *Every basis has the same cardinality, called the* rank $r(M)$ *of M;*

(2) *For each basis B and for each $j \in E \backslash B$, there exist a unique circuit, called the* fundamental circuit *$X(B, j) = X_M(B, j)$, such that $X(B, j)_j = +$ and $\underline{X(B, j)} \subseteq B \cup \{j\}$;*

(3) *A subset B of E is a basis of M if and only if $E \backslash B$ is a cobasis, i.e., a basis of the dual M^*.*

We will usually denote by N the cobasis $E \backslash B$, and denote by $Y(N,i)$ the fundamental cocircuit $X_{M^*}(N,i)$ for each $i \in B$

For each basis B and the cobasis $N = E \backslash B$, the *tableau* $T(B) = (t_{ij} : i \in B, j \in N)$ is the $\{+,0,-\}$-matrix defined by

$$t_{ij} \;=\; X(B,j)_i \text{ for } i \in B, j \in N. \tag{2.34}$$

Clearly, the tableau $T(B)$ contains not only all information about the fundamental circuits $X(B,j), j \in N$, but also contains that of the fundamental cocircuits $Y(N,i), i \in B$ through the following well-known relationship:

$$t_{ij} \;=\; -Y(N,i)_j \text{ for } i \in B, j \in N. \tag{2.35}$$

Therefore, the row vectors $(t_{ij} : j \in N), i \in B$ represent the essential part of the fundamental cocircuits with opposite sign, while the column vectors $(t_{ij} : i \in B), j \in N$ represent that of the fundamental circuits with same sign.

The following property is well-known and can be easily verified from the oriented matroid axioms (OM1)$\sim$(OM4) and the definitions.

Lemma 2.7. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Let B be a basis, and let $t_{rs}, r \in B, s \in N$ be any nonzero entry. Then the set $B' = B \backslash \{r\} \cup \{s\}$ is a basis, and the tableaux $T(B) = (t_{ij})$ and $T(B') = (t'_{ij})$ are related as follows: for each $i \in B \backslash \{r\}$ and $j \in N \backslash \{s\}$,*

$$\begin{aligned} t'_{sr} &= t_{rs}\,, \\ t'_{sj} &= -t_{rs} \cdot t_{rj}\,, \\ t'_{ir} &= t_{rs} \cdot t_{is}\,, \text{ and} \\ t'_{ij} &= t_{ij} \circ (-t_{rs} \cdot t_{is} \cdot t_{rj}) \quad \text{unless } t_{ij} = t_{rs} \cdot t_{is} \cdot t_{rj} \neq 0. \end{aligned}$$

The replacement of a tableau $T(B)$ by an "adjacent" tableau $T(B')$ is called a *pivot operation on the position* (r,s).

Now we come back to the complementarity problem. Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid with $n \geq 0$. For the complementarity problems for which we give an algorithm, we assume that

Assumption 2.8. *$M = (\hat{E}_{2n}, \mathcal{F})$ contains a basis B with $\mid B \mid = n$, $g \notin B$ and complementary, i.e., $B \cap \overline{B} = \emptyset$.*

This assumption is always satisfied for the case of LCP (1.3) where $\mathcal{F}$ arises from the linear subspace $V(A,b)$ in (2.1) the set $\{n+1, n+2, \ldots, 2n\}$ is a complementary basis.

We define here two types of basis. A basis B or tableau $T(B)$ is called *feasible* if $g \notin B$ and $X(B,g) \geq 0$, and called *cofeasible* if $g \notin B$, there exists $r \in B$ such that $t_{rg} < 0$, and $t_{rj} \leq 0$ for all $j \in N$, where $N = \hat{E}_{2n} \backslash B$. (See Figure 2.4.1 where the symbols $\oplus$ and $\ominus$ indicate a nonnegative entry and a nonpositive entry.)

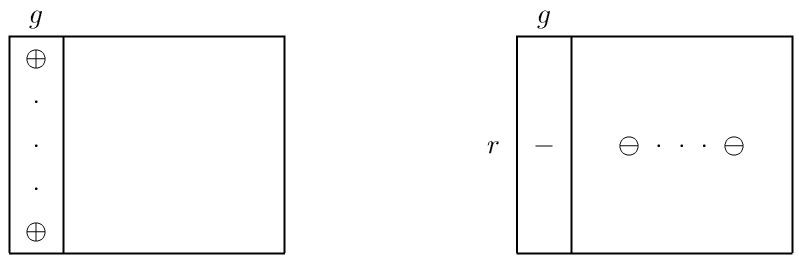


Figure 2.1: feasible and cofeasible tableaux

Property 2.9. *Let B be a basis with $g \notin B$, and let N be the cobasis $\hat{E}_{2n} \backslash B$.*

(1) *If B is feasible, then $X(B, g)$ is feasible, and if in addition complementary then $X(B, g)$ is complementary as well and solves the OMCP.*

(2) *If B is cofeasible, then $Y(N, r)$ is cofeasible, and if in addition B is complementary and $Y(N, r)_{\overline{r}} = t_{r\overline{r}} = 0$, then $Y(N, r)$ solves the dual OMCP, where r is a basic element for which $t_{rg} < 0$, and $t_{rj} \leq 0$ for all $j \in N$.*

Theorem 2.10 (Basis-Form Duality Theorem) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying Assumption 2.8 and the assumptions (2.28) and (2.29) of Theorem 2.2. Then exactly one of the following statements holds:*

(a) *there exists a complementary feasible basis;*

(b) *there exists a complementary cofeasible basis.*

A basis B or tableau $T(B)$ is called *terminal* if B is complementary and either feasible or cofeasible. We shall prove the theorem above using the following algorithm, called the criss-cross method, which will be shown to find a terminal tableau after a finite number of steps if the assumptions are satisfied.

In order to describe the algorithm, the following lemma is important.

Lemma 2.11. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the assumptions of Theorem 2.10. Let B be a complementary basis with $g \notin B$. Then, for each $r, s \in B$ the following statements hold:*

(1) $t_{rg} \neq 0$ *implies* $t_{r\overline{r}} \geq 0$*;*

(2) $t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{r}} = 0$ *and* $t_{r\overline{s}} = +$ *imply* $t_{sg} = -$ *and* $t_{s\overline{s}} = +$*;*

(3) $t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{s}} = 0$ *and* $t_{r\overline{s}} = +$ *imply* $t_{s\overline{r}} = -$.

Proof. For the proof, we set $X^0 = X(B, g)$, $X^1 = X(B, \overline{r})$, $X^2 = X(B, \overline{s})$, $Y^1 = Y(N, r)$, and $Y^2 = Y(N, s)$.

(1) Assume $t_{rg} \neq 0$. Let $t_{rg} \neq 0$ for some $r \in B$, and suppose $t_{r\overline{r}} = -$. Then X^1 is s.s.r. and $X_g^1 = 0$. On the other hand, the vector $Z = (t_{rg}X^0) \circ X^1 \in \mathcal{F}$ is s.s.p. and $Z_g \neq 0$. This contradicts to the assumption (2.28).

(2) Assume $t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{r}} = 0$ and $t_{r\overline{s}} = +$. Let $\gamma = t_{sg}$ and $\beta = t_{s\overline{s}}$. First we show $\beta = +$. Suppose $\beta \leq 0$. Then the vector $Z = X^2 \circ (-X^1) \in \mathcal{F}$ is s.s.r. with $Z_g = 0$, and the vector $W = X^0 \circ (-X^1) \in \mathcal{F}$ is s.s.p with $W_g \neq 0$. This contradicts to the assumption (2.28). Therefore $\beta = +$.

Now, we show $\gamma = -$. Suppose $\gamma \geq 0$. Clearly, the vector $W = Y^1 \circ (-Y^2) \in \mathcal{F}^*$ is s.s.p. with $W_g \neq 0$. It suffices to show that there is an s.s.r. vector, say Z in $\mathcal{F}^*$ with $Z_g = 0$, since this contradicts to the assumption (2.29). There are two cases to consider.

Case: $\gamma = 0$. In this case, the vector $Z = Y^2$ is an s.s.r. vector with $Z_g = 0$, and hence a contradiction.

Case: $\gamma = +$. In this case, by eliminating the g-component from the vectors Y^1 and Y^2 using the axiom (OM4), we have a vector $Z \in \mathcal{F}^*$, which is s.s.r. with $Z_g = 0$.

This completes the proof.

(3) Assume $t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{s}} = 0$ and $t_{r\overline{s}} = +$. Let $\delta = t_{s\overline{r}}$. We have to show $\delta = -$. Suppose $\delta \geq 0$. Then the vector $W = X^0 \circ (-X^1) \in \mathcal{F}$ is s.s.p. with $W_g \neq 0$ and the vector $Z = X^2 \circ (-X^1)$ is s.s.r. with $Z_g = 0$. A contradiction to the assumption (2.28).

∎

Figure 2.4.2 illustrates the meaning of the lemma.

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\overline{r}} \\ \cline{2-3} r & \bullet & \alpha \\ \cline{2-3} \end{array} \qquad \Longrightarrow \ \alpha = \oplus$$

(• indicates a nonzero entry)

$$\begin{array}{c|c|cc|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \overline{r} & \multicolumn{1}{c}{\overline{s}} \\ \cline{2-4} r & - & 0 & + \\ s & \gamma & 0 & \beta \\ \cline{2-4} \end{array} \qquad \Longrightarrow \ \gamma = -, \beta = +$$

$$\begin{array}{c|c|cc|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \overline{r} & \multicolumn{1}{c}{\overline{s}} \\ \cline{2-4} r & - & 0 & + \\ s & & \delta & 0 \\ \cline{2-4} \end{array} \qquad \Longrightarrow \ \delta = -$$

Figure 2.2: Sign Pattern of Complementary Tableaux

Criss-Cross method

The criss-cross method starts with any complementary basis B such that $g \notin B$. The algorithm assumes that the n complementary pairs $\{\{e, \overline{e}\} : e \in E_{2n}\}$ are linearly ordered, and this order induces a linear order on any complementary subset of E_{2n}. We denote by $\min S$ the smallest element in S, for any complementary subset S.

If it terminates then it outputs a pair (B, e) where B is a terminal basis, and $e = g$ if B is feasible, $e = r$ if B is detected cofeasible because of the r-row of $T(B)$.

```
procedure Criss-Cross(B:complementary basis);
repeat
    if B is feasible then
        STOP: return the pair (B; g)              {feas. compl. basis found }
    else
        r := min{i ∈ B | t_ig < 0};
        if t_{r ī} ≤ 0 for all i ∈ B then
            STOP: return the pair (B; r)          {cofeas. compl. basis found}
        else
            s := min{i ∈ B | t_{r ī} > 0};
            p := max{r, s};
            if t_{p p̄} ≠ 0 then
(*)             DIAGONAL PIVOT: B := B\{p} ∪ {p̄}
                                                  {B is again a basis, by Lemma 2.7}
            else
(**)            EXCHANGE PIVOTS: B := B\{r, s} ∪ {r̄, s̄}
                                                  {B is again a basis, by Lemma 2.12 }
            endif
        endif
    endif
until true=false.
```

Lemma 2.12. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (2.28) and (2.29) of Theorem 2.2. When the criss-cross method is applied to M, the exchange pivots (**) always produce a complementary basis.*

Proof. Let B be a complementary basis produced by the criss-cross method for which the exchange pivots (**) are executed. Let r, s, p be indices chosen for B by the algorithm. First consider the case $p = r$. Since (**) is executed, $r > s$, $t_{rg} = -$, $t_{r\overline{r}} = 0$ and $t_{r\overline{s}} = +$. Also, we have $t_{sg} \geq 0$. By Lemma 2.11 (2), $t_{s\overline{r}} \neq 0$. This together with Lemma 2.7 implies $B\backslash\{r, s\} \cup \{\overline{r}, \overline{s}\}$ is a basis again.

Now consider the case $p = s$. Since (**) is executed, $s > r$, $t_{rg} = -$, $t_{r\overline{s}} = +$ and $t_{s\overline{s}} = 0$. Also, by the description of the algorithm and Lemma 2.11 (1), $t_{r\overline{r}} = 0$. By Lemma 2.11 (3), we have $t_{s\overline{r}} = -$. This implies $B\backslash\{r, s\} \cup \{\overline{r}, \overline{s}\}$ is a basis again. ∎

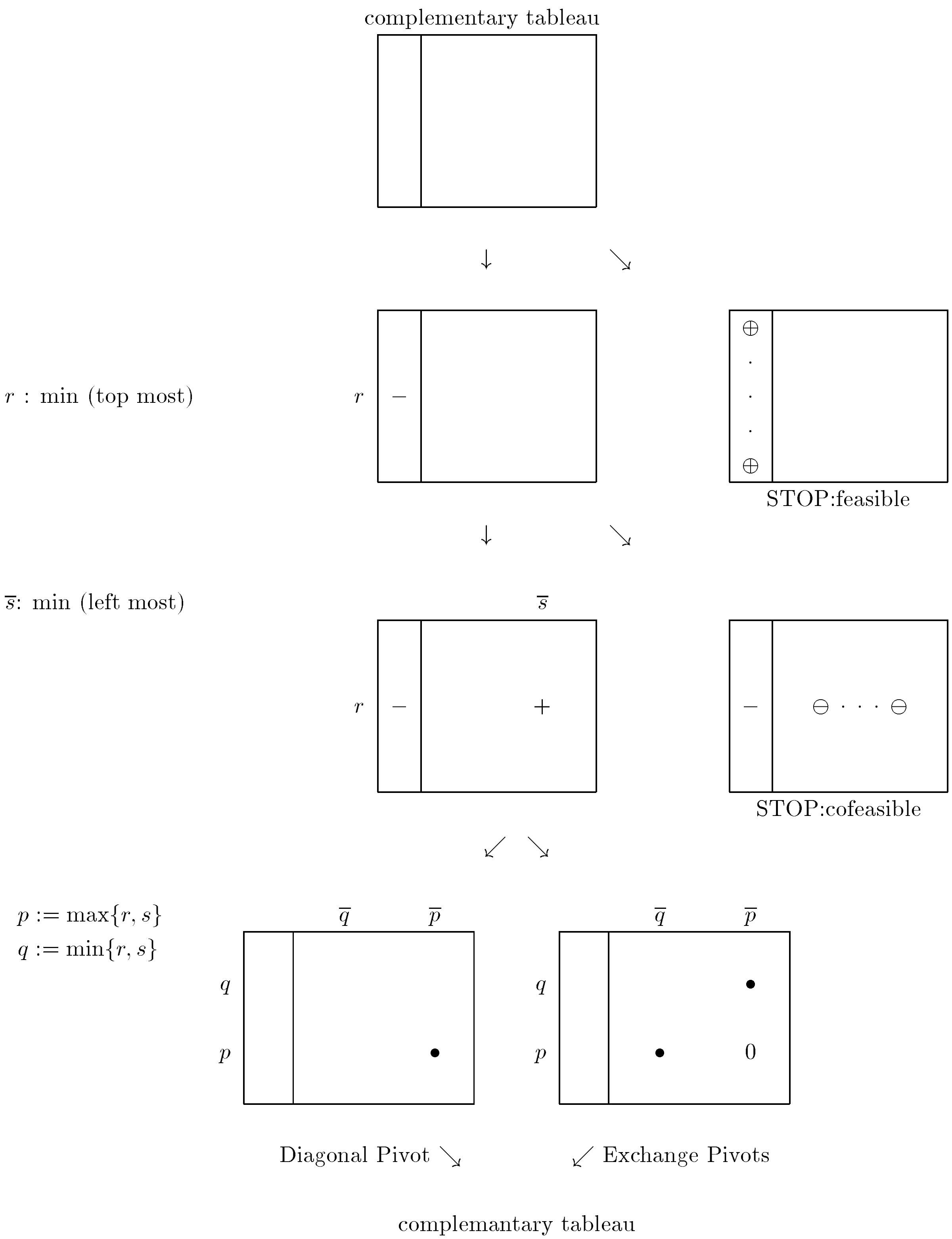


Figure 2.3: Criss-Cross Algorithm for the Complementarity Problem

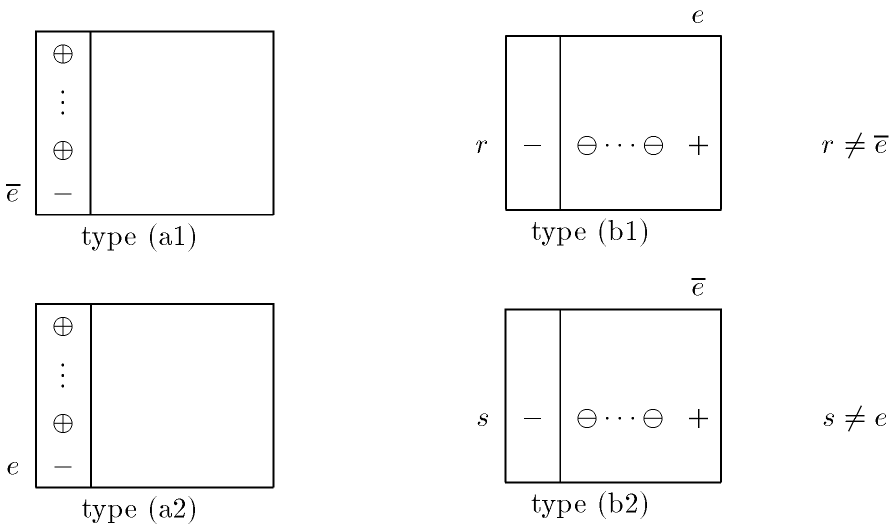


Figure 2.4: Four types of tableaux – Only one type can exist!

Lemma 2.13. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (2.28) and (2.29) of Theorem 2.2. Then for each $e \in E_{2n}$, at most one of the following four statements holds:*

(a1) *$\exists$ a complementary basis B^1 not containing g such that $\overline{e} \in B^1, t^1_{ig} \geq 0$ for all $i \in B^1 \backslash \{\overline{e}\}$ and $t^1_{\overline{e}g} = -$;*

(a2) *$\exists$ a complementary basis B^2 not containing g such that $e \in B^2, t^2_{ig} \geq 0$ for all $i \in B^2 \backslash \{e\}$ and $t^2_{eg} = -$;*

(b1) *$\exists$ a complementary basis B^3 not containing g such that $\overline{e} \in B^3, t^3_{rg} < 0, t^3_{r\overline{i}} \leq 0$ for all $i \in B^3 \backslash \{\overline{e}\}$ and $t^3_{re} = +$, for some $r \in B^3$ with $r \neq \overline{e}$;*

(b2) *$\exists$ a complementary basis B^4 not containing g such that $e \in B^4, t^4_{sg} < 0, t^4_{s\overline{i}} \leq 0$ for all $i \in B^4 \backslash \{e\}$ and $t^4_{s\overline{e}} = +$, for some $s \in B^4$ with $s \neq e$,*

where $T(B^k) = (t^k_{ij})$ for $k = 1, 2, 3, 4$.

Proof. Let $X^i = X(B^i, g)$ for $i = 1, 2$, $Y^1 = Y(N^3, r)$ and $Y^2 = Y(N^4, s)$, where $N^j = \hat{E}_{2n} \backslash B^j$. By Lemma 2.11 (1), $t^3_{r\overline{r}} = t^4_{s\overline{s}} = 0$, and therefore the dual vectors Y^1 and Y^2 are both complementary. It is easy to see that the vectors X^1, X^2, Y^1 and Y^2 satisfy the conditions in (a1), (a2), (b1) and (b2) of Lemma 2.5, respectively. By Lemma 2.5, at most one of the four vectors exists. Hence, at most one of the four bases B^1, B^2, B^3, B^4 exists. ∎

The four types of tableaux described in Lemma 2.13 are illustrated in Figure 2.4.4.

Theorem 2.14 (Finiteness of the Criss-Cross Method) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying Assumption 2.8 and the assumptions (2.28) and (2.29) of Theorem 2.2. Then the criss-cross method solves the OMCP associated with M, that is, it terminates after a* *finite number of pivots, and obtains either a complementary feasible basis or a complementary cofeasible basis.*

Proof. Suppose the criss-cross method does not terminate for some OMCP problem satisfying the conditions of the theorem, and let $M = (\hat{E}_{2n}, \mathcal{F})$ be such an example with minimal number of elements. Since there are finitely many bases in M, it must produce a cycle of adjacent bases: $B^0, B^1, \ldots, B^k = B^0$. Because of the minimality of number of elements, within this cycle every element in E_{2n} must leave and enter some basis. Let $\{e, \overline{e}\}$ be the largest complementary pair in E_{2n}. Consider the situation when e enters a basis. By the description of the algorithm, there are two possible cases:

Case E1: $r = \overline{e}$ and $t_{\overline{e}g}$ is the only negative entry in the g-column of the tableau;

Case E2: $r \neq \overline{e}$, $s = \overline{e}$ and t_{re} is the only positive entry in the r-row of the tableau.

When e leaves a basis, there are also two possible cases:

Case L1: $r = e$ and t_{eg} is the only negative entry in the g-column of the tableau;

Case L2: $r \neq e$, $s = e$ and $t_{r\overline{e}}$ is the only positive entry in the r-row of the tableau.

These four cases (E1), (E2), (L1), (L2) fit exactly the cases (a1), (b1), (a2), (b2) of Lemma 2.13, respectively. By the same lemma, no two of the four statements can occur, and no two of (E1), (E2), (L1) and (L2) can occur, a contradiction. Therefore, the criss-cross method must terminate. Clearly, when it terminates, the final basis is either a complementary feasible basis or a complementary cofeasible basis. ∎

The duality theorem, Theorem 2.10, is a straightforward consequence of Theorem 2.14.

It is interesting to note that in [Gra67] Grave gave a finite pivot method for LCPs with PSD-matrices, which is somewhat similar to our algorithm. The critical difference is that Grave's algorithm depends on the actual values of tableau entries and it assumes nondegeneracy.

2.5 Another Finiteness Proof

The criss-cross method we described in the previous section is merely one of finite algorithms for the OMCP (satisfying the conditions of Theorem 2.14). As for the special case of the Oriented Matroid Programming case, one can design different finite pivoting algorithms. In some sense, the criss-cross method is a most rigid algorithm among them, since each pivot operation is uniquely determined and a sequence of complementary bases generated is completely determined once a linear order of complementary pairs $\{e, \overline{e}\}$ is fixed.

One way to see the possibility of relaxing the algorithm is to prove the finiteness of the criss-cross method more constructively as in [FMa91]. In fact, we can show that some binary number associated with each iteration of the algorithm increases monotonically. For each $e \in E_{2n}$, we denote by $[e]$ the complementary pair $\{e, \overline{e}\}$. Clearly $[e] = [\overline{e}]$. Let $[E_{2n}] = \{[e] \mid e \in E_{2n}\}$. Thus $[E_{2n}]$ is the set E_{2n} with each complementary pairs e, $\overline{e}$ identified. As it was assumed for the criss-cross method, we assume that the n complementary pairs $\{\{e, \overline{e}\} : e \in E_{2n}\}$ are linearly ordered, and this order is just a linear order on $[E_{2n}]$. For a $\{0,1\}$-vector L in $\{0,1\}^{[E_{2n}]}$, and for $j \in [E_{2n}]$, let $L + j$ denote the vector in $\{0,1\}^{[E_{2n}]}$

$$(L+j)_i = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ L_i & \text{if } i > j \end{cases} \qquad (i \in [E_{2n}]).$$

We say L' is *lex greater* than L if $L \neq L'$ and L' is larger than L in the last subscript j for which they differ. Obviously, if $L_j = 0$ for some j, then $L + j$ is lex greater than L.

Let us consider the following updating of L at each iteration of the criss-cross method:

$$\begin{array}{l} \text{set L=0 initially, and} \\ \text{update L by L+p} \end{array} \tag{2.36}$$

where p is the larger element $\max\{r, s\}$ between the two basic elements r and s chosen at each iteration.

Theorem 2.15 (Monotonic Improvements) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying Assumption 2.8 and the assumptions (2.28) and (2.29) of Theorem 2.2. If the criss-cross method is applied to M with the vector L updated by (2.36), then L strictly increases at each iteration, and therefore the algorithm must terminate in $2^n - 1$ iterations.*

We omit the proof of this theorem. One just have to show $L_p = 0$ holds before being updated by $L + p$, and the fundamental lemma Lemma 2.12 is essential again as it was for the proof of Theorem 2.14.

One implication of the theorem is that as long as L increases at each iteration, the algorithm is finite. Consider L at a general stage of the criss-cross method. A *zero interval* of L is a set of consecutively ordered indices: $j_1, j_2, \ldots, j_k \in [E_{2n}]$ such that $L_{j_1} = L_{j_2} = \ldots = L_{j_k} = 0$. One can easily see that the linear order among the indices (complementary pairs) is not essential at this stage. So one can pretend that each zero intervals are not ordered, and one can just set any linear ordering on a zero interval whenever necessary.

For the case of linear programming, interesting experimental results supporting the practicality of some flexible version of the criss-cross method have been reported in [NM90].

3 Convex Polyhedra

3.1 Polyhedra and Faces

The intersection of a finite number of closed halfspaces in R^d is called a *convex polyhedron* or simply a *polyhedron*. Equivalently, a polyhedron is a subset P of R^d which can be represented as the solution set to a system of linear inequalities:

$$P = \{x \in R^d : Ax \leq b\}$$

where A is an $m \times d$ real matrix and b is a real m dimensional vector.

One can easily show that every polyhedron is *convex*, i.e., if x^1 and x^2 are points in a polyhedron P, then any *convex combination* of x^1 and x^2, $x = \lambda x^1 + (1-\lambda)x^2$ ($0 \leq \lambda \leq 1$), is also in P, or equivalently the whole line segment $[x^1, x^2]$ is contained in P.

There are natural substructures of a polyhedron, called faces, which are themselves polyhedra. The notion of faces plays an essential role in the combinatorial study of polyhedra. We shall now define faces and study some basic properties.

Let P be a polyhedron. For a real d-vector c and a real number d, a linear inequality $c^T x \leq \delta$ is called *valid* for P if $c^T x \leq \delta$ holds for all $x \in P$. A subset F of a polyhedron P is called a *face* of P if it is represented as

$$F = P \cap \{x : c^T x = \delta\}$$

for some valid inequality $c^T x \leq \delta$.

By this definition, the polyhedron itself and the empty set are faces of any polyhedron. But from the definition, it is not clear how many faces a polyhedron can have. Even it is not clear whether the number of faces of a polyhedron is finite. The first theorem states that it is in fact finite and the faces are generated by any system linear inequalities to represent the polytope.

A linear inequality system $Ax \leq b$ is said to *represent* P if $P = \{x : Ax \leq b\}$.

Theorem 3.1. *Let $Ax \leq b$ be a linear inequality system representing a polyhedron P. Then a nonempty subset F of P is a face of P if and only if F is represented as the set of solutions to an inequality system obtained from $Ax \leq b$ by setting some of inequalities to equalities, i.e.,*

$$F = \{x : A^1 x = b^1 \text{ and } A^2 x \leq b^2\},$$

where $A = \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$ *and* $b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$.

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let $Ax \leq b$ be any linear inequality system representing a polyhedron P, and let $c^T x \leq \delta$ be a valid inequality for P such that there exists a point $x \in P$ with $c^T x = \delta$. Define*

$$S = \{i \in \{1, 2, \ldots, m\} : x \in P \text{ and } c^T x = \delta \text{ imply } A_i x = b_i\}.$$

Then the following statements hold.

(a) *There exists a point $x' \in P$ such that $c^T x' = \delta$, $A_i x' = b_i$ for all $i \in S$, and $A_i x' < b_i$ for all $i \in \{1, 2, \ldots, m\} \setminus S$.*

(b) *There exists no point $x'' \in P$ such that $c^T x'' < \delta$ and $A_i x'' = b_i$ for all $i \in S$.*

Proof. Exercise. ∎

Proof of Theorem 3.1 Let $Ax \leq b$ be a linear inequality system representing a polyhedron P, and let F be a nonempty subset of P.

To prove the "only if" part, we assume that F is represented by $F = \{x : A^1 x = b^1 \text{ and } A^2 x \leq b^2\}$ where $A = \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$ and the vector $b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$. Define $c = \mathbf{1}^T A^1$ and $\delta = \mathbf{1}^T b^1$, where $\mathbf{1}$ denotes the vector of all 1's (of an appropriate size). Then one can easily verify that $F = P \cap \{x : c^T x = \delta\}$, implying F is a face of P.

For the "if" part, we assume F is a face of P and

$$F = P \cap \{x : c^T x = \delta\}$$

for a valid inequality $c^T x \leq \delta$. Let

$$S = \{i \in \{1, 2, \ldots, m\} : x \in P \text{ and } c^T x = \delta \text{ imply } A_i x = b_i\}.$$

By Lemma 3.2 (b), we have the equality

$$P \cap \{x : c^T x = \delta\} = P \cap \{x : A_i x = b_i \quad \forall i \in S\}.$$

This implies that the face F is the set of solutions to the inequality system obtained from $Ax \leq b$ by setting the inequalities $A_i x \leq b_i$, $i \in S$ to equalities. This completes the proof. ∎

3.2 Theorem of Minkowski-Weyl

By definition, every convex polyhedron is represented as a finite number of linear inequalities. The well-known theorem of Minkowski and Weyl states that a convex polyhedron has another "finite" representation.

In order to state the theorem, we need some definitions. For vectors $v_1, v_2, \ldots, v_k \in R^d$, we define $conv(v_1, v_2, \ldots, v_k)$ as their *convex hull*:

$$conv(v_1, v_2, \ldots, v_k) \quad = \quad \{x : x = \sum_{j=1}^{k} \lambda_j v_j \ , \ \sum_{j=1}^{k} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \ \forall j = 1, 2, \ldots, k\},$$

and define $nonneg(v_1, v_2, \ldots, v_k)$ as their *nonnegative hull*:

$$nonneg(v_1, v_2, \ldots, v_k) \quad = \quad \{x : x = \sum_{j=1}^{k} \lambda_j v_j \text{ and } \lambda_j \geq 0 \ \forall j = 1, 2, \ldots, k\}.$$

For two subsets P and Q of R^d, $P + Q$ denotes the *Minkowski sum* of P and Q:

$$P + Q \quad = \quad \{p + q : p \in P \text{ and } q \in Q\}.$$

Theorem 3.3 (Minkowski-Weyl's Theorem) : *For a subset P of R^d, the following statements are equivalent:*

(a) *P is a polyhedron, i.e., for some matrix A and vector b, $P = \{x : Ax \leq b\}$;*

(b) *There are vectors $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_s$ such that*
$P = conv(v_1, v_2, \ldots, v_n) + nonneg(r_1, r_2, \ldots, r_s)$.

This theorem suggests us to consider two special types of polyhedra, one of form $conv(v_1, v_2, \ldots, v_n)$ and the other of form $nonneg(r_1, r_2, \ldots, r_s)$.

The first one is widely known as polytope. A *convex polytope* or simply *polytope* is the convex hull $conv(v_1, v_2, \ldots, v_n)$ of a finite set $\{v_1, v_2, \ldots, v_n\}$of points in R^d. The Minkowski-Weyl theorem implies that the convex polytopes are exactly the bounded convex polyhedra.

The second one is very important for our study. A subset P of R^d is called *homogeneous cone* if it can be represented as the solution set to a system of homogeneous linear inequalities:

$$P = \{x : Ax \leq 0\}$$

where A is an $m \times d$ real matrix and 0 is the m dimensional zero vector. Clearly, the homogeneous cones are a special type of polyhedra, and in fact they are equivalent to the polyhedra of the second kind, as stated in the following variation of Minkowski-Weyl's theorem.

Theorem 3.4 (Minkowski-Weyl's Theorem for Homogeneous Cones) : *For a subset P of R^d, the following statements are equivalent:*

(a) *P is a homogeneous cone, i.e., for some matrix A, $P = \{x : Ax \leq 0\}$;*

(b) *There are vectors $r_1, r_2, \ldots, r_s$ such that $P = nonneg(r_1, r_2, \ldots, r_s)$.*

Although this second theorem, the cone theorem, might look to be a special case of the first theorem, one can easily derive the first theorem from the cone theorem (Exercise). Furthermore the cone theorem is more elegant and nicer to prove. Thus we will give a proof of Theorem 3.4.

The proof will be algorithmic. We will give two algorithms, one transforming a representation of form (a) to that of form (b) and the other transforming (b) to (a). The former algorithm is called Double Description Method [MRTT53] and the latter is called Fourier-Motzkin Elimination. In this sense, these algorithms are a sort of dual to each other. But one is not quite the inverse of the other.

It should be noted that both algorithms are not only of theoretical importance but of practical importance as well: they can be the most efficient algorithms available presently for a number of applications.

Double Description Method

Double Description Method [MRTT53], abbreviated by DD method, is a finite algorithm to produce a nonnegative hull representation of a homogeneous cone from its inequality system representation. We give a most primitive version of the algorithm first, and then explain how to obtain a much more efficient version of the algorithm for practical purposes.

Suppose that an $m \times d$ matrix A and a m-vector b are given, and let $P = \{x : Ax \leq 0\}$. The main idea of DD method is to represent the polyhedron P in *double description form* (abbreviated by DD form):

$$P = \{x : \; A_K\, x \leq 0, \tag{3.1}$$

$$x = \sum_{j \in J} \lambda_j r_j \text{ and } \lambda_j \geq 0 \; \forall j \in J\}, \tag{3.2}$$

where J is a nonempty (index) set, r_j is an d-vector for all $j \in J$, K is a subset of row indices $\{1, 2, \ldots, m\}$ of A, and A_K is a submatrix of A indexed by K. We denote by R the $d \times |J|$ matrix of column vectors r_j's. The algorithm starts with any such DD form (K, J, R), selects any index $h \in K$, transforms it to another DD form (K', J', R') with $K' = K - h$, and repeats. When $K = \emptyset$, we have a nonnegative hull description of P we seek for.

One simple (and perhaps most efficient) way to obtain an initial DD form of P is by selecting a maximal submatrix A_B of A consisting of linearly independent rows of A. Then we set K be the complement $\{1, 2, \ldots, m\} \setminus B$, and set $J = \{1, 2, \ldots, |B|\}$. The vectors r_j's are obtained by solving the system of equations:

$$A_B\, R \;=\; -I,$$

where I is the identity matrix of size $|B|$, R is a matrix of unknown column vectors r_j, $j \in J$. One can easily verify that this system has a solution. When $|B| = d$, the triple (K, J, R) represents a DD form for P (exercise). When $|B| < d$, the triple is not yet a DD form for P, but by adding few more vectors to R we obtain a DD form, namely a set of vectors to span the null space of A_B and their negatives (why? exercise).

Now it is left to explain how to transform a DD form with $|K|$ inequalities to a DD form with $|K| - 1$ inequalities to represent the same set P. (Here we ignore the nonnegativity inequalities $\lambda_j \geq 0$ from the counting.) One can easily see that it is enough to show how to transform a DD form with one inequality (i.e., $|K| = 1$) to a DD form without any inequality.

Lemma 3.5 (Main Lemma for Double Description Method) : *Let P be a subset of R^d represented by a DD form:*

$$P = \{x : \; a^T\, x \leq 0,$$

$$x = \sum_{j \in J} \lambda_j r_j \text{ and } \lambda_j \geq 0 \; \forall j \in J\},$$

for d-vectors a and r_j. Then the set P is represented by:

$$P = \{x : \; x = \sum_{j \in \hat{J}} \lambda_j r_j \text{ and } \lambda_j \geq 0 \; \forall j \in \hat{J}\}, \tag{3.3}$$

where

$$\begin{aligned} \hat{J} &= J^{+} \cup J^{0} \cup (J^{+} \times J^{-}) \\ J^{+} &= \{j \in J : a^T r_j < 0\} \\ J^{0} &= \{j \in J : a^T r_j = 0\} \\ J^{-} &= \{j \in J : a^T r_j > 0\} \end{aligned}$$

and

$$r_{jj'} = -(a^T r_{j'}) r_j + (a^T r_j) r_{j'} \quad \textit{for each } (j, j') \in J^{+} \times J^{-}.$$

Proof. Let the RHS of equality (3.3) be $\hat{P}$. Since the inclusion $\hat{P} \subset P$ is obvious, it is sufficient to show $P \subset \hat{P}$. We leave this proof for exercise. ∎

Here we write the DD method in procedural form:

> **procedure** DoubleDescriptionMethod(A);
> **begin**
>   Obtain any initial DD form (K, J, R) for $P = \{x : Ax \leq 0\}$
>   **while** $K \neq \emptyset$ do
>   **begin**
>   Let h be any index in K;
>   Obtain a DD form (K', J', R') of P with $K' = K - h$;
>   /* by using Lemma 3.5 */
>   replace (K, J, R) with (K', J', R')
>   **end**;
>   Output R
> **end**

It is now clear that DD method produces a nonnegative hull representation of any homogeneous cone $\{x : Ax \leq 0\}$ in finite steps, and this completes a half of proof for Theorem 3.4.

The DD method given here is very primitive, and the straightforward implementation will be quite useless, because the size of J increases very fast and goes beyond any tractable limit. One reason for this is that many (perhaps, most) of the vectors $r_{jj'}$ the algorithm generates (defined in Lemma 3.5), are unnecessary. For a practical implementation, one has to keep at least two things in mind: (1) The algorithm should not create any redundant vectors $r_{jj'}$ at each stage, and (2) Some good heuristics must be used for selecting an inequality index h at each stage to minimize the growth of $|J|$. The first point can be nicely taken care of: if the initial DD form is minimal and if a new vector $r_{jj'}$ is created if and only if the vectors r_j and $r_{j'}$ define "adjacent rays" of the cone $\{x : A_i x \leq 0 \text{ for all } i \in \{1, 2, \ldots, m\} \setminus K\}$, then the algorithm won't generate any redundant vectors. The adjacency checking can be correctly done combinatorially. The second point is difficult to explain, because we have no theoretical argument which shows one way of selecting h is more efficient than another. Yet, our extensive experiences indicate that a selection rule can make an enormous and critical difference in how the size of J increases. One successful way to order the rows A is simply the lexicographic ordering. Another good way is to select dynamically a row index h such that the size of J^- is maximum.

An pascal code for the DD method has been developed by the author [Fu93]. This code was used successfully in solving relatively large problems, see [CGAF93]. A new project to implement the DD method and the reverse search algorithm, a new algorithm for the same purpose, is in progress [AF93].

The Fourier-Motzkin Elimination

In order to complete the proof for Theorem 3.4, we must show that the nonnegative hull of a finite set of vectors can be represented as $\{x : Ax \leq 0\}$.

We assume that a set P is given as the nonnegative hull of a finite set of generators $r_1, r_2, \ldots, r_s$:

$$\begin{aligned} P &= nonneg(r_1, r_2, \ldots, r_s) \\ &= \{x : x = \sum_{j=1}^{s} \lambda_j r_j \text{ and } \lambda_j \geq 0 \; \forall j = 1, 2, \ldots, s\}. \end{aligned}$$

In order to transform this representation to a linear inequality system representation, we simply have to reduce the system

$$x = \sum_{j=1}^{s} \lambda_j r_j \text{ and } \lambda_j \geq 0 \; \forall j = 1, 2, \ldots, s \tag{3.4}$$

to an equivalent system of linear inequalities by eliminating the variables λ_j, $j = 1, 2, \ldots, s$. More precisely, we need to reduce the system to what we call an x-equivalent system.

Here, two systems $S_1(\lambda, x)$ and $S_2(x)$ of linear inequalities with the variable vectors (λ, x) and x, respectively are said to be *x-equivalent* if

(a) If (λ, x) is a solution to S_1, then x is a solution to S_2.

(b) If x is a solution to S_2 then there is λ such that (λ, x) is a solution to S_1; and

Geometrically, if two systems $S_1(\lambda, x)$ and $S_2(x)$ of linear inequalities are x-equivalent, then the set of solutions to S_2 is exactly the projection of the set of solutions of S_1 onto the space of x variables. Thus, the Fourier-Motzkin algorithm we shall describe can be interpreted geometrically.

Before we describe the algorithm, it is better to do some "cosmetic" preprocessing to the system 3.4. Since there are d equalities in the system, we can first eliminate some d of λ_j's by simple substitutions to obtain an equivalent system of inequalities without any equalities. This means that what we need is to have an algorithm transforming a system of form:

$$Ax \leq 0$$

to an equivalent system with one of any prescribed variable x_j deleted. The purpose of the Fourier-Motzkin elimination is exactly this.

Lemma 3.6 (Main Lemma for the Fourier-Motzkin Elimination) : *Let A' be a real $m \times (d+1)$ matrix :*

$$\begin{bmatrix} a_{10} & A_1 \\ a_{20} & A_2 \\ \vdots & \vdots \\ a_{m0} & A_m \end{bmatrix},$$

and let $S_1(x_0, x = (x_1, \dots, x_d))$ be the system of linear inequalities:

$$A' \begin{bmatrix} x_0 \\ x \end{bmatrix} \le 0$$

Then the system $S_1(x_0, x)$ is x-equivalent to $S_2(x)$ given by:

$$\hat{A}x \le 0.$$

where $\hat{A}$ is the matrix of row vectors A_i $(i \in \hat{E})$,

$$\begin{aligned} \hat{E} &= E^0 \cup (E^+ \times E^-) \\ E^+ &= \{i \in \{1, 2, \dots, m\} : a_{i0} > 0\} \\ E^0 &= \{i \in \{1, 2, \dots, m\} : a_{i0} = 0\} \\ E^- &= \{i \in \{1, 2, \dots, m\} : a_{i0} < 0\}, \end{aligned}$$

and

$$A_{(i,i')} = \frac{A_i}{a_{i0}} + \frac{A_{i'}}{-a_{i'0}} \quad \text{for each } (i, i') \in E^+ \times E^-.$$

Proof. The (a) part of x-equivalence is obvious from the construction. To verify the (b) part, let $\overline{x}$ be a solution to S_2, and then we must show the existence of $\overline{x}_0$ such that $(\overline{x}_0, \overline{x})$ is a solution to S_1. This part is left for exercise. ∎

3.3 Face Lattice of Polytopes

Let P be a polytope in R^d. The set of all faces of P, denoted by $Faces(P)$, ordered by set inclusion is called the *face lattice* of the polytope. Clearly the least element and the greatest element of $Faces(P)$ are $\emptyset$ and P, respectively. This set is finite by Theorem 3.1. A trivial bound for the cardinality of $Faces(P)$ is 2^m where m is the number of inequalities to represent P. A large part of convex polytope theory has been devoted to study the set $Faces(P)$, and there are just too many interesting results to mention.

In this section, we state some of the fundamental results on $Faces(P)$ without proofs. Many of the results given here can be proved nicely in broader context of oriented matroid framework, but to present these proofs will go beyond the scope of this lecture. (This lecture is intended to be neither complete nor self-contained.)

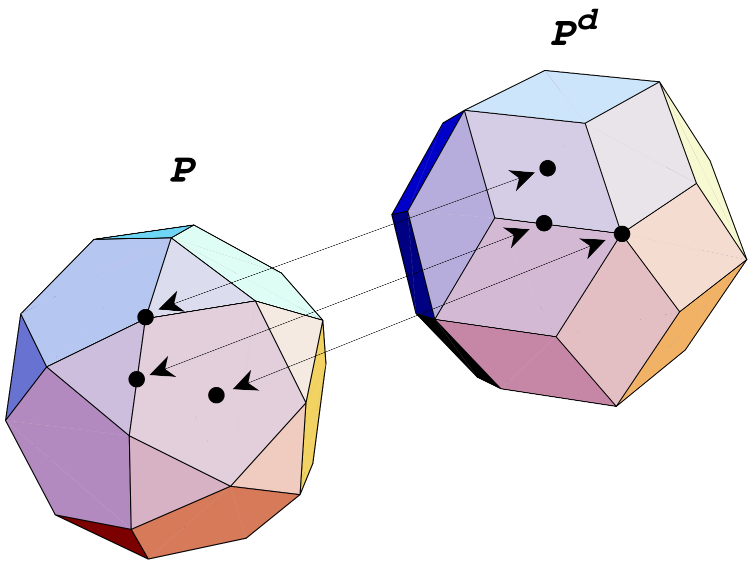
Proposition 3.7. *Let P be a convex polytope. Then its face lattice $Faces(P)$ satisfies Jordan-Dedekind chain property, i.e., every maximal chain: $F = F_1 \subset F_2 \subset \cdots \subset F_k = F'$ between any two ordered elements $F \subset F'$ in $Faces(P)$ has the same length.*

This proposition is a corollary of a more general statement in oriented matroids, Proposition 5.2. This property implies that the *dimension* of each face F can be defined combinatorially. The *dimension* $dim(F)$ of a face F in a polytope P is one less than the length of any maximal chain from the smallest face to F. Consequently, the empty face has dimension -1. The dimension of a polytope P is the dimension of P as a face. A face of dimension i will be called i-face, and a polytope of dimension i is called a i-polytope. We shall denote by i-$Faces(P)$ the set of all i-faces of P.

The standard definition of dimension of a face F is the maximum number of affine independent vectors in F. This is not convenient for our purpose because this cannot be extended to abstract combinatorial polytopes.

There are convenient names for faces of particular dimensions. The faces of dimension 0 are called the *vertices*, the faces of dimension 1 are called the *edges*, and the faces of dimension $dim(P) - 1$ are called the *facets*.

Theorem 3.8 (Polarity of Polytopes) : *For any polytope P, there exists a polytope P^d whose face lattice is isomorphic to the polar of the face lattice of P.*



Any polytope P^d satisfying the property of the theorem above is called a *dual polytope* of P. There is a natural construction of a dual polytope. Let P be a d-dimensional polytope in R^d, and assume that P contains the origin 0 in its interior. (If not, one can translate P to satisfy this property.) Then it can be proved that the following set P^*, called the *polar* of P,

$$P^* = \{y \in R^d : x^T\ y \leq 1 \text{ for all } x \in P\}$$

is a polytope, and a dual polytope of P. In fact, by using the Minkowski-Weyl's Theorem, Theorem 3.3, we know that the polytope P can be represented by the convex hull of a finite set of points as: $P = conv(v_1, v_2, \ldots, v_n)$. Then, it is easy to see that P^* is a polytope because it has a finite description, namely,

$$P^* = \{y \in R^d : v_i^T\ y \leq 1 \text{ for all } i = 1, 2, \ldots, n\}.$$

What remained to be proved is the existence of a one-to-one correspondence between $Faces(P)$ and $Faces(P^*)$ which reverses the set inclusion. We leave this as an exercise for the reader.

For a d-polytope P, we denote by $f_i(P)$ the number of i-faces of P, for each $i = -1, 0, 1, \ldots, d$, i.e., $f_i(P) = |i\text{-}Faces(P)|$. The f-vector $f(P)$ of a d-polytope P is $(f_0(P), f_1(P), \ldots, f_{d-1}(P))$.

The following is a classical theorem:

Theorem 3.9 (The Euler-Poincaré Relation) : *For any d-polytope P, the following equality holds:*

$$\sum_{i=-1}^{d} (-1)^i f_i(P) = 0.$$

It is interesting to read the book of Grünbaum [Gru72] (page 141) for the history of wrong proofs of the theorem. This can be considered as a serious warning that intuitive statements on higher dimensional geometry might be often incorrect or not so obvious. In order to prove the Euler-Poincaré Theorem correctly, the axiomatic framework of oriented matroids has been shown to be useful, see [CLM82]. This theorem is also a direct consequence of the shellability theorem of convex polytopes, which was first stated by L. Sshläfli in 1852 and proved only recently by [BM71]. The shallability theorem has been generalized in oriented matroids by Edmonds-Fukuda-Mandel, see [Fu82, Ma82].

Another fundamental result on f-vectors of polytopes is the upper bound theorem. Consider the moment curve in R^d given by $x(t) = (t, t^2, t^3, \ldots, t^d)$. For n real numbers, $t_1 < t_2 < t_3 < \cdots < t_n$ with $n > d$, we define $C(n, d)$ be the convex hull of the n points $\{x(t_1), x(t_2), \ldots, x(t_n)\}$. This polytope, known to be the *cyclic polytope*, has been shown to have the largest number of i-faces among all d-polytopes with n vertices. This theorem, known to be the upper bound theorem for convex polytopes, was first formulated as a conjecture by Motzkin [Mo53] and proved by McMullen [Mc70].

Theorem 3.10 (Upper Bound Theorem) : *For any d-polytope P with n-vertices, the following inequality holds for each $-1 \leq i \leq d$:*

$$f_i(P) \leq f_i(C(n, d)).$$

For each i, the number $f_i(C(n,d))$ can be written as a function of n and d explicitly (see, e.g. [MS71]). Among them, the most important one is for $i = d-1$,

$$f_{d-1}(C(n,d)) = \binom{n - \lfloor (d+1)/2 \rfloor}{n-d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n-d},$$

which is the maximum number of facets in a d-polytope with n-vertices. Using the polarity, Theorem 3.8, this number is also the maximal number of vertices in a d-polytope with n-facets.

This number is of course an exponential function of n and d. To see how fast it can grow, we can use a symbolic computational system such as *Mathematica* to compute this number exactly. Here is the table of $f_{d-1}(C(n,d))$ for some "small" polytopes.

$n \setminus d$	1	2	3	4	5	6	7	8	9	10
10	2	10	16	35	42	50	40	25	10	–
20	2	20	36	170	272	800	1120	2275	2730	4004
30	2	30	56	405	702	3250	5200	17250	25300	63756
100	2	100	196	4850	9312	152000	285760	3460375	6367090	60990020

Finally we mention a couple of nice results on the graph structure of the polytope.

Proposition 3.11 (Diamond Property) : *Let P be a polytope. For any two ordered faces $F \subset F'$ of P with $dim(P') = dim(P) + 2$, there are exactly two faces of P between F and F'.*

This proposition implies that in any polytope P each edge contains exactly two vertices. This defines the graph $G(P)$ of a polytope P. The following theorem of Balinski [Ba61] is well known.

Theorem 3.12 (d-Connectivity) : *The graph $G(P)$ of a d-polytope P is d-connected.*

A d- polytope P is called *simple* if each vertex of P is adjacent to exactly d-vertices, and *simplicial* if its dual polytope P is simple.

Blind and Mani [BM71] has recently shown that the graph structure of a simple polytope contains the complete information of its combinatorial structure.

Theorem 3.13. *The face lattice of any simple d-polytope is uniquely determined by its graph $G(P)$.*

See also Kalai [K88] for a simple and elegant proof for this theorem. Unfortunately, the proof does not yield any efficient way to reconstruct the face lattice of a simple polytope from its graph. The complexity of the problem is still unknown and it remains to be a challenging open problem.

4 Arrangements

4.1 Arrangements of Hyperplanes and Spheres

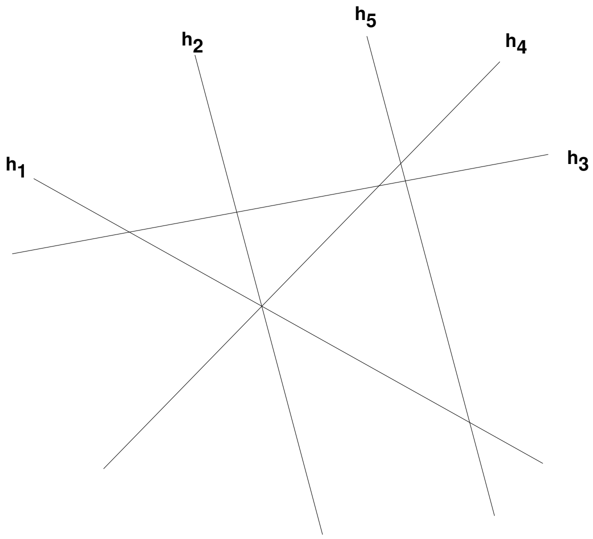
Arrangemet of Hyperplanes

A finite family $\mathcal{A} = \{h_i : i = 1, 2, \ldots, m\}$ of hyperplanes in R^d is called an *arrangement of hyperplanes*. To eliminate nonessential generalities, we always assume the following regularity assumption.

Assumption 4.1. *There is a subset of $\mathcal{A}$ whose intersection is a single point.*

If this assumption is not satisfied, the intersection of all hyperplanes is an affine space of dimension at least 1, and there is a hyperplane orthogonal to this affine space which can serve as the ground Euclidean space R^d for smaller d. The combinatorial structure of such nonregular arrangement, the notion to be defined precisely later in this section, will be isomorphic to that of a lower dimensional regular arrangement.

The following drawing illustrates an arrangement of hyperplanes in R^2, which satisfies the assumption.



A representation of an arrangement $\mathcal{A}$ is the pair (A, b) where A is a real $m \times d$ matrix, b is a real m-vector and $h_i = \{x : A_i\, x = b_i\}$ for $i = 1, 2, \ldots, m$. The arrangement represented by (A, b) is denoted by $\mathcal{A}(A, b)$.

Let $\mathcal{A}$ be an arrangement of hyperplanes. Associated with each hyperplane h_i in the arrangement, there are two open halfspaces bounded by the hyperplane, which will be denoted by h_i^+ and h_i^-. We call h_i^+ (h_i^-) the *plus side* (*minus side*) of the hyperplane. Here it is not important which side is the plus side. The only importance is to fix the assignment of $+$ and $-$. Accordingly, we shall use h_i^0 for the hyperplane itself h_i, which can be called the *zero side* of the hyperplane. When the arrangement is represented by (A, b), there is a canonical way to assign the sides as

$$h_i^+ = \{x : A_i\, x < b_i\}, \qquad h_i^- = \{x : A_i\, x > b_i\}$$

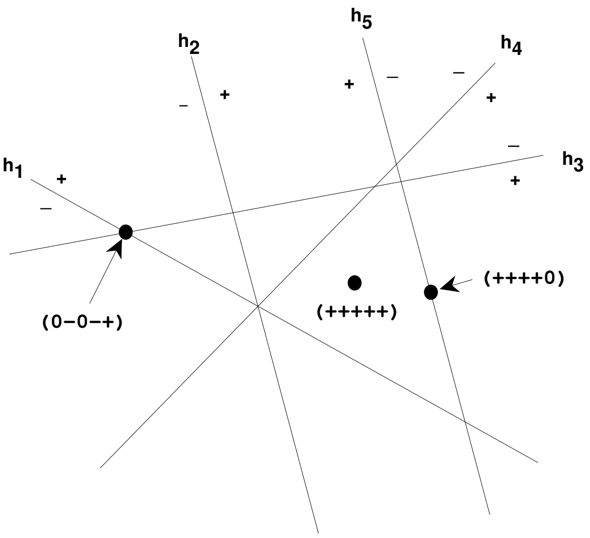
For each point $x \in R^d$, we shall associate with the sign vector $\sigma(x)$ on $\{1, 2, \ldots, m\}$ defined by

$$\sigma(x)_i = \begin{cases} + & \text{if } x \in h_i^+ \\ 0 & \text{if } x \in h_i^0 \\ - & \text{if } x \in h_i^- \end{cases} \qquad i \in \{1, 2, \ldots, m\}.$$

Since the vector $\sigma(x)$ encodes the position of a point x with respect to each hyperplanes, it is called the *position vector* of x. We denote by $\mathcal{F}(\mathcal{A})$ the set of all position vectors, i.e.,

$$\mathcal{F}(\mathcal{A}) = \{\sigma(x) : x \in R^d\}.$$

A *face* of the arrangement is a subset F of R^d which can be represented as $F = \cap_{i=1}^m h_i^{Y_i}$ for some sign vector $Y \in \mathcal{F}(\mathcal{A})$. It is important to note that the definition of faces does not depend on any particular assignment of "+" and "−" to the halfspaces of each hyperplane.

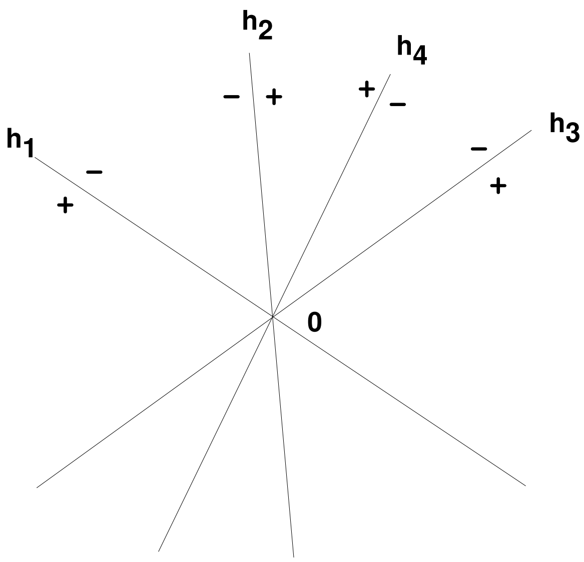


The set $Faces(\mathcal{A})$ of all faces of $\mathcal{A}$ is denoted by $Faces(\mathcal{A})$ and called the complex of the arrangement. The complex $Faces(\mathcal{A})$ with its elements ordered by set inclusion is defined to be the *combinatorial structure* of the arrangement.

What is the relation between the set $\mathcal{F}(\mathcal{A})$ and $Faces(\mathcal{A})$? Even though the set $\mathcal{F}(\mathcal{A})$ contains only finite information as opposed to infinite information of the complex $Faces(\mathcal{A})$, it has the complete information about the combinatorial structure of $\mathcal{A}$. The complex $Faces(\mathcal{A})$ ordered by set inclusion is isomorphic to the set $\mathcal{F}(\mathcal{A})$ ordered by *facial incidence relation* on sign vectors: for two sign vectors Y and Y' in $\{+, 0, -\}^m$, Y is said to be a *face* of Y' (denoted by $Y \preceq Y'$) if $Y_i \neq 0$ implies $Y_i = Y'_i$.

Central Arrangement of Hyperplanes and Arrangement of Spheres

An arrangement of hyperplanes in which all its hyperplanes contain the origin 0 is called a *central arrangement of hyperplanes*. By Assumption 4.1, this condition is equivalent to $\cap_{i=1}^m h_i = \{0\}$.



A central arrangement is a special type of hyperplane arrangements. However, any arrangement of hyperplanes can be considered as the intersection of a central arrangement with another hyperplane. Furthermore, central arrangements are much easier to deal with mathematically because the faces are all homogeneous cones, unlike noncentral arrangements where two different types of faces exist, bounded and unbounded. Furthermore, the set of position vectors of any central arrangement $\mathcal{A}$:

$$\mathcal{F}(\mathcal{A}) = \{\sigma(x) : x \in R^d\}$$

satisfies the axiom of oriented matroids, suggesting a new axiomatic way to study the combinatorics of central arrangements. To see this, let $(A, 0)$ be a representation of a central arrangement $\mathcal{A}$, and let V be the column space of A, i.e.,

$$V = \{y \in R^m : y = Ax, x \in R^d\}$$

Clearly V is a vector subspace of R^m. Now let $\delta(V)$ be the set of the sign vectors of vectors in V. Recall that the sign vector $\delta(y)$ of a real m-vector y is simply the vector in $\{+, 0, -\}^m$ with $\delta(y)_j = sign(y_j)$. Observing that $\sigma(x) = \delta(Ax)$ for $x \in R^d$, we have

$$\mathcal{F}(\mathcal{A}) = \delta(V).$$

This implies that the set $\mathcal{F}(\mathcal{A})$ of all sign vectors is an oriented matroid.

Therefore, any general statement on oriented matroids can be interpreted as a general statement on the combinatorial structure of central arrangements. Furthermore, it appears that the axioms of oriented matroids captures the essence of the combinatorial structure of arrangements. This fact will be verified in the next two sections, Section 5 and 6

Finally, we define an *arrangement of spheres* on the unit d-spheres S^d (in R^{d+1}) is a set $\mathcal{A} = \{S_i : i = 1, 2, \ldots, m\}$ of unit $(d-1)$-spheres on S^d. An arrangement of spheres can be considered as the cut section of a central arrangement in R^{d+1} with the unit sphere S^d. A *representation* of an arrangement of spheres is simply a representation $(A, 0)$ of the associated central arrangement. With a representation $(A, 0)$, each sphere S_i is merely the sets

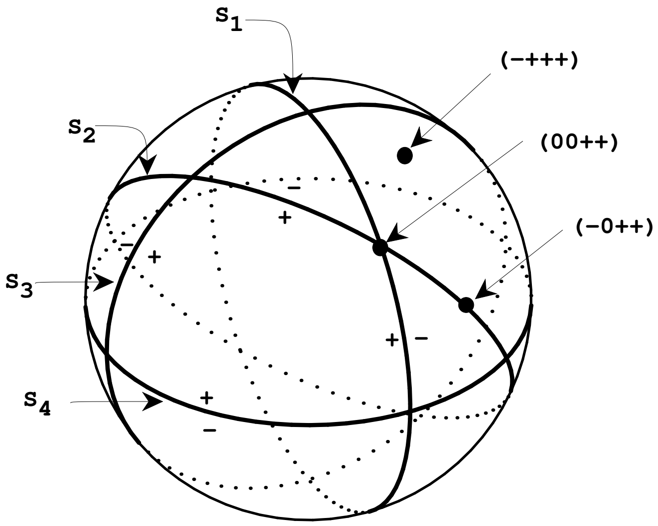
$$S_i = \{x : A_i\, x = 0\} \cap S^d$$

for each $i = 1, 2, \ldots, m$.

The regularity assumption for arrangements of spheres, corresponding to Assumption 4.1, is

Assumption 4.2. *The intersection of all spheres S_i in $\mathcal{A}$ is empty.*

The faces of an arrangement of spheres are defined accordingly, i.e., a subset F of S^d is a face if it is the intersection of S^d and some face of the corresponding central arrangement. Thus, as combinatorial objects, an arrangement of spheres is equivalent to some central arrangement.



There are two reasons that we prefer arrangements of spheres to central arrangements of hyperplanes. The first reason is practical. A central arrangement of hyperplanes in R^3 is very difficult to visualize, but the corresponding arrangement of spheres in S^2 can be visualized easily as we see above. The second reason is more important: as we see in the Section 5.1, every oriented matroid can be represented by an arrangement of topological spheres, which is a generalization of arrangements of spheres. Therefore, arrangements of spheres are a more natural structure to deal with in the study of oriented matroids. We shall define this general structure later.

4.2 Polyhedral Realization of Arrangements

In this section we show that every arrangement of spheres (and hence every central arrangement of hyperplanes) is combinatorially equivalent to some convex polytope.

The following figure is a simple example to illustrate this claim.

Let $\mathcal{A}$ be an arrangement of spheres $\{S_i : i = 1, 2, \ldots, m\}$ in S^d. Since the face poset $\mathcal{F}(\mathcal{A})$ of the arrangement does not contain the greatest element, it is convenient to use $\hat{\mathcal{F}}(\mathcal{A})$ to denote the set $\mathcal{F}(\mathcal{A})$ with the artificial largest element *mbone* added. We shall call this set $\hat{\mathcal{F}}(\mathcal{A})$ the *face lattice* of the arrangement A.

Now suppose that the arrangement $\mathcal{A}$ has a representation $(A, 0)$, and $S_i = \{x : A_i\, x = 0\}$. Consider the following polytope:

$$P_A = \{x : y\ A\ x \leq 1, \forall\ y \in \{-1, +1\}^m\}$$

Theorem 4.3. *The face lattice $\hat{\mathcal{F}}(\mathcal{A})$ is isomorphic to the face lattice of the polytope P_A.*

Observing that the face lattice of every d-dimensional polytope in R^d is isomorphic to a substructure (lower ideal) of the face lattice of an arrangement of spheres in S^d, the polyhedral representation of an arrangement is rather surprising. Each of two structures are "specializations" of the other. This two way relationship is important in study of these fundamental structures. One simple application of the theorem is that every combinatorial theorem on general convex polytopes are true for arrangements.

We call an inequality $cx \leq \delta$ *facet-defining* for a polytope P if the set $P \cap \{x : cx \leq \delta\}$ is a facet of P_A. The key lemma for proving the theorem above is:

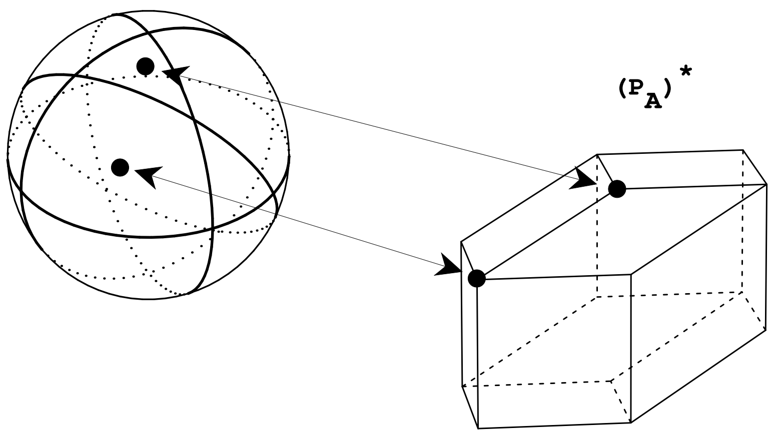
Lemma 4.4. *Let y be any vector in $\{-1, +1\}^m$. Then, the sign vector $\delta(y)$ of y is in $\mathcal{F}(\mathcal{A})$ if and only if the inequality $y\ A\ x \leq 1$ is facet-defining for P_A.*

The polar of the polytope P_A is the polytope

$$\begin{aligned} (P_A)^* &= conv(\{yA \in R^d : y \in \{-1, +1\}^m\}) \\ &= \{yA \in R^d : y \in [-1, +1]^m\} \\ &= L_1 + L_2 + \cdots + L_m, \end{aligned}$$

where L_i is the line segment $[-A_i, A_i]$. This set, which is the Minkowski sum of a finite set of line segments, is known to be a *zonotope*. By the polarity of polytopes, Theorem 3.8, we have the following correspondance:

Lemma 4.5. *Let y be any vector in $\{-1,+1\}^m$. Then, the sign vector $\delta(y)$ of y is in $\mathcal{F}(\mathcal{A})$ if and only if the vector yA is a vertex of the zonotope $(P_A)^*$.*



5 OM Lattices and OM Polyhedra

Let us recall the definition of oriented matroids.

An *oriented matroid on* E is a pair $M = (E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a set of sign vectors on E satisfying the following axioms:

(OM1) $\mathbf{0} \in \mathcal{F}$; (where $\mathbf{0}$ denotes the zero vector)
(OM2) $Y \in \mathcal{F} \Longrightarrow -Y \in \mathcal{F}$;
(OM3) $Y, Y' \in \mathcal{F} \Longrightarrow Y \circ Y' \in \mathcal{F}$;
(OM4) $Y, Y' \in \mathcal{F}$ and $f \in D(Y, Y') \Longrightarrow$ there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z_j = (Y \circ Y')_j$ for all $j \notin D(Y, Y')$.

Here we call each member of $\mathcal{F}$ a *face* of the oriented matroid. The faces are sometimes called *covectors*.

It is also important to recall that for any linear subspace V of R^E, the set $\mathcal{F} = \delta(V)$ of all sign vectors coming from V satisfies the axioms (OM1)$\sim$(OM4). Such an oriented matroid is called *representable* or *linear*.

As we remarked in Section 4.1, the set $\mathcal{F}(\mathcal{A})$ of positions vectors of any central arrangement of hyperplanes satisfies the axioms as well.

Now we introduce a partial order among the faces of an oriented matroid. For two sign vectors Y and Y' in $\{+, 0, -\}^E$, Y is said to be a *face* of Y' (denoted by $Y \preceq Y'$) if $i \in E$ and $Y_i \neq 0$ imply $Y_i = Y'_i$. This partial order is a direct abstraction of the facial incidence relation in arrangement of hyperplanes and spheres. For a set of sign vectors $\mathcal{S}$, the set of all minimal (maximal) members of $\mathcal{S}$ is denoted by $Min\mathcal{S}$ ($Max\mathcal{S}$).

For an oriented matroid $M = (E, \mathcal{F})$, we always consider the set of faces $\mathcal{F}$ as a poset ordered by facial relation $\preceq$. The maximal faces of M are called the *topes* and the nonzero minimal faces are called the *vertices*. The set of topes and the set of vertices are denoted by $\mathcal{T}(M)$ and $\mathcal{V}(M)$, respectively, i.e.,

$$\begin{aligned} \mathcal{T}(M) &= Max\mathcal{F} \\ \mathcal{V}(M) &= Min\mathcal{F} \backslash \{\mathbf{0}\}. \end{aligned}$$

Proposition 5.1. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Then the following statements hold:*

(a) *The set $\mathcal{T} = \mathcal{T}(M)$ uniquely determines the oriented matroid by*

$$\mathcal{F} = \{Y \in \{+, 0, -\}^E : Y \circ T \in \mathcal{T} \ \forall T \in \mathcal{T}\}.$$

(b) *The set $\mathcal{V} = \mathcal{V}(M)$ uniquely determines the oriented matroid by*

$$\mathcal{F} = \{Y \in \{+, 0, -\}^E : Y = V_1 \circ V_2 \circ \cdots \circ V_k,\ V_i \in \mathcal{V} \ \forall i = 1, 2, \ldots, k \ \textit{and}\ 1 \leq k \leq |E|\}.$$

Since $\mathcal{F}$ does not contain the greatest element, it is convenient for us to denote by $\hat{\mathcal{F}}$ the set $\mathcal{F} \cup \{\mathbf{1}\}$, where $\mathbf{1}$ is the artificial greatest element. The set $\hat{\mathcal{F}}$ is clearly a lattice.

Proposition 5.2. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Then, the following statements hold:*

(a) *The lattice $\hat{\mathcal{F}}$ has the Jordan-Dedekind chain property, i.e. every maximal chain between two ordered faces Y and Y' in $\mathcal{F}$ has the same length.*

(b) *For any ordered faces Y, Y' in $\hat{\mathcal{F}}$ with distance two, there are exactly two faces between them.*

The dimension $dim(Y)$ of a face Y in an oriented matroid M is one less than the length of any maximal chain from $\mathbf{0}$ to Y. The dimension of an oriented matroid is defined as the dimension of its topes.

5.1 OM and Topological Representation Theorem

One of the most important theorems in oriented matroid theory is the topological representation theorem [FL78, Ma82]. By this theorem, every oriented matroid can be considered as a simple topological object as well as a combinatorial object. This fact is extremely important when one tries to formulate a conjecture on oriented matroids, because the axioms (OM1)-(OM4) of oriented matroids do no give much imagination to guess what can be true for the structure.

Let S^d be the unit d-sphere in R^{d+1} and denote by $S_{(d-1)}$ the $(d-1)$-sphere $\{x \in S^d : x_1 = 0\}$ embedded in S^d. The sphere $S_{(d-1)}$ partitions S^d into three sets $\{S^+_{(d-1)}, S^0_{(d-1)}, S^-_{(d-1)}\}$ where $S^+_{(d-1)} = \{x \in S^d : x_1 > 0\}$ and $S^-_{(d-1)} = \{x \in S^d : x_1 < 0\}$.

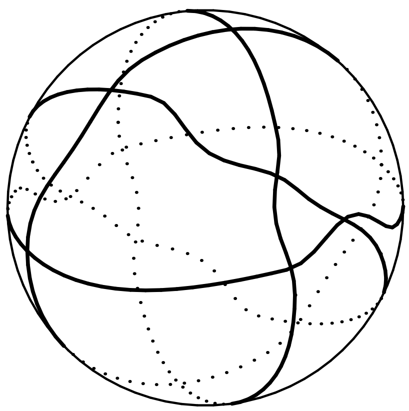
A subset S of S^d is called a *pseudosphere* if there is a homeomorphism $f : S^d \to S^d$ such that $f(S_{(d-1)}) = S$. Every pseudosphere S in S^d partitions S^d into three sets $\{S^+, S^0, S^-\}$ which are the images of $\{S^+_{(d-1)}, S^0_{(d-1)}, S^-_{(d-1)}\}$ under f. The sets S^+ and S^- are called the *sides* of S.

A *sphere system* is a triple $(E, S^d, \mathcal{A})$ where E be a finite set and $\mathcal{S}$ is a family $\mathcal{A} = \{\{S_i^+, S_i^0, S_i^-\} : i \in E\}$ satisfying the following axioms:

(S1) Each S_i^0 is a pseudosphere in S^d whose sides are S_i^+ and S_i^-, for each $i \in E$;

(S2) For each subset S of E, $\cap_{i \in S} S_i^0$ is a sphere, called a flat of $\mathcal{S}$;

(S3) For each flat F and any $i \in E$, either $F \subset S_i^0$ or $F \cap S_i^0$ is a pseudosphere in F whose sides are $F \cap S_i^+$ and $F \cap S_i^-$.



For a sphere system $(E, S^d, \mathcal{A})$, there is a natural way to define the position vector $\sigma(x)$ of each point $x \in S^d$:

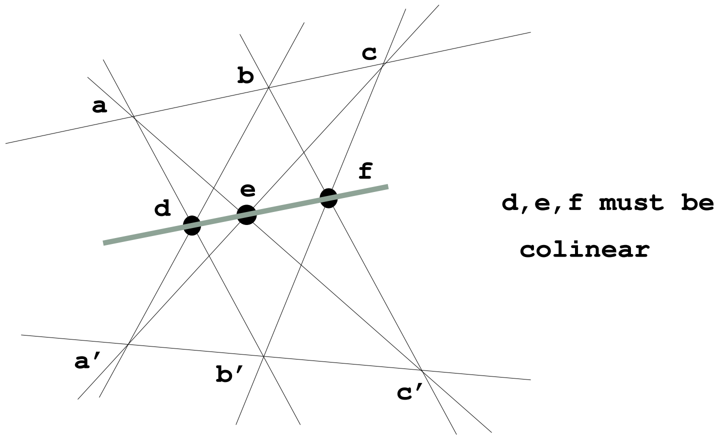
$$\sigma(x)_i = \begin{cases} + & \text{if } x \in S_i^+ \\ 0 & \text{if } x \in S_i^0 \\ - & \text{if } x \in S_i^- \end{cases} \qquad i \in E.$$

Theorem 5.3. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Then there exists a sphere system $(E, S^d, \mathcal{A})$ such that $\sigma(S^d) = \mathcal{F}$. Conversely, if $(E, S^d, \mathcal{A})$ is a sphere system, then $(E, \sigma(S^d))$ is an oriented matroid.*

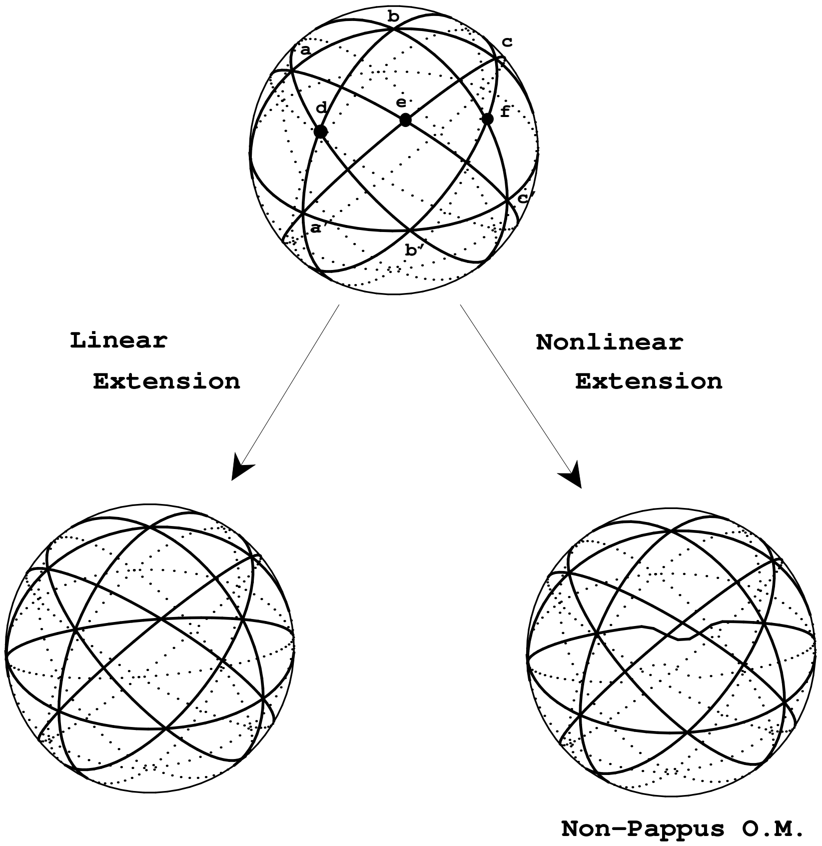
5.2 Construction of Nonlinear OMs

As we mentioned, there are infinitely many oriented matroids that are nonlinear. Several different ways to construct nonlinear oriented matroids are known. The easiest way is perhaps to use one of the well known theorems on projective geometry such as Pappus Theorem and Desargues Theorem. Another construction is to use the abstraction of linear programming: oriented matroids of rank 4 or more can be constructed so that the simplex algorithm applied to the associated program will produce a cycle of nondegenerate pivots, see [Fu82, BLSWZ93].

Here we explain the main idea to construct a non-Pappus oriented matroid which has rank 3. (It is quite easy to see that every oriented matroid of rank 2 or less is always linear.) The following drawing illustrates the Pappus Theorem, which states that if three points a, b and c in the plane are colinear and other three points a', b' and c' are also colinear, then the intersection points d, e and f in the figure must be colinear.



In order to construct a nonlinear oriented matroid, we first construct a linear arrangement of 8 spheres in S^2. Then, we add one more topological 1-sphere so that it passes through the points d and f but e. The resulting sphere arrangement and the associated set of signed vectors cannot be linear because of the Pappus theorem.



5.3 Complexity of Oriented Matroids and Arrangements

For an oriented matroid $M = (E, \mathcal{F})$ of dimension d, $f_i(M)$ denotes the number of i-dimensional faces in M, for each $-1 \le i \le d$.

Theorem 5.4 ([FST91]) : *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then*

$$f_i(M) \le \binom{d}{i} f_d(M) \qquad \forall 0 \le i \le d.$$

Some special cases of these inequalities include:

$$\begin{aligned} f_{d-1}(M) &\le d f_d(M) \\ f_0(M) &\le f_d(M). \end{aligned}$$

These inequalities were used to design a polynomial algorithm to check whether a given set of sign vectors is the set of topes of an oriented matroid [FST91].

One can generalize the theorem in various different ways, see [FTT91].

6 Graphs Associated with OM

6.1 Tope Graphs and Cocircuit Graphs

By Proposition 5.2, there are two graphs naturally associated with any oriented matroid. The first one is the *tope graph* $TG(M) = (\mathcal{T} \equiv \mathcal{F}_d, \mathcal{F}_{d-1})$ where the nodes are the topes and two topes T, T' are adjacent if they have a common $(d-1)$-face. The second one is the *cocircuit graph* $CG(M) = (\mathcal{V} \equiv \mathcal{F}_0, \mathcal{F}_1)$ where the nodes are the vertices (cocircuits) of M and two vertices are adjacent if they are the two 0-faces of some 1-face.

Two oriented matroids $M = (E, \mathcal{F})$ and $M' = (E', \mathcal{F}')$ are called *isomorphic* if the two posets $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic.

We shall list below some fundamental theorems on tope graphs and cocircuit graphs.

Theorem 6.1 ([BEZ90, CFG91]) : *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then the following statements hold:*

(a) *The tope graph $TG(M)$ determines the oriented matroid up to isomorphism.*

(b) *The cocircuit graph $CG(M)$ determines the oriented matroid up to isomorphism if the oriented matroid is uniform, i.e., $|E \setminus \underline{C}| = d$ for any cocircuit C. Furthermore, there is a non-uniform oriented matroid whose cocircuit graph does not determine the oriented matroid.*

Theorem 6.2 ([CF93]) : *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension d. Then the following statements hold:*

(a) *The tope graph $TG(M)$ is $(d+1)$-connected.*

(b) *The cocircuit graph $CG(M)$ is $2d$-connected.*

The *n-cube* Q_n is the graph with vertex set $\{1,-1\}^n$ where two vertices are adjacent if they are different in exactly one component.

Theorem 6.3 ([FH93]) : *Let $M = (E, \mathcal{F})$ be an oriented matroid of dimension $d \geq 3$. Let $n = |E|$. Then the following statements hold:*

(a) *The tope graph $TG(M)$ is bipartite, and isometrically embeddable in the n-cube Q_n.*

(b) *The cocircuit graph $CG(M)$ contains a K_3 (triangle) as a vertex-induced subgraph.*

A much stronger statement can be proved for the statement (b) if the oriented matroid is linear: the cocircuit graph $CG(M)$ contains a complete graph K_{d+1}, see [Sh79]. Las Vergnas [La80] conjectured that this is true for nonlinear oriented matroids as well.

A graph G is called *antipodal* if for each vertex v there is a unique vertex $\overline{v}$ such that $distance(v,u) < distance(v,\overline{v})$ for all neighbor u of $\overline{v}$.

Proposition 6.4 ([FH93]) : *The tope graph of an oriented matroid is antipodal.*

Theorem 6.5 ([FH93]) : *A graph G is isomorphic to the tope graph of an oriented matroid of dimension $d \leq 2$ if and only if G is isometrically embeddable in an n-cube, antipodal and planar.*

References

[AF92a] Avis, D. and Fukuda, K., "A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra," *Discrete Comput Geometry* 8 (1992), 295-313.

[AF92b] Avis, D. and Fukuda, K., "Reverse search for enumeration," Research Report No. 92-5, Graduate School of Systems Management, University of Tsukuba, Tokyo (1992), submitted.

[AF93] Avis, D. and Fukuda, K., "RS+DD: Vertex enumeration codes in C, ReverseSearch and DoubleDescription," project in progress (1993).

[Ba61] Balinski., "On the graph structure of convex polyhedra in n-space," *Pacific J. Math.* 11 (1961), 431-434.

[BEZ90] Björner, A., Edelman, P.H. and Ziegler, G.M., "Hyperplane arrangements with a lattice of regions," *Discrete Comput Geom* 5 (1990), 263-288.

[BLSWZ93] Björner, A., Las Vergnas, M., Sturmfels, B., White, N., and Ziegler, G., Oriented Matroids, Cambridge University Press, 1993.

[B77] Bland, R.G., "A combinatorial abstraction of linear programming," *J. Combin. Theory, Ser. B* 23 (1977), 33-57.

[BL78] Bland, R.G., and Las Vergnas, M., "Orientability of matroids," *J. Combin. Theory, Ser. B* 24 (1978), 94-123.

[BM87] Blind, R. and Mani, P. "On puzzles and polytope isomorphism," *Aeq. Math.* 34 (1987), 287-297.

[BS89] Bokowski, J., and Sturmfels, B., "An infinite family of minor-minimal nonrealizable-chirotopes," *Mathematishe Zeitshrift* Vol. 200 (1989), 583-589.

[BM71] Bruggesser, H. and Mani, P.., "Shellable decomposition of cells and spheres," *Math. Scand.* 29 (1971),197-205.

[CGAF93] Ceder, G., Garbulski, G.D., Avis, D. and Fukuda, K., "Ground states of a ternary lattice model with nearest and next-nearest neighbor interactions," preprint, Department of Materials Science and Engineering, MIT (1993).

[Chv83] Chvátal, V., *Linear Programming*, W.H. Freeman, 1983.

[Chu89] Chung, S.J., "NP-completeness of the linear complementarity problem," *J. Optim. Theory Appl.*, Vol. 60 (1989), 393-399.

[CFG91] Cordovil, R., , K. and Guedes de Oliveira, A., "On the cocircuit graph of an oriented matroid," submitted to J. Combin. Theory, Ser. B (1991).

[CF93] Cordovil, R. and Fukuda, K., "Oriented matroids and combinatorial manifolds," Europ. J. Combinatorics 14 (1993), 9-15.

[CLM82] Cordovil, R. and Las Vergnas, M., Mandel,. A., "Euler's relation, Möbius functions and matroid identities," *Geometriae Didicata* 12 (1982), 147-162.

[Co63] Cottle, R.W., "Symmetric dual quadratic programs. *Quart. Appl. Math.*, Vol. 21 (1963), 237-243.

[Co68] Cottle, R.W., "The principal pivoting method of quadratic programming," in Dantzig, G.B., and Veinott, A.F., (eds.) *Mathematics of Decision Sciences, Part 1,* American Mathematical Society, Providence, R.I. (1968), 142-162.

[Co89] Cottle, R.W., "Completely-Q matrices. *Math. Programming* 19 (1989), 347-351.

[Co90] Cottle, R.W., "The principal pivoting method revisited," *Mathematical Programmng*, Series B 48, 369-385 (1990).

[CD68] Cottle, R.W., and Dantzig, G.B., 'MComplementary pivot theory of mathematical programming," *Linear Algebra and Its Applications*, Vol. 1 (1968), 103-125.

[CPS92] Cottle, R.W., Pang, J.-S., and Stone, R.E., *The linear complementarity problem*, Academic Press, 1992.

[CPV89] Cottle, R.W., Pang, J.-S., and Venkateswaran, V., "Sufficient matrices and the linear complementarity problem," *Linear Algebra and Its Applications* 114/115, 231-249 (1989).

[DRT90] Den Hertog, D., Roos, C., and Terlaky, T., 'MThe linear complementarity problem, sufficient matrices and the criss-cross method," Reports of The Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1990.

[DFL89] M. Deza, K. Fukuda and M. Laurent, "The Inequicut Cone," Research Report No. 89-04, Graduate School of Systems Management, University of Tsukuba, Tokyo (1989), to appear.

[FL78] Folkman, J., and Lawrence, J., "Oriented matroids," *J. Combin. Theory, Ser. B*, Vol. 25 (1978), 199-236.

[Fu82] Fukuda, K., Oriented Matroid Programming, Ph.D. Thesis, University of Waterloo, 1982.

[Fu93] Fukuda, K., "DD.p: A Pascal Implementation of the Double Description Method," Graduate School of Systems Management, University of Tsukuba, Version 2.0 (1993).

[FH93] Fukuda, K. and Handa, K., "Antipodal Graphs and Oriented Matroids," *Discrete Mathematics* 111 (1993), 245-256.

[FMa91] Fukuda, K., and Matsui, T., "On the finiteness of the criss-cross method," *Europ. J. Oper. Res.*, Vol. 52 (1991), 119-124.

[FMi91] Fukuda, K. and Mizukoshi, I., Mathematica package: Vertex enumeration for convex polyhedra and hyperplane arrangements, Version 0.41 Beta, Graduate School of Systems Management, University of Tsukuba, Tokyo (1991), available via anonymous ftp from cs.sunysb.edu (directory pub/Combinatorica) and from MathSource (Wolfram Research).

[FR93] Fukuda, K. and Rosta, V., "Combinatorial face enumeration in convex polytopes," Research Report No. 93-1, Graduate School of Systems Management, University of Tsukuba (1993), submitted.

[FTa90] Fukuda, K. and Tamura, A., "Dualities in signed vector systems," Portugaliae Mathematica, Vol. 47 (1990), 151-165.

[FST91] Fukuda, K., Saito, S. and Tamura, A., Combinatorial face enumeration in arrangements and oriented matroids, Discrete Applied Mathematics 31 (1991), 141-149.

[FSTT91] Fukuda, K., Saito, S., Tamura, A. and Tokuyama, T., Bounding the number of k-faces in arrangements of hyperplanes, Discrete Applied Mathematics 31 (1991), 151-165.

[FTT91] Fukuda, K., Tamura, A. and Tokuyama, T., A theorem on the average number of subfaces in arrangements and oriented matroids, Research Reports on Information Sciences, B-242, Tokyo Institute of Technology (1991), Geometriae Dedicata, to appear.

[FTe89] Fukuda, K., and Terlaky, T.: A general recursion algorithm for quadratic programming. preprint (1989).

[FTe92] Fukuda, K. and Terlaky, T., "Linear complementarity and oriented matroids," Journal of the Operations Research Society of Japan, 35, (1992), 45-61.

[GN65] Gale, D and Nikaido, H. , "The Jacobian matrix and global univalence of mappings," *Math. Ann.* 159 (1965), 81-93.

[Gra67] Graves, R.L., "A principal pivoting simplex algorithm for linear and quadratic programming," *Operations Research*, Vol. 15 (1967), 482-494.

[Gru67] Grünbaum, B., *Convex Polytopes*, Interscience, N.Y., 1967.

[Gru72] Grünbaum, B., *Arrangements and spreads*, CBMS Lecture Notes 10, Amer. Math. Soc., 1972.

[K88] Kalai, G., "A simple way to tell a simple polytope from its graph," *J. Combin. Theory, Ser. A* 49 (1988), 381-383.

[KT88] Klafszky, E., and Terlaky, T., 'MSome generalizations of the criss-cross method for quadratic programming," *Linear Algebra and Its Applications*, to appear.

[KT89] Klafszky, E., and Terlaky, T., 'MSome generalizations of the criss-cross method for the linear complementarity problem of oriented matroids," *Combinatorica* Vol. 9 (1989), 189-198.

[KMNY91] Kojima, M., Meggiddo, N., Noma, T. and Yoshise, A., *A unified approach to interior point algorithms for linear complementarity problems*, Lecture Notes in Computer Science 538, Springer-Verlag, 1991.

[La80] Las Vergnas, M., (1980), Convexity in oriented matroids, *J. Combin. Theory, Ser. B* 29, 231-243.

[L65] Lemke, C.E., "Bimatrix equilibrium points and mathematical programming," *Management Science*, Vol. 11 (1965), 681-689.

[Mc70] McMullen, P. , "The maximum number of faces of a convex polytope," *Mathematika* 17 (1970), 179-184.

[MS71] McMullen, P. and Shephard, G.C., *Convex Polytopes and the Upper Bound Conjecture*, Cambridge University Press, 1971.

[Ma82] Mandel, A., "Topology of Oriented Matroids. Ph.D. Thesis, University of Waterloo, 1982,"

[Mo53] Motzkin, T.S., "Comonotone curves and polyhedra," Abstract 111, *Bull. Amer. Math. Soc.* 63 (1957), p. 35.

[MRTT53] Motzkin, T.S. , Raiffa, H., Thompson, G.L. and Thrall, R.M., "The double description method," in "Contribution to the Theory of Games, Vol. II" (H.W. Kuhn and A.W. Tucker, eds.), Annals of Math. Studies 28, Princeton University Press, 1953, pp.81-103.

[NM90] Namiki, M., and Matsui, T., "Some modifications of the criss-cross method," Research Report, Department of Information Sciences, Tokyo Institute of Technology, Tokyo, Japan, 1990.

[STW58] Samelson, H., Thrall, R.M., and Wesler, O. "A partition theorem for Euclidean n-space," *Proc. Amer. Math. Soc.* 9, (1958), 805-807.

[Sc86] Schrijver, A., *Theory of Linear and Integer Programming*, John Wiley & Sons, 1986.

[Sh79] Shannon, R.W., (1979), Simplicial cells in arrangements of hyperplanes, *Geom. Dedic.* 8, 179-187.

[St34] Steinitz, E., *Polyeder und Raumeinteilungen*, Enzykl. mah. Wiss., Vol.3 (geometrie), Part 3AB2, (1922), 1-139.

[SR85] Steinitz, E. and Rademacher, H., Vorlesungen über die Theorie der Polyeder, Springer, Berlin, 1934.

[Te85] Terlaky, T., "A convergent criss-cross method," *Mathematische Operationsforschung und Statistics ser. Optimization*, Vol. 16 (1985), 683-690.

[Te87] Terlaky, T., "A finite criss-cross method for oriented matroids," *J. of Combin. Theory, Ser. B*, Vol. 42 (1987), 319-327.

[To84] Todd, M.J., "Complementarity in oriented matroids," *SIAM J. Alg. Disc. Meth.*, Vol. 5 (1984), 467-485.

[To85] Todd, M.J., "Linear and quadratic programming in oriented matroids," *J. of Combin. Theory, Ser. B*, Vol. 39 (1985), 105-133.

[Wa87] Wang, Zh., "A conformal elimination free algorithm for oriented matroid programming," *Chinese Annals of Mathematics*, 8 B 1 (1987).

[YP89] Ye, Y., and Pardalos, P., "A class of linear complementarity problems solvable in polynomial time," Department of Management Sciences, The University of Iowa, Iowa City, Iowa, 1989.

[Za75] Zaslavski, T. , *Facing up to arrangements*, Memoirs Amer. Math. Soc., No. 154, 1975..

[Zi93] Ziegler, G. M., "Letures on Polytopes," Technical Report TR 93-6, Konrad-Zuse-Zentrum für Informationstechnik Berlin (1993).