A List Heuristic for Vertex Cover

Happy Birthday Vasek!

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Operations Research Letters (to appear)

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revised: 2006.11.16

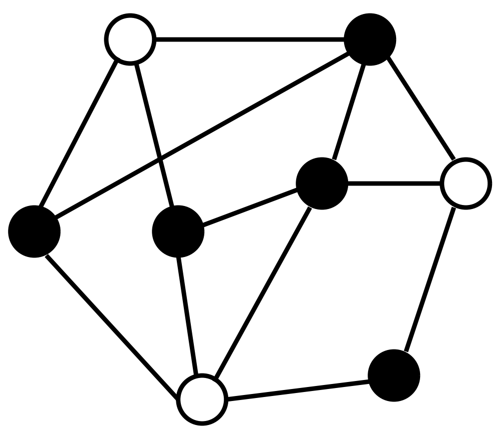
$\boxed{\text{Outline}}$

- The Vertex Cover Problem
- Integer Programming Formulation
- LP-Relaxation
- Heuristics: LP rounding, matching & greedy
- List Heuristic
- Analysis of List Heuristic

Introduction

Vertex Cover

A subset of vertices C s.t. every edge has one of its endpoints in C.



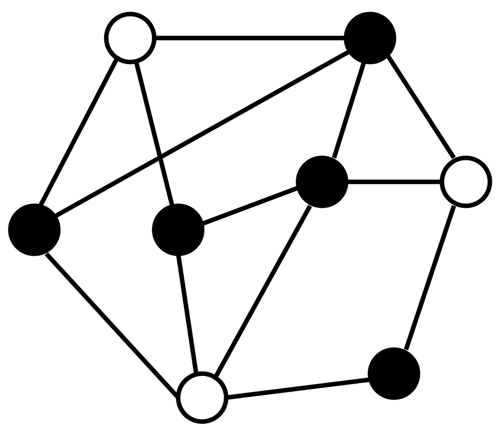
Minimum Vertex Cover Problem

Find a minimum cardinality vertex cover.

ILP and LP

ILP Formulation of Vertex Cover

$$
\begin{array}{ll}
\text{min} & \sum_{v \in V} x_v \\
\text{s.t.} & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\
 & x_v \in \{0,1\} \quad \forall v \in V
\end{array}
$$



LP Relaxation of Vertex Cover(primal)

$$
\begin{array}{ll}
\text{min} & \sum_{v \in V} x_v \\
\text{s.t.} & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\
 & x_v \geq 0 \quad \forall v \in V
\end{array}
$$

Non-integer solution is a fractional vertex cover.

Primal and Dual LPs

Primal LP

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\ & x_v \geq 0 \quad \forall v \in V \end{aligned}$$

Primal variables are vertex weights.

Dual LP

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V \\ & y_e \geq 0 \quad \forall e \in E \end{aligned}$$

Dual variables are edge weights.

$\delta(v)$ is the set of edges with endpoint v.

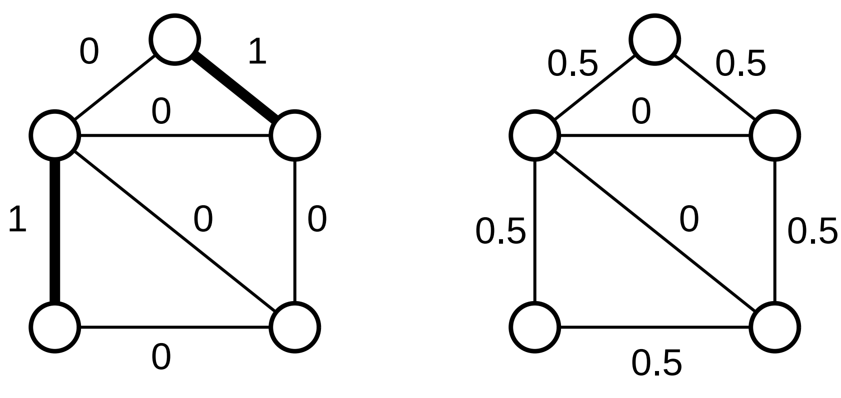
Dual: Matching

Dual of LP Relaxation

$$\begin{array}{ll} \max & \sum_{e \in E} y_e \\ \text{s.t.} & \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V \\ & y_e \geq 0 \quad \forall e \in E \end{array}$$

An integer feasible solution is a **matching**.

A non-integer solution is called a **fractional matching**.



LP duality

$\boldsymbol{x} = \{x_v\}$: any $\{0,1\}$ primal feasible solution
$\boldsymbol{y} = \{y_e\}$: any feasible fractional matching
Opt : size of min vertex cover.

By weak LP-duality,

$$\sum_{e \in E} y_e \leq \sum_{v \in V} x_v$$

Since min cover is optimum ILP solution,

$$\boxed{\sum_{e \in E} y_e \leq \text{Opt} \leq \sum_{v \in V} x_v}$$

$\boxed{\text{LP heuristic}}$

$\boldsymbol{x}^* = \{x_v^*\}$: optimal for LP relaxation.

Define $\{0,1\}$-solution $\boldsymbol{x} = \{x_v\}$ by rounding:

$$x_v = \begin{cases} 1 & (x_v^* \geq 1/2) \\ 0 & (x_v^* < 1/2) \end{cases}$$

Proposition.
$\boldsymbol{x}$ has approx. ratio 2.

- By definition, $\forall v \; x_v \leq 2x_v^*$.
- Since x^* is an optimum fractional cover,

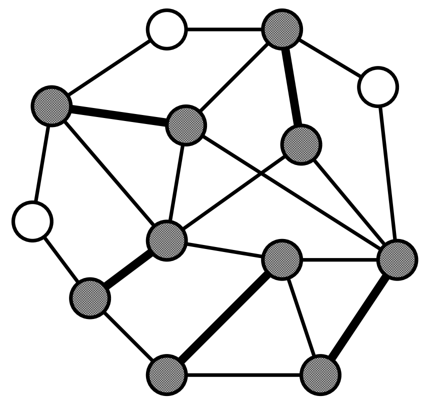
$$\sum_{v \in V} x_v^* \leq \text{Opt}.$$

Thus,

$$\frac{\sum x_v}{\text{Opt}} \leq \frac{\sum 2x_v^*}{\text{Opt}} \leq \frac{2\text{Opt}}{\text{Opt}} = 2$$

Matching heuristic

1. Take any maximal matching M.
2. Let C be set of the vertices incident with some matching edge.



Proposition.
C has approx. ratio 2.

- $|C| = 2|M|$.
- M is a feasible solution of the dual.

Thus, by LP-duality, Opt $\geq |M| = |C|/2$ and

$$\frac{|C|}{\text{Opt}} \leq \frac{|C|}{|C|/2} = 2.$$

$\boxed{\text{Greedy heuristic}}$

while G has an edge **do**
  let v be the highest degree vertex.
  select v.
  remove v and all edges incident to it.
end
Output the selected vertices.

Has tight approx. ratio

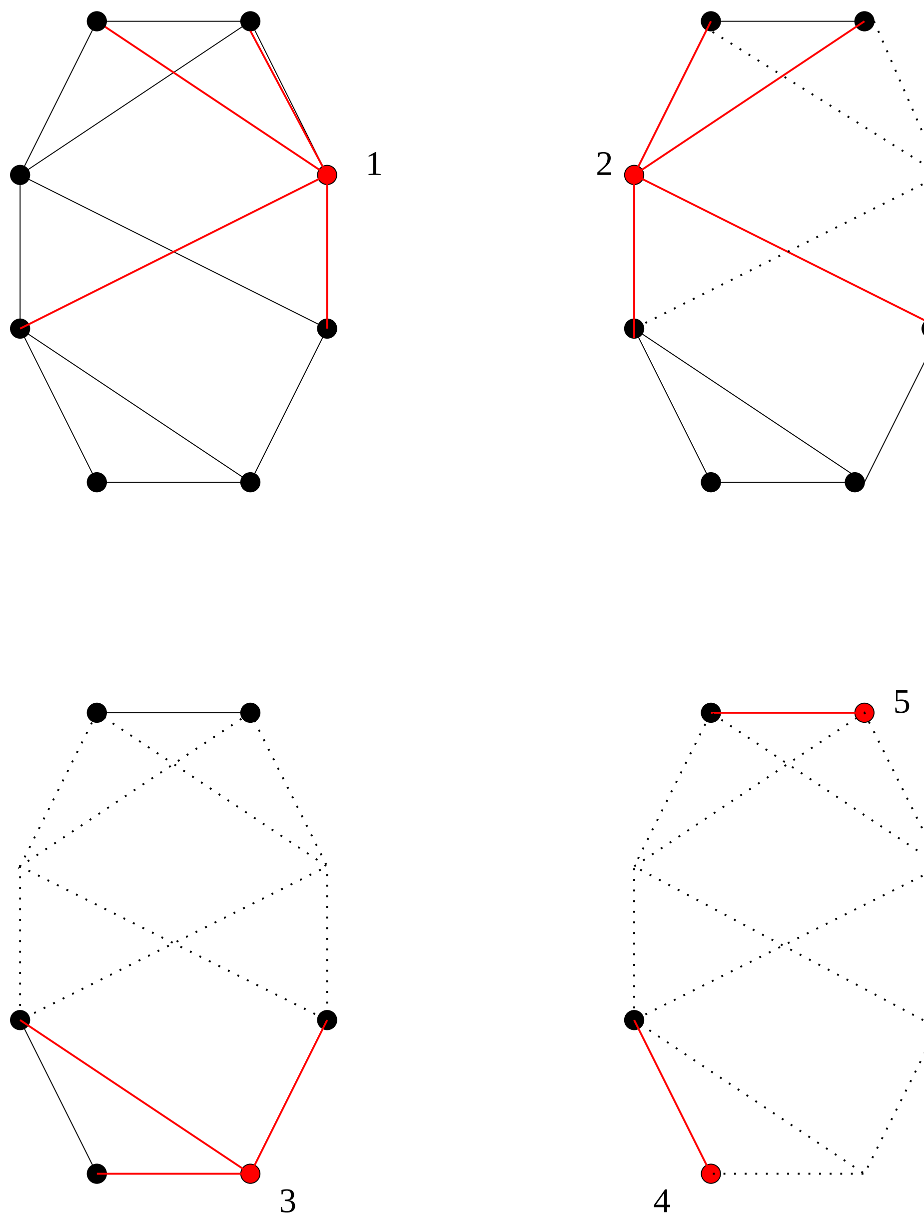
$$H(\Delta) = 1 + \frac{1}{2} \ldots + \frac{1}{\Delta}$$

($\Delta =$ max vertex degree)

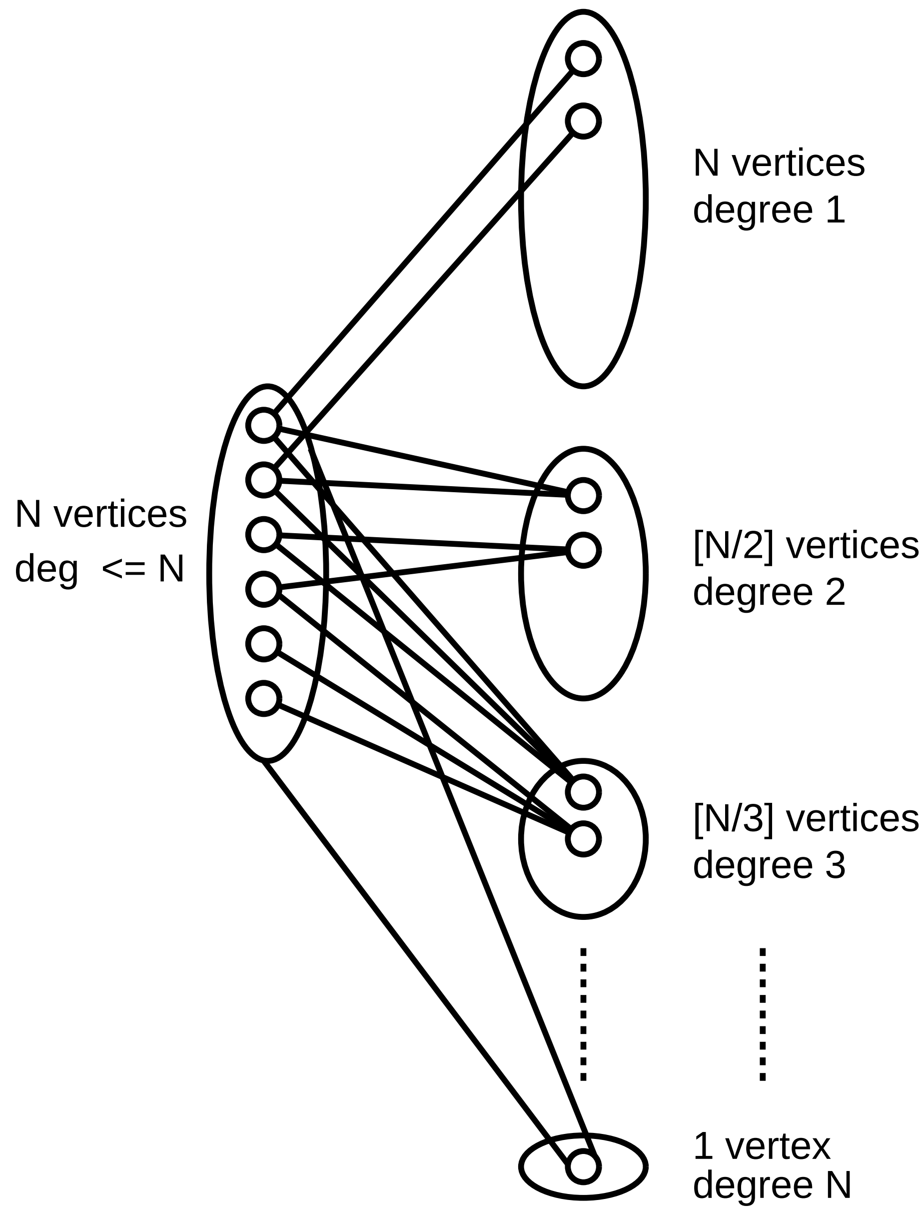
Unweighted case : Johnson '74, Lovasz '75

Weighted case: Chvátal '79

Example of Greedy Heuristic



Worst case for Greedy heuristic



Opt $= N$

$C_{\text{GRE}} = N + \lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{3} \rfloor + \cdots + 1 \simeq N * H(N)$

So $\frac{|C_{\text{GRE}}|}{\text{Opt}} \simeq H(N) = H(\Delta)$

$\boxed{\text{Proof of Greedy upper bound}}$

$C = \{v_1, v_2, \ldots, v_t\}$: vertices chosen by greedy.

u_i: # of uncovered edges when v_i chosen.

Assign each of these edges weight $1/u_i$

Total edge weight assigned is $|C|$

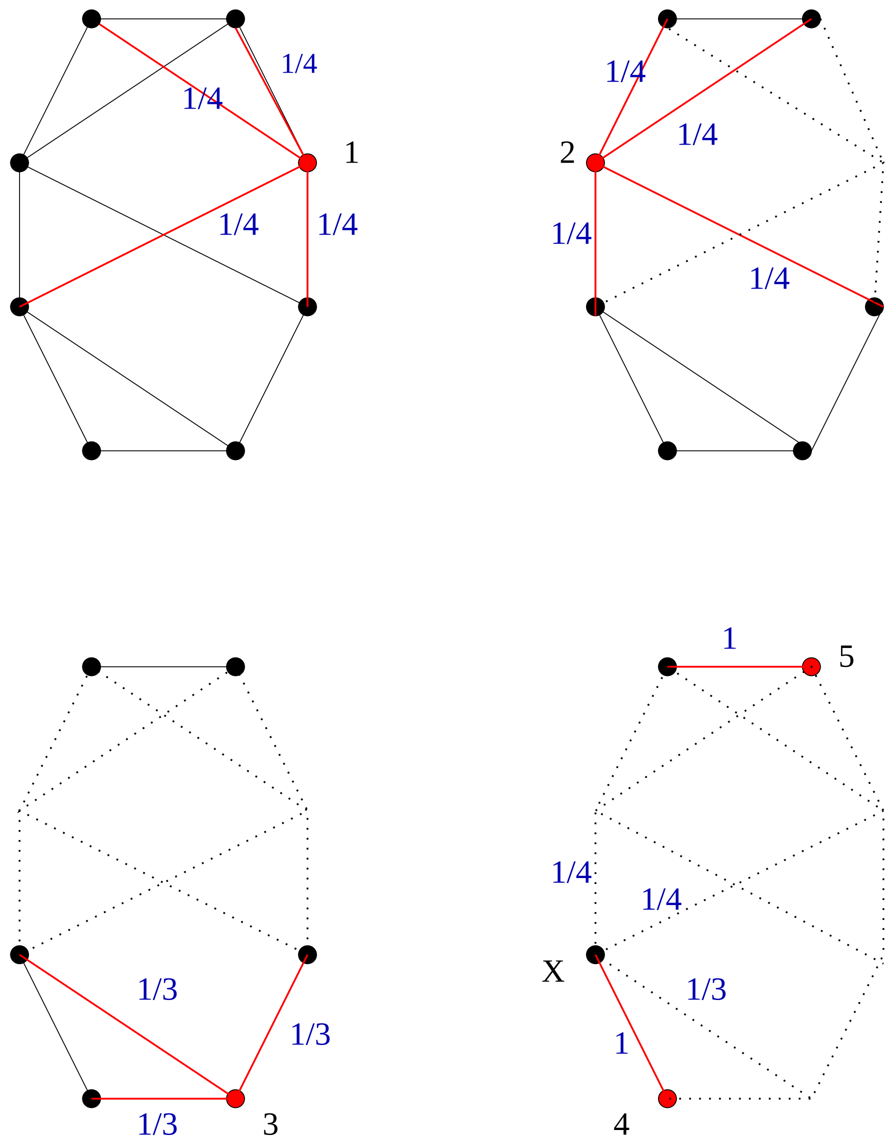
At any vertex v:
At most k incident edges have weight $\geq 1/k$
(otherwise weight of first labelled is $< 1/k$)

So maximum weight of edges at v is $H(\Delta)$

Dividing all edge weights by $H(\Delta)$ gives a fractional matching of total weight $|C|/H(\Delta)$

So: $\quad OPT \geq \frac{|C|}{H(\Delta)}$

Edge Weight Assignment by Greedy



Weight of each vertex is at most $H(\Delta)$, eg. $w(x) = 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = H(\Delta)$.

Total edge weight is $|C| = 5$.

$\boxed{\text{List heuristic}}$

Sort vertices in non-increasing order
by degrees.

for each vertex v in this order
  if there is an edge incident to v
    select v.
    remove v and all its edges.

Output the selected vertices.

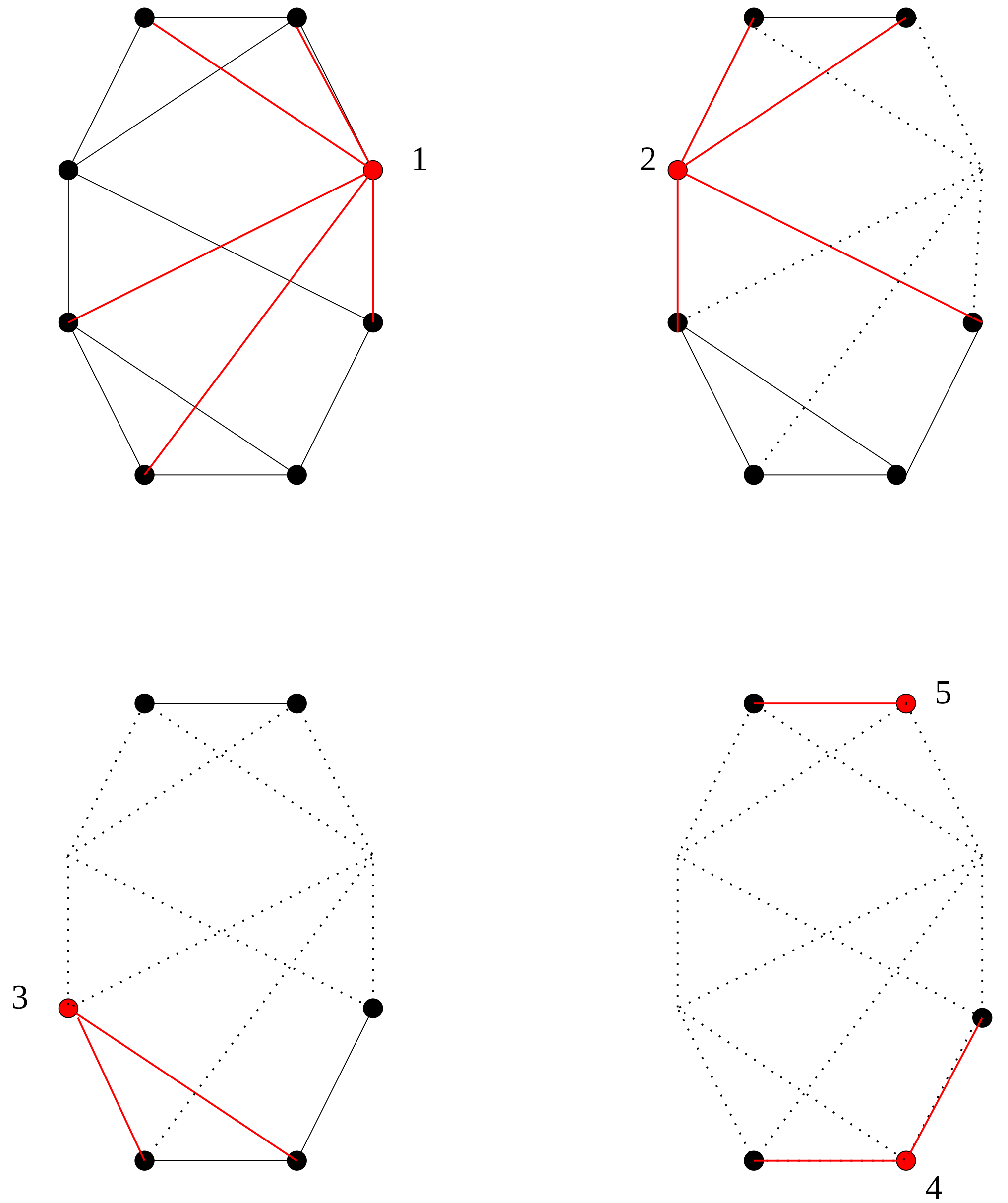
List heuristic: static ordering

Sort vertices in non-increasing order
by degrees.
for each vertex v in this order
  if there is an edge incident to v
    select v.
    remove v and all its edges.
Output the selected vertices.

Greedy heuristic: dynamic ordering

while G has an edge **do**
  let v be the highest degree vertex.
  select v.
  remove v and all edges incident to it.
end
Output the selected vertices.

Example of List Heuristic



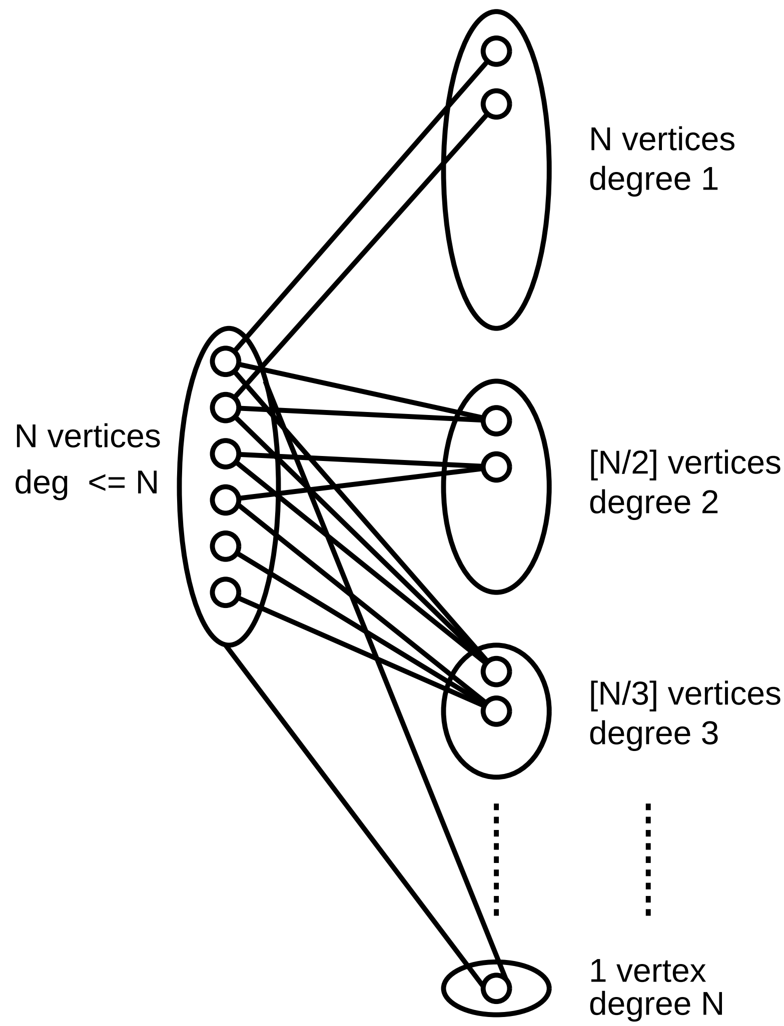
Greedy would choose vertex 4 instead of 3

$\boxed{\text{Motivating Example}}$

- Nodes submit secret bids to supply connectivity to other nodes for a fixed price K
- Node i offers to connect to a subset $\delta(i)$ of other nodes.
- Regulator must accept bids in decreasing order by $d_i = |\delta(i)|$, as long as each bid connects to at least one new node.

List beats greedy on this example

$$\text{Opt} = N, \quad |C_{\text{GRE}}| \simeq N * H(N)$$



$$|C_{\text{List}}| \leq 3N/2$$

since min degree on LHS $> N/2$ and $\leq N/2$ vertices on the RHS have degree $> N/2$
but...........

Bounds for List heuristic

Theorem 1.

$$\frac{\text{List}}{\text{Opt}} \leq \frac{\sqrt{\Delta}}{2} + \frac{3}{2}.$$

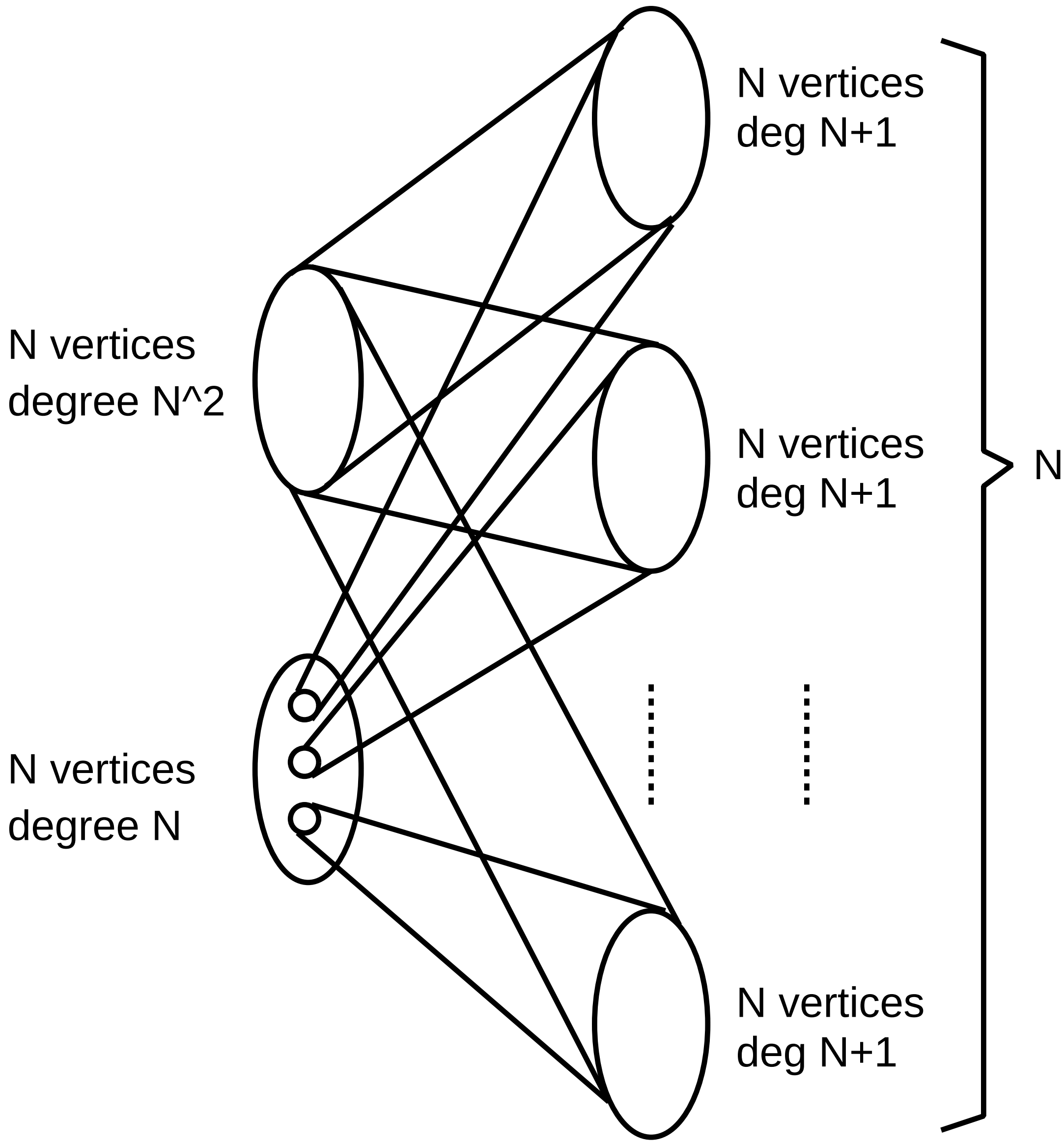
This bound is tight up to the constant.

Theorem 2.
There is a class of graphs such that

$$\frac{\text{List}}{\text{Opt}} \geq \frac{\sqrt{\Delta}}{2}.$$

The above bound holds for any fixed vertex order based on the degree sequence.

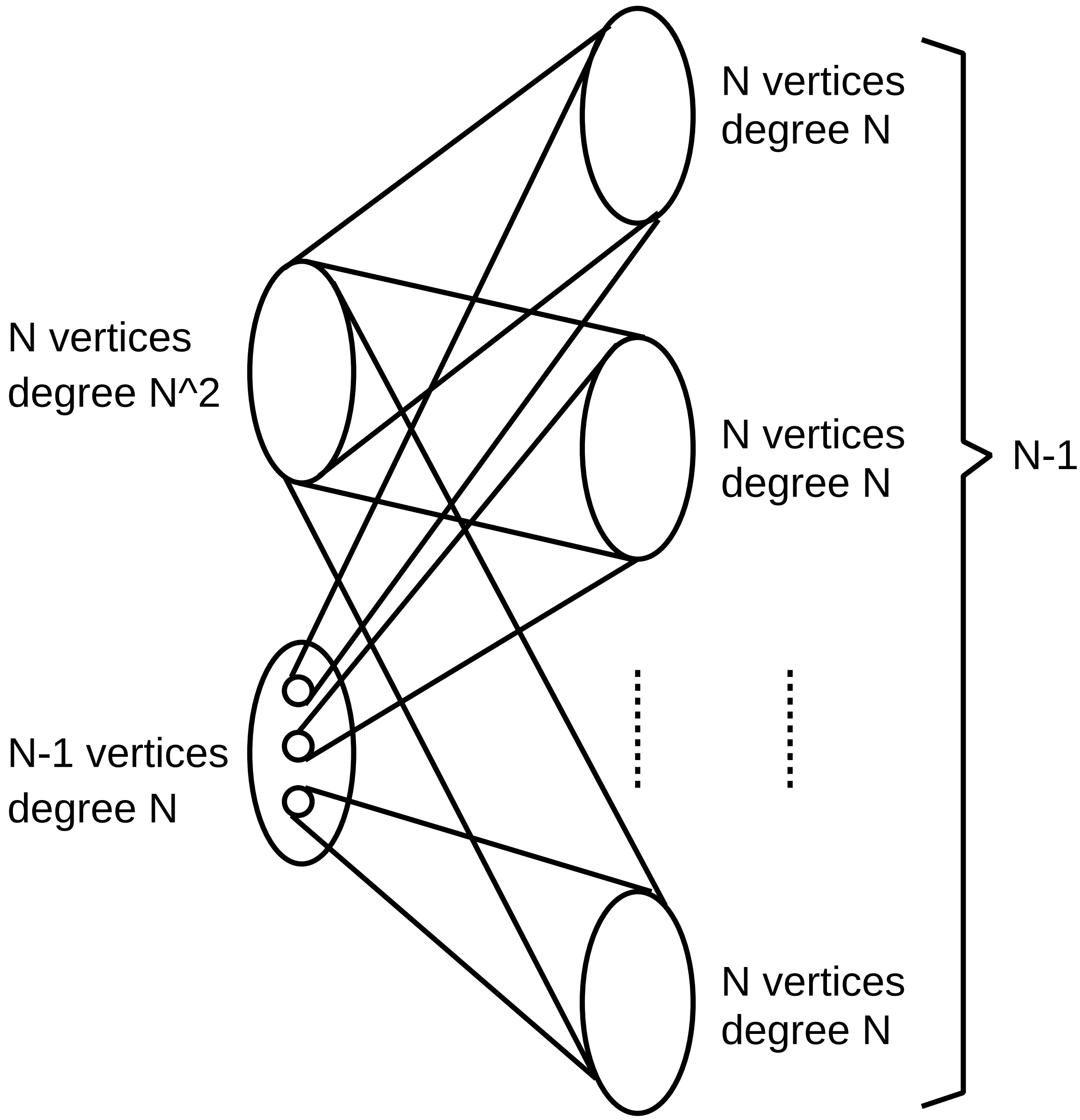
Worst case for List decreasing



$$\text{Opt} = 2N, \quad \text{List} = N^2 + N,$$

$$\frac{\text{List}}{\text{Opt}} = \frac{N+1}{2} = \frac{\sqrt{\Delta}}{2} + \frac{1}{2}.$$

Worst case for any List heuristic



$$\text{Opt} = 2N - 1, \quad \text{List} = N^2,$$

$$\frac{\text{List}}{\text{Opt}} = \frac{N^2}{2N-1} \geq \frac{\sqrt{\Delta}}{2}.$$

Proof of a weaker upper bound (1/2)

Theorem 3.
$$\frac{\text{List}}{\text{Opt}} \leq \sqrt{2\Delta}.$$

For $i = 1, ..., t$ suppose List selects vertex v_i which has degree d_i.

Assign edge weights as follows:
for $i = 1, ...t$, assign **one** of v_i's uncovered edges weight

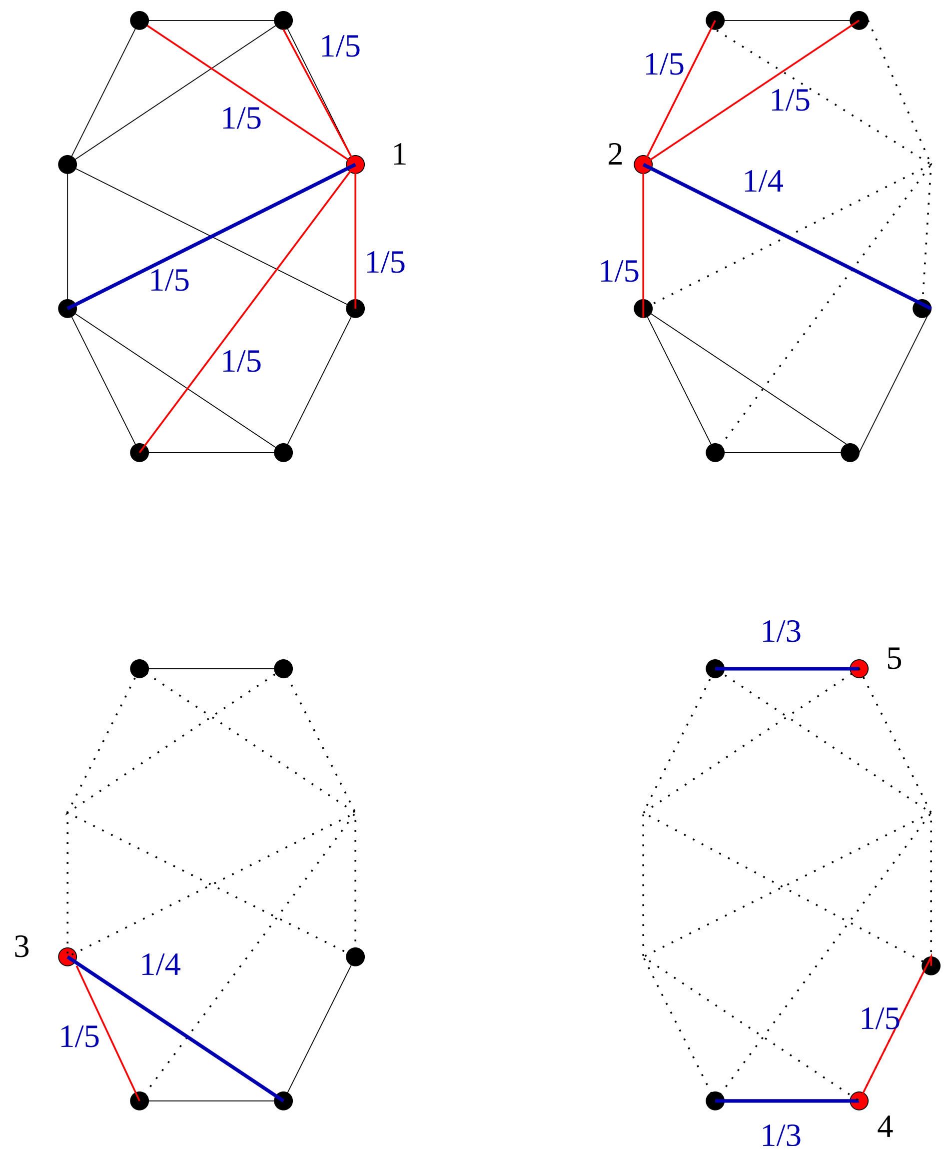
$$y_e = \frac{1}{d_i}.$$

All unassigned edges get weight

$$y_e = \frac{1}{\Delta}.$$

This is a feasible fractional matching for G.

Edge Weight Assignment by List



For each v_i selected one (blue) edge gets weight $1/d_i$.
The other (red) edges get weight $1/\Delta = 1/5$.
Total edge weight is $3\frac{1}{6}$, so Opt ≥ 4

Proof of upper bound (2/2)

Lemma (Cauchy-Schwartz)
If $d_i \geq 0, i = 1, ..., t$ have $\sum_{i=1}^{t} d_i \leq 2m$ then

$$\sum_{i=1}^{t} d_i \geq \frac{t^2}{2m}.$$

$$\begin{aligned}
Opt &\geq \sum_{e \in E} y_e \\
&= \sum_{i=1}^{t} \frac{1}{d_i} + \frac{m-t}{\Delta} \\
&\geq \frac{t^2}{2m} + \frac{m}{\Delta} - \frac{t}{\Delta} \quad \text{(Lemma)} \\
&\geq 2\sqrt{\frac{t^2}{2m} \cdot \frac{m}{\Delta}} - \frac{t}{\Delta} \quad (a + b \geq 2\sqrt{ab}) \\
&= \frac{2t}{\sqrt{2\Delta}} - \frac{t}{\Delta} \\
&\geq \frac{t}{\sqrt{2\Delta}} \quad (\Delta \geq 2) \\
&= \frac{List}{\sqrt{2\Delta}}
\end{aligned}$$

Conclusion

Four heuristics analyzed by LP methods.

Ranked by perfomance ration $PR = \frac{HEUR}{OPT}$:

- LP-rounding: $PR = 2$
- Matching: $PR = 2$
- Greedy: $PR = H(\Delta) = O(log(\Delta))$
- List: $PR = \sqrt{\Delta}/2 + 3/2$

In practice?