Linear and Integer Linear Programming

In a linear programming problem, we are asked to minimize or maximize a linear function subject to a set of linear constraints. A fuller definition can be found in Chapter 1 of Chvatal's book Linear Programming and on pages 770-773 of Cormen. These references also contain some simple examples. Chvatal's chapter contains many exercises which involve formulating simple problems as linear programs (the solutions to many of these exercises are given at the back of the book).

However, our favourite reference for this topic is Professor Fukuda's notes (pages 1-13) to which there is a link fron the lecture summary page. This is in part because his first example involves wine production but mostly because we will use his discussion of the dual problem for a linear program (touched on in class) and so if a student prefers to read only one reference they should read this one.

In an Integer Linear Program, we have the additional restriction that the variables must take integer values.

In class, we formulated some of the problems we had studied previously as LPs and IPs. Specifically, we defined

The Max Flow LP

$max \quad \sum_{e=su} x_e - \sum_{e=us} x_e$
subject to
$\quad \forall v \in V - s - t, \quad \sum_{e=vu} x_e - \sum_{e=uv} x_e = 0$
$\quad \forall e \in E, \quad x_e \leq c_e$
$\quad \forall e \in E, \quad x_e \geq 0$

If we want an integer valued flow then we must insist that the x_e are integer and we have an IP.

We also disucussed:

The Min Cut IP

$min \quad \sum_{e=uv} c_e y_e$
subject to
$\quad \forall e = uv \in E, \quad y_e - z_v + z_u \geq 0$
$\quad \forall v \in V, \quad z_e \geq -1$
$\quad \forall v \in V, \quad z_e \leq 0$
$\quad z_s = -1$
$\quad z_t = 0$
$\quad \forall e \in E, \quad y_e \geq 0$
$\quad$ all y_e and z_v integer.

Note that our constraints ensure that every z_v is -1 or 0. So every feasible solution to the IP points out an $s-t$ cut $S = \{v | z_v = -1\}$. For any given choice of the z_v, our constraints ensure that for every edge $e = uv$ with $u \in S$ and $v \in V - S$, $y_e \geq 1$. So, for any feasible solution to the IP, the objective function value is at least the capacity of the cut S. Hence the optimum solution value for the IP is at least the capacity of a minimum capacity $s - t$ cut.

On the other hand, given a minimum capacity $s - t$ cut S we can obtain a solution whose objective function value is the capacity of S by setting $z_v = -1$ for $v \in S$, setting $z_v = 0$ for $v \notin S$, setting $y_e = 1$ if $e = uv$ for some $u \in S, v \notin S$, and setting $y_e = 0$ for all other e. Thus, we see that solving the IP does tell us the capacity of a minimum capacity $s - t$ cut.

Finally we tried to define an IP for the shortest path problem. We ran into some difficulties, so we decided to assume that there were no negative cycles. This allowed us to formulate the problem as finding the minimum weight integer $s-t$ flow of volume 1, since our results on nice digraphs tell us that such a flow can be decomposed into an $s - t$ path and a set of cycles. Since there are no negative weight cycles, a minimum weight flow of volume 1 uses only cycles of cost zero, and hence the flow and the path have the same cost.

The Shortest Path + Cycles LP

min $\sum_e w_e x_e$
subject to
$\sum_{e=ut} x_e - \sum_{e=tu} x_e = 1$
$\forall v \in V - s - t, \quad \sum_{e=uv} x_e - \sum_{e=vu} x_e = 0$
$\forall e \in E, \quad x_e \geq 0$
all x_e integer.