Branch and Bound for Knapsack Problem (KP)

Input: Positive integers $a_1, \cdots, a_n, c_1, \cdots, c_n, b, n$

(KP)

$$\begin{array}{llllllll} \max & c_1x_1 & + & c_2x_2 & + & \cdots & + & c_nx_n \\ & a_1x_1 & + & a_2x_2 & + & \cdots & + & a_nx_n \leq b \end{array} \tag{1}$$

$$x_j = \begin{cases} 1 \\ 0 \end{cases}, j = 1, \cdots, n.$$

(LP) We get an LP relaxation for KP by replacing (**??**) by

$$0 \leq x_j \leq 1, j = 1, \cdots, n \tag{2}$$

Assume that we have labelled input so that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}$$

Delete any a_j, c_j for which $a_j > b$ (item j cannot be chosen)
LPsolution: Choose largest k s.t.

$$\sum_{i=1}^{k} a_i \leq b$$

Let

$$x_1 = x_2 = \cdots = x_k = 1$$

$$x_{k+1} = \frac{b - \sum_{i=1}^{k} a_i}{a_{k+1}}$$

$$z = \sum_{j=1}^{k} c_j + c_{k+1}x_{k+1}$$

(If $x_{k+1} = 0$ we have an optimum solution to KP).

B&B method

L = list of problems to solve (all integer KPs)
Initially $L = KP$, $z_{LB} = -\infty$, where z_{LB} is the best known integer solution.

1. Take a problem P from L. If none, stop.

2. Solve the LP relaxation of P.

3. (Bound) If solution (x, z) is integer set $z_{LB} = \max\{z_{LB}, z\}$ and go to 1.
   If LP infeasible go to 1.

4. (Branch) Choose x_j which is fractional. Create two new problems P_1 and P_2 and put them on L:

$$P_1 = P \text{ and } x_j = 0$$
$$P_2 = P \text{ and } x_j = 1$$

Example

(KP)

$$\begin{array}{lccccccccc} \max & 24x_1 & + & 17x_2 & + & 12x_3 & + & 6x_4 & & \\ & 10x_1 & + & 8x_2 & + & 6x_3 & + & 5x_4 & \leq & 15 \end{array}$$

$$x_j = \begin{cases} 1 \\ 0 \end{cases}, j = 1, 2, 3, 4$$

$$z_{LB} = -\infty$$

LP solution $x_1 = 1$, $x_2 = \frac{5}{8}$, $z = 34\frac{5}{8}$

<u>Iteration</u> (See next page for details)

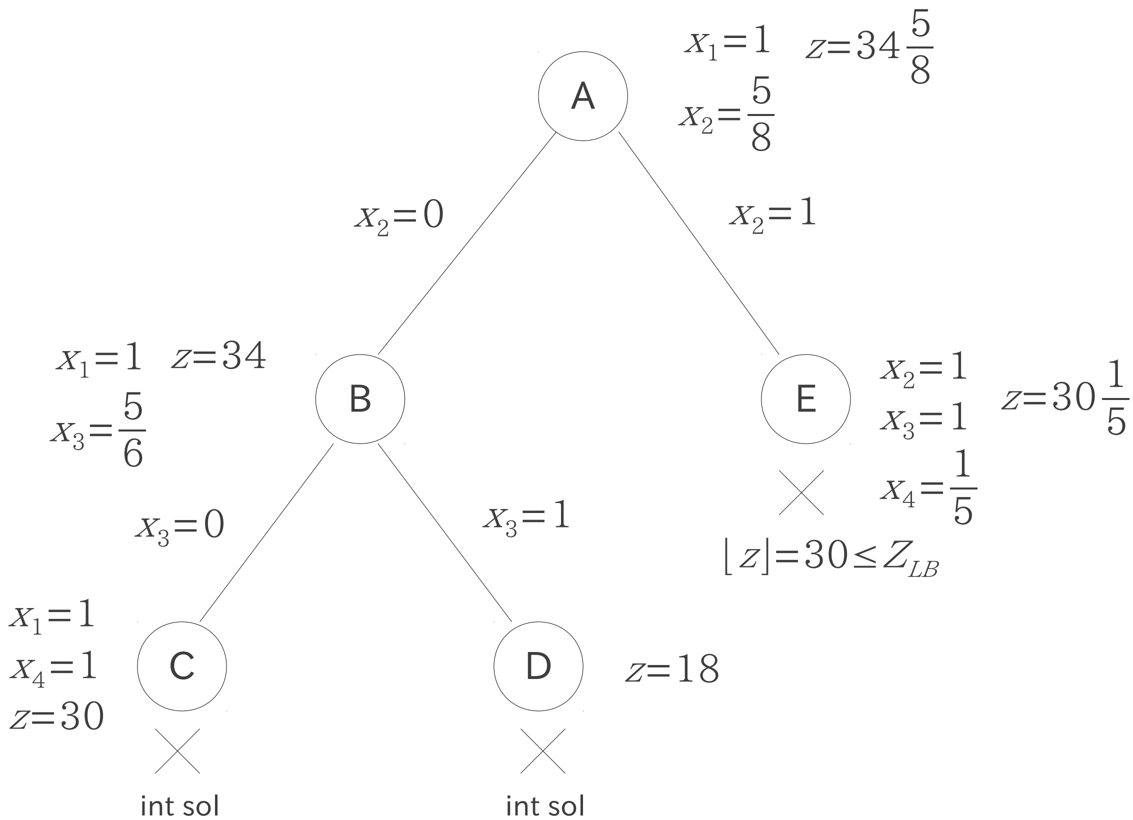
1. $L = A$. A selected.

2. $L = BE$. B selected.

3. $L = C, D, E$
   C selected, integer, $z_{LB} = 30$
   D selected, integer, $z = 18$, $z_{LB} = 30$

4. $L = E$. E selected.
   $z = 30\frac{1}{5} \implies \lfloor z \rfloor = 30 \leq z_{LB}$
   So no need to branch.

5. $L = \phi$. Stop.



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$$\begin{array}{llllllllll} \max & 24x_1 & + & 17x_2 & + & 12x_3 & + & 6x_4 & & \\ & 10x_1 & + & 8x_2 & + & 6x_3 & + & 5x_4 & \leq & 15 \end{array}$$

$$0 \leq x_j \leq 1, j = 1, \cdots, 4$$

$$x_1 = 1$$

$$x_3 = \frac{5}{8}$$

$$z = 24 + \frac{17 \times 5}{8} = 34\frac{5}{8}$$

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$$\begin{array}{llllllll} \max & 24x_1 & + & 12x_3 & + & 6x_4 & & \\ & 10x_1 & + & 6x_3 & + & 5x_4 & \leq & 15 \end{array}$$

$$x_1 = 1$$

$$x_3 = \frac{5}{6}$$

$$z = 24 + \frac{12 \times 5}{6} = 34$$

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$$\begin{array}{lccccc} \max & 24x_1 & + & 6x_4 & & \\ & 10x_1 & + & 5x_4 & \leq & 15 \end{array}$$

$$x_1 = x_4 = 1$$

$$z = 30$$

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$$\begin{array}{lccccccc} \max & \cancel{24x_1} & + & 6x_4 & + & 12 \\ & \cancel{10x_1} & + & 5x_4 & \leq & 9 \\ & & & x_4 & = & 1 \end{array}$$

$$x_3 = x_4 = 1$$

$$z = 18$$

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$$\begin{array}{lccccccccc} \max & \cancel{24x_1} & + & 12x_3 & + & 6x_4 & + & 17 \\ & \cancel{10x_1} & + & 6x_3 & + & 5x_4 & \leq & 7 \end{array}$$

$$x_3 = 1$$

$$x_4 = \frac{1}{5}$$

$$z = 17 + 12 + \frac{6}{5}$$