Proposal & Area Exam

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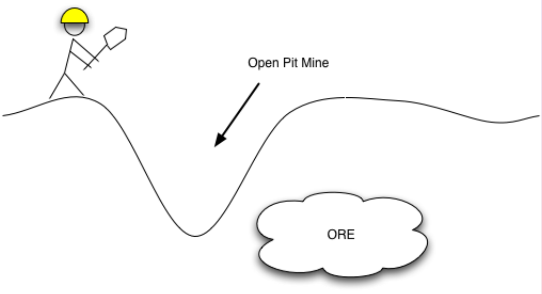
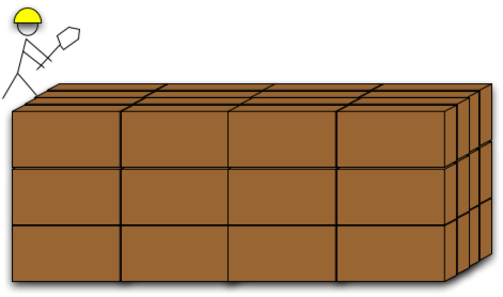
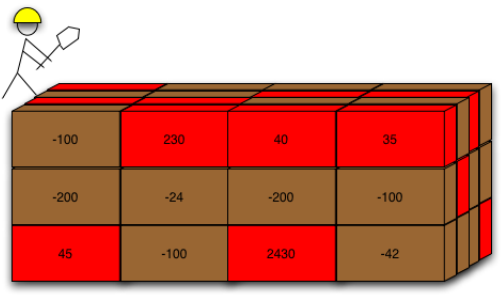
An open pit mine.



- The ground is broken up into sections

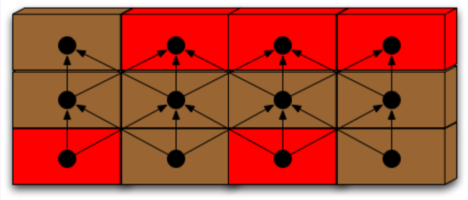


- Using estimation or simulation techniques from drill hole data, economic values are produced for each block



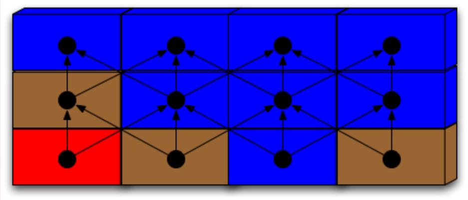
- Ore blocks can return a profit when mined
- Waste blocks cost money to remove

- Each block is considered as a node of a graph
- Arcs are added to represent slope requirements



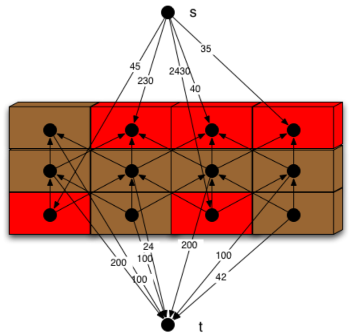
Graph Closure

- A graph closure is a subset S of nodes such that no arcs leave S
- A maximum weight graph closure is known as “the ultimate pit”



Maximum Network Flow

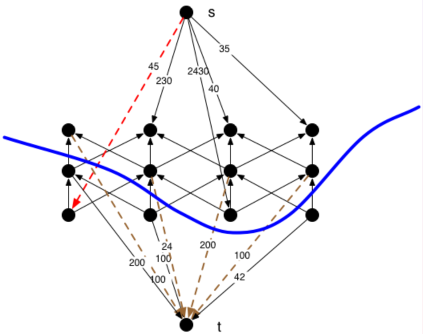
- source node *s* with arcs to each ore node
- sink node *t* with arcs from each waste node



- Capacities on the arcs are the absolute value of the blocks
- Slope arcs have infinite capacity

Minimum Cut

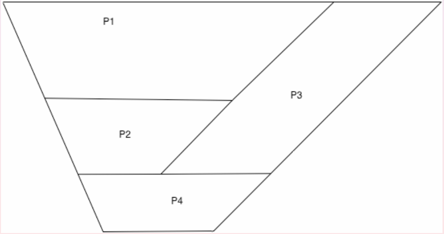
The minimum cut represents the maximum weight graph closure



- Minimize the waste inside and the ore outside the pit

Pushbacks

- The ultimate pit is much to large to produce short term schedules on
- The pit is broken up into smaller more manageable pieces called pushbacks



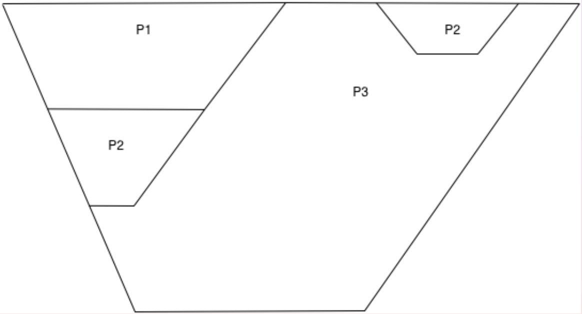
Pushback Design

- There are a number of techniques used to produce pushbacks
- The most popular is to scale some factor affecting the economic block model and run an ultimate pit algorithm

- With an artificially low price of metal - a small pit will be produced

Problems with Existing Pushback Design Methods

- This process is somewhat add-hoc and successive pits may have drastically different sizes and not connected



- Such problems are termed “gap” problems in mining literature

Partially ordered knapsack

- One would like a way to produce a pit with a given knapsack constraint

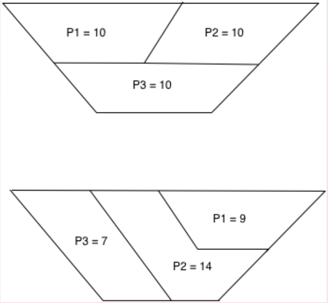
$$\begin{aligned} \max \quad & \sum_{i=1}^{n} w_i x_i \\ s.t. \quad & x_i \leq x_j \qquad \textit{for block } i \textit{ above } j \\ & \sum_{i=1}^{n} c_i x_i \leq b \qquad (1) \\ & x_i \in \{0,1\} \forall i \end{aligned}$$

- Constraint (1) ruins total unimodularity
- No natural way to add a knapsack constraint to the min cut formulation

Discounting

Another problem with existing methods is that they are greedy and don't consider economic discounting

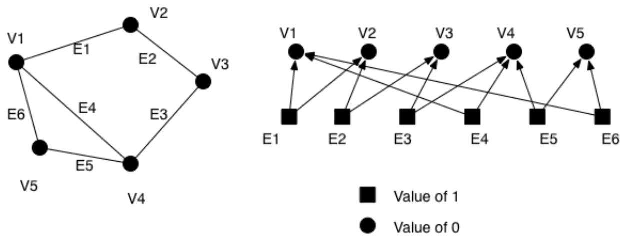
- Discount rate of 10%



- NPV of Design 1 = 27.36
- NPV of Design 2 = 27.51

Complexity of POK

- The POK problem can be shown to be NP-complete from a reduction from maximum clique

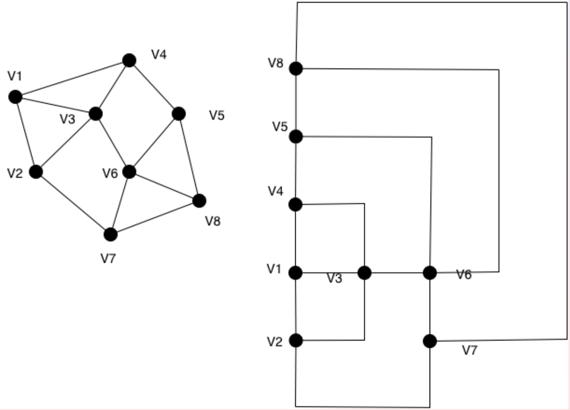


- The graph has a clique of size s if and only if the directed graph has a graph closure of weight $\binom{s}{2}$ with at most $b = \binom{s}{2} + s$ nodes

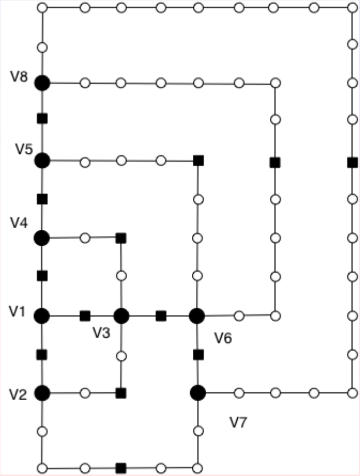
Complexity of connectivity

- This reduction needs doesn't work in the context of the open it problem, the nodes have bounded degree.
- Requiring the blocks removed to be physically connected make the problem NP-complete even for one level (relates to underground).
- Reduction from "Connected node cover in planar graphs of maximum degree 4" (Garey and Johnson)
  - a node cover is a subset of nodes such that each edge has at least one endpoint in the subset
  - a node cover is connected if the graph it induces is connected

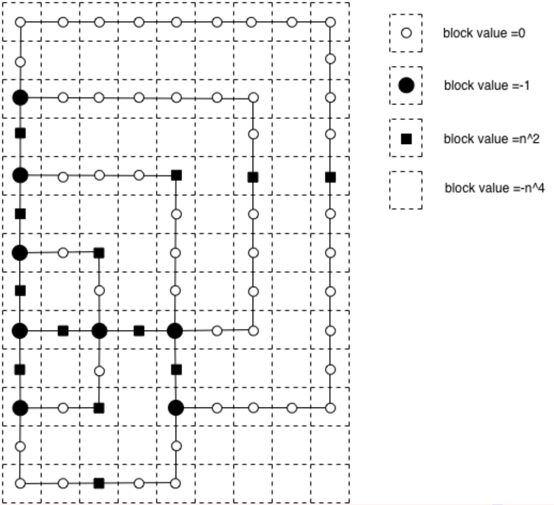
- Given a planar graph of maximum degree 4, Tamassia and Tollis gave an algorithm to embed the graph in a grid of size $O(n^2)$ in linear time



- Bisect the edges to form grid nodes, and identify a special node corresponding to the edge in each path



- The maximum valued subset of connected blocks defines the minimum connected node cover



Pipage Rounding - IP formulation

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} w_i x_i + \sum_{j=1}^{n} p_j y_j \\
s.t. \quad & x_j \leq 1 - y_i \qquad \forall i \in DownCone(j) \\
& \sum_{i=1}^{n} c_j y_j \leq b \\
& x_i, y_j \in \{0, 1\} \qquad \forall i, j
\end{aligned}
$$

- $x_i = 1$ if block i is left in the ground
- $y_i = 1$ if block i is sent to the mill
- c_i, p_i and w_i are respectively the knapsack size, profit and cost associated with block i

We can relax the IP and rewrite it as:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} w_i(1 - \max\{y_j : j \in Cone(i)\}) + \sum_{j=1}^{n} p_j y_j \\ s.t. \quad & \sum_{i=1}^{n} c_i y_i \le b \\ & 0 \le y_i \le 1 \end{aligned}$$

Let $F(x) = \sum_{i=1}^{n} w_i(\prod_{k \in Cone(i)}(1 - y_i)) + \sum_{j=1}^{n} p_j y_j$

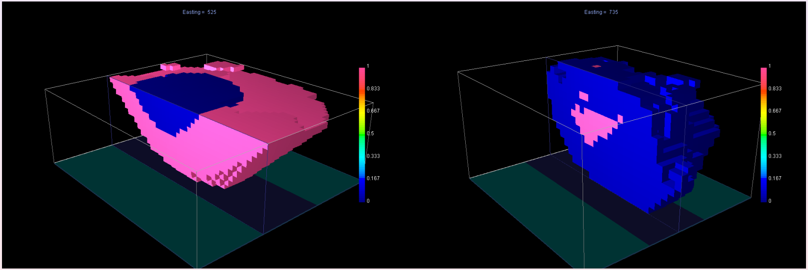
- $F(x)$ equals the objective function at integral vectors (strictly below elsewhere).

- Solve the LP relaxation, to obtain a fractional solution y^*.
- Choose two indices, i' and i'', such that $0 < y^*_{i'}, y^*_{i''} < 1$.
- Set $y^*_{i'} + \epsilon$ and $y^*_{i''} - \epsilon \frac{c_{i'}}{c_{i''}}$ where ϵ is an endpoint of the interval:

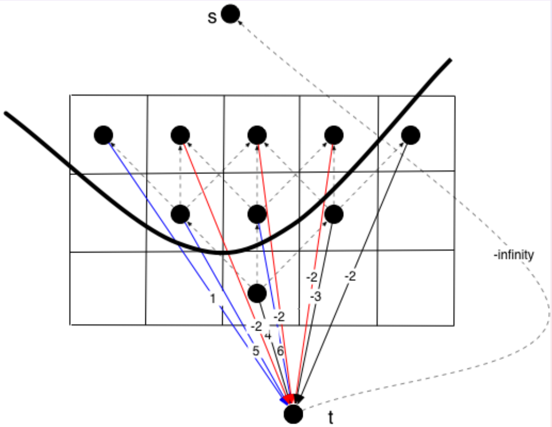
$$[-\min\{y_{i'}, (1 - y_{i''})\frac{c_{i''}}{c_{i'}}\}, \min\{1 - y_{i'}, y_{i''}\frac{c_{i''}}{c_{i'}}\}]$$

- Choose the endpoint such that $F(y(\epsilon)) \geq F(y^*)$

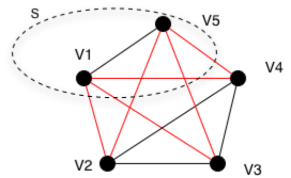
This algorithm performed well on a real data set (within 6.9% of optimal).



The POK problem can be formulated naturally as a maximum directed cut problem with a knapsack constraint.



Maximum Cut Polytope

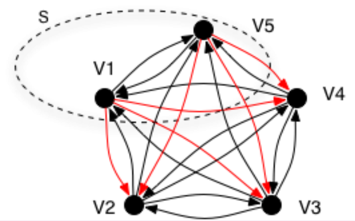


The cut vector for S is:

$$\begin{aligned}\delta(S) &= (x_{12}, x_{13}, x_{14}, x_{1,5}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}) \\ &= (1, 1, 1, 0, 0, 1, 0, 1, 1)\end{aligned}$$

The cut polytope, $\mathrm{CUT}_n^{\square}$, is the convex hull of all cut vectors for K_n.

Maximum Directed Cut Polytope



The directed cut vector for S is:

$$\begin{aligned}\delta^+(S) &= (x_{(1,2)}, x_{(1,3)}, \ldots, x_{(5,3)}, x_{(5,4)}) \\ &= (1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)\end{aligned}$$

The directed cut polytope, $\mathrm{DCUT}_n^{\square}$ is the convex hull of all directed cut vectors of the complete directed graph.

Triangle Inequalities

It's known that for any three nodes i, j, k of K_n the following inequalities are facet inducing for $\mathrm{CUT}_n^{\square}$:

$$x_{ik} - x_{ij} - x_{jk} \leq 0 \quad (2)$$

$$x_{ij} + x_{jk} + x_{ki} \leq 2 \quad (3)$$

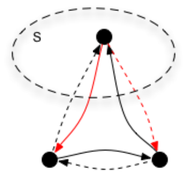
These inequalities for every triple define what is known as the semi-metric polytope $\mathrm{MET}_n^{\square}$. Inequalities (2) define the semi-metric cone MET_n.

- We can prove similar results in the directed case:

$$\begin{aligned} x_{(i,k)} - x_{(i,j)} - x_{(j,k)} &\leq 0 \\ x_{(i,j)} + x_{(j,k)} + x_{(k,i)} &\leq 1 \end{aligned}$$

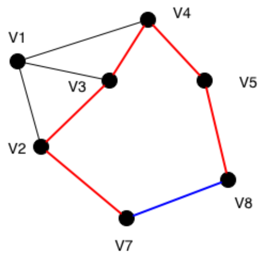
  are facet inducing for $\mathrm{DCUT}_n^\square$.
- We define the directed semi-metric polytope, $\mathrm{DMET}_n^\square$, by the triangle inequalities and:

$$x_{(i,j)} + x_{(j,k)} + x_{(k,i)} = x_{(j,i)} + x_{(k,j)} + x_{(i,k)}.$$



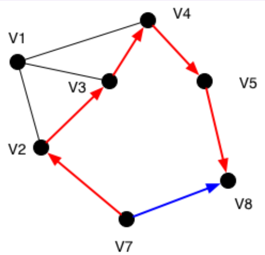
Projecting the Triangle Inequalities

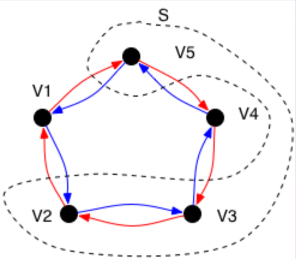
For general graph G, a linear description of the projection of $\text{MET}_n^{\square}$ and MET_n onto $E(G)$ is well understood.



$$\text{MET}(G) = \{x \in \mathbb{R}_+^E | x_e - x(C \setminus \{e\}) \leq 0 \textit{ for C cycle of } G, e \in C\}$$

We have a similar characterization for the projection of DMET_n onto the $A(G)$ for an arbitrary digraph.





$$x_{(7,8)} \leq x_{(7,2)} + x_{(2,3)} + x_{(3,4)} + x_{(4,5)} + x_{(5,8)}$$

$$x_{(1,2)} + x_{(2,3)} +, \ldots, + x_{(5,1)} = x_{(2,1)} + x_{(3,2)} +, \ldots, + x_{(1,5)}$$

Since we can optimize over $\mathrm{DMET}_n^{\square}$ in polynomial time, we can assign an objective function value of 0 to edges not appearing in G and optimize over $\mathrm{DMET}(G)$.

$$\begin{aligned} \max \quad & \sum_{(i,j)\in A(G)} c_{(i,j)} x_{(i,j)} \\ s.t. \quad & x \in DMET_n^{\square} \\ & \sum_{(i,j)\in A(G)} w_{(i,j)} x_{(i,j)} \leq b \end{aligned}$$

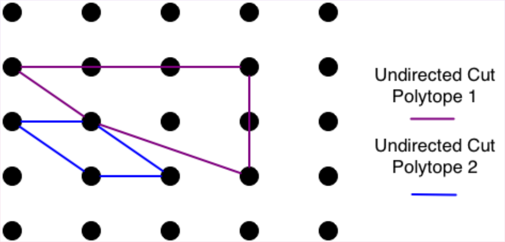
Natural relaxation of the POK problem.

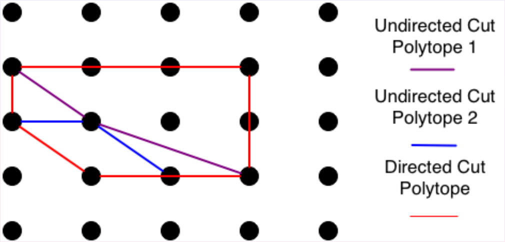
Other results related to the directed cut polytope.

- The dimension of the $\text{DMET}_n^{\square}$ and $\text{DCUT}_n^{\square}$ is $\binom{n}{2} + n - 1$.
- Other facet inducing inequalities: directed versions of hypermetric inequalities (pure, pentagonal,...).
- Bijection between the convex hull of two cut polytopes and the directed cut polytope.
- Switching, permutation and lifting operations for valid inequalities.

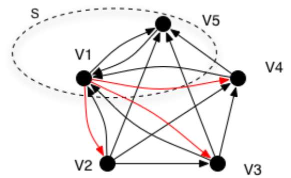
Further Work

- Study the structure of $\mathrm{DMET}^{\square}(G)$ intersected with a knapsack constraint.
- Characterization of when $\mathrm{DMET}^{\square}(G) = \mathrm{DCUT}^{\square}(G)$, for undirected graphs $\mathrm{MET}^{\square}(G) = \mathrm{CUT}^{\square}(G)$ if G is K_5-minor free.
- Complete the linear description of $\mathrm{DMET}^{\square}(G)$.
- Combinatorial algorithm for finding violated projected inequalities for $\mathrm{DMET}(G)$ and $\mathrm{DMET}^{\square}(G)$.


Undirected Cut
Polytope 1
Undirected Cut
Polytope 2


Undirected Cut Polytope 1
Undirected Cut Polytope 2
Directed Cut Polytope

The dimension of the $\mathrm{DCUT}_n^{\square}$ is $\binom{n}{2} + n - 1$



Let $\mathcal{A}$ be the family of all cut vectors, if

$$CUT_n = \{x \in \mathbb{R}^{E_n} | v_i^T x \leq 0 \; for \; i = 1, ..., m\}$$

then

$$CUT_n^{\square} = \{x \in \mathbb{R}^{E_n} | (v_i^{\delta(S)})^T x \leq -v_i(\delta(S)) \; for \; i = 1, ..., m \; and \; \delta(S) \in \mathcal{A}$$

where $v(\delta(S)) = v^T \delta(S) = \sum_{ij \in \delta(S)} v_{ij}$ and $v_e^{\delta(S)} = -v_e$ if $e \in \delta(S)$ and v_e otherwise