All Meals for a Dollar

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All Meals for a Dollar

Find the Cheapest Menu!

	Food	Serv. Size	Energy (kcal)	Protein (g)	Calcium (mg)	Price (cents)	Max Serv.
x_1	Oatmeal	28g	110	4	2	3	4
x_2	Chicken	100g	205	32	12	24	3
x_3	Eggs	2 large	160	13	54	13	2
x_4	Whole Milk	237cc	160	8	285	9	8
x_5	Cherry Pie	170g	420	4	22	20	2
x_6	Pork w. beans	260g	260	14	80	19	2
	Min. Daily Amt.		2000	55	800		

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Opt. Solution: $x_1 = 4, x_4 = 9/2, x_5 = 2$. Cost:92.5¢. (Chvatal–1980)

Linear Programming (LP) Formulation

$$\begin{array}{ll} \text{minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\ \text{subject to} & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 3 \\ & 0 \leq x_3 \leq 2 \\ & 0 \leq x_4 \leq 8 \\ & 0 \leq x_5 \leq 2 \\ & 0 \leq x_6 \leq 2 \end{array}$$

$$\begin{aligned} 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\geq 2000 \\ 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\geq 55 \\ 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\geq 800 \end{aligned}$$

Her problem is known as a *diet problem*.

Linear Programming (LP) Formulation

$$
\begin{array}{ll}
\text{minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{subject to} & 0 \le x_1 \le 4 \\
& 0 \le x_2 \le 3 \\
& 0 \le x_3 \le 2 \\
& 0 \le x_4 \le 8 \\
& 0 \le x_5 \le 2 \\
& 0 \le x_6 \le 2
\end{array}
$$

$$
\begin{array}{rcl}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 & \ge & 2000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 & \ge & 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 & \ge & 800
\end{array}
$$

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All Meals for a Dollar

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \leq 100$$
$$0 \leq x_1 \leq 4$$
$$0 \leq x_2 \leq 3$$
$$0 \leq x_3 \leq 2$$
$$0 \leq x_4 \leq 8$$
$$0 \leq x_5 \leq 2$$
$$0 \leq x_6 \leq 2$$

$$\begin{aligned} 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\geq 2000 \\ 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\geq 55 \\ 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\geq 800 \end{aligned}$$

Vertex Enumeration Problem:
Compute all vertices of this polyhedron.

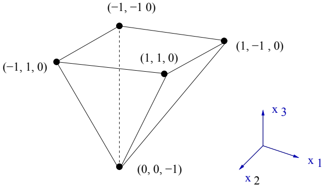
All menus for a $1

All (17) Extreme

Solutions to the Diet Problem with Budget $1.00

Cost	Oat-meal	Chicken	Eggs	Milk	Cherry Pie	Pork Beans
92.5	4.	0	0	4.5	2.	0
97.3	4.	0	0	8.	0.67	0
98.6	4.	0	0	2.23	2.	1.40
100.	1.65	0	0	6.12	2.	0
100.	2.81	0	0	8.	0.98	0
100.	3.74	0	0	2.20	2.	1.53
100.	4.	0	0	2.18	1.88	1.62
100.	4.	0	0	2.21	2.	1.48
100.	4.	0	0	5.33	2.	0
100.	4.	0	0	8.	0.42	0.40
100.	4.	0	0	8.	0.80	0
100.	4.	0	0.50	8.	0.48	0
100.	4.	0	1.88	2.63	2.	0
100.	4.	0.17	0	2.27	2.	1.24
100.	4.	0.19	0	8.	0.58	0
100.	4.	0.60	0	3.73	2.	0
100.	4.	0	1.03	2.21	2.	0.78

Example in R^3



H-representation:

$$1 - x_1 \quad + x_3 \geq 0$$

$$1 \quad - x_2 + x_3 \geq 0$$

$$1 + x_1 \quad + x_3 \geq 0$$

$$1 \quad + x_2 + x_3 \geq 0$$

$$- x_3 \geq 0$$

V-representation:

$$v_1 = (-1, 1, 0), \quad v_2 = (-1, -1, 0), \quad v_3 = (1, -1, 0),$$

$$v_4 = (1, 1, 0), \quad v_5 = (0, 0, -1)$$

Convex Hull and Vertex Enumeration

A convex polyhedron P in R^d has two representations:

H-representation:

A set of m *facet* generating inequalities.

$$P = \{x \in R^d \mid b + Ax \geq 0\}$$

V-representation:

A set of *vertices* $v_1, \cdots, v_s$ and *extreme rays* $z_1, \cdots, z_u$.

$$P = \{x \in R^d \mid x = \sum_{i=1}^{s} \lambda_i y_i + \sum_{j=1}^{u} \mu_j z_j,$$

$$\lambda_i \geq 0, \mu_j \geq 0, \sum_{i=1}^{s} \lambda_i = 1\}.$$

Vertex Enumeration Problem:

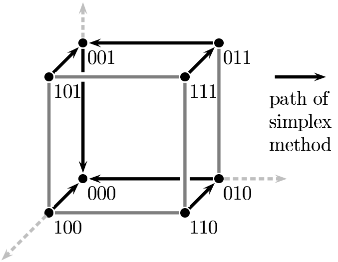
H-representation => V-representation

Facet Enumeration Problem:

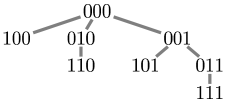
V-representation => H-representation

Reverse search algorithm

http://cgm.cs.mcgill.ca/ avis/C/lrs.html



(a) The "simplex tree" induced by the objective $(-\sum x_i)$.



(b) The corresponding reverse search tree.