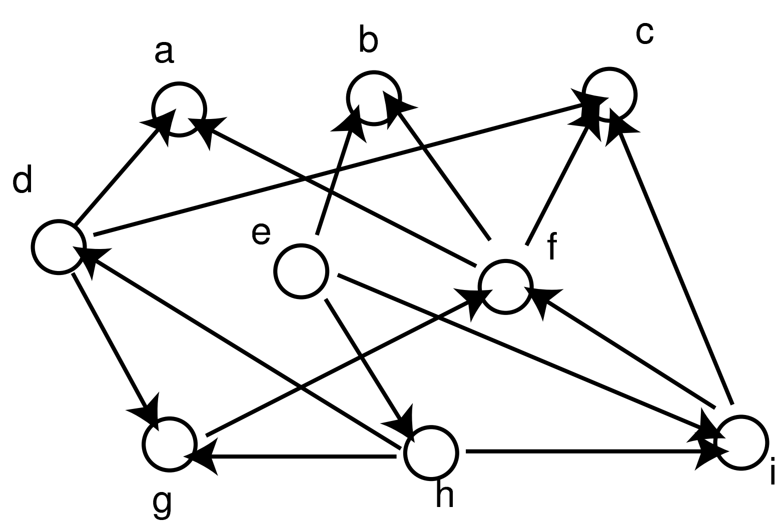
1. Initialise $L[v] \leftarrow -1$ for all v.
2. Construct the $Pred$ tables for G.
4. Call $LongPath(v)$ for each v.
5. Return the maximum of $L[v]$ for $i = 1, 2, \ldots, n$.

$LongPath(v)$
1. **If** $L[v] \geq 0$ **then**
2.   **return** $L[v]$
3. **else**
4.   **if** $Pred[v]$ is empty **then**
5. $L[v] \leftarrow 0$
6.   **else**
7. $L[v] \leftarrow 1 + \max\{LongPath(u) : u \in Pred[v]\}$
8.   **return** $L[v]$.



How to extract a solution

Suppose that $L[v]$ contains the length of the longest path ending in v. How to find the longest path?

Solution: we use a recursive procedure.

1. Compute $L[v]$ for all v.
2. Find v^* such that $L[v^*]$ is maximized.
3. Let P be an empty path.
4. $FindLong(P, L, v^*)$.
5. Output P.

$FindLong(P, L, v)$

6. Add v to the beginning of P.
7. **If** v is not a source **then**
8.   find $u \in Pred[v]$ such that $L[v] = L[u] + 1$
9. $FindLong(P, L, u)$

Problem: step 8 can take mroe time than necessary...

Storing information about optimal solutions

For each v, store a vertex $u \in Pred[v]$ that preceeds v in a longest path.

$LongPath2(v)$

If $L[v] \geq 0$ **then**
  return $L[v]$
else
  if $Pred[v]$ is empty **then**
    $L[v] \leftarrow 0$
  else
    Find $u^* \in Pred[v]$ that maximizes $LongPath(u^*)$.
    $\pi[v] \leftarrow u^*$
    $L[v] \leftarrow 1 + LongPath(u^*)$
  return $L[v]$.

Then use the table π to construct the optimal (replacing earlier procedure $FindLong$).

$FindLong2(v)$

Add v to the beginning of P.
If v is not a source **then**
  $u \leftarrow \pi[v]$
  $FindLong(P, L, u)$

Takes time linear in the number of vertices.

How many optimal solutions are there?

We can also quickly compute the number of optimal solutions (e.g. the number of longest paths).

Let $m(v)$ denote the number of longest paths finishing with vertex v.

If v is a source (no incoming edges) then $m(v) = 1$. Otherwise, the longest path ending in v must have entered v via one of the vertices $u \in Pred[v]$. If this is the case, then the length of the longest path to u must be $L[v] - 1$. So....

$$m(v) = \sum_{u \in Pred[v] : L[u] = L[v]-1} m(u),$$

that is, the sum of $m(u)$ over all $u \in Pred[v]$ such that

$$L[v] = L[u] + 1.$$

Exercise: Come up with an algorithm that computes $m(v)$ for all v. Answers in tutorials.

CASE STUDY 1 - Matrix Chain Multiplication

See also Cormen chpt 16. (chpt 15 in the 2nd edition).

Let A and B be two matrices.

- The product AB is defined only if the number of columns of A equals the number of rows of B.

- If A has dimensions $p \times q$ and B has dimensions $q \times r$ then AB has dimensions $p \times r$, and it takes pqr scalar multiplications to compute AB.

- Given three matrices A, B, C (with compatible dimensions)
$$((AB)C) = (A(BC))$$

- Even though $(AB)C$ and $A(BC)$, they can take quite different times to compute.

- e.g. if A is 100×10, B is 10×50, C is 50×5 then computing $(AB)C$ takes $100 \cdot 10 \cdot 50 + 100 \cdot 50 \cdot 5 = 75000$ operations while computing $A(BC)$ takes $10 \cdot 50 \cdot 5 + 100 \cdot 10 \cdot 5 = 7500$ operations.

Matrix-chain muliplication problem

We are given matrices $A_1 A_2 \cdots A_n$.
Matrix A_i has dimensions $p_{i-1} \times p_i$.

Problem: What is the minimum number of scalar multiplications needed to evaluate
$A_1 A_2 A_3 \cdots A_n$?

Subproblem:
For each i, j such that $1 \leq i \leq j \leq n$:
What is the minimum number of scalar multiplications needed to evaluate
$A_i A_{i+1} A_{i+2} \cdots A_j$?

Note: the matrix $A_i A_{i+1} \cdots A_j$ has p_{i-1} rows and p_j columns.

Looking for the recursion

Let $m[i,j]$ denote the number of scalar multiplications needed to evaluate $A_i A_{i+1} \cdots A_j$.

If $i = j$ then $m[i,j] = 0$ since we just have to get the matrix A_i.

If $k \geq i$ and $k < j$ then the minimum number of scalar multiplications needed to evaluate

$$(A_i A_{i+1} \ldots A_k)(A_{k+1} \cdots A_j)$$

is

(the min. number needed to compute $A_i A_{i+1} \ldots A_j$)

+(the min. number needed to compute $A_i A_{i+1} \ldots A_j$)

+(the operations needed to multiply the two matrices)

That is,

$$m[i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

In order to find the minimum, we choose the k that minimizes this expression.

Recursion and algorithm

- If $i = j$ then $m[i,j] = 0$.
- If $i < j$ then

$$m[i,j] = \min_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\}$$

We can evaluate this using memoization.

1. $m[i,j] \leftarrow -1$ for all $1 \le i \le j \le n$.
2. Output $MCR(1,n)$

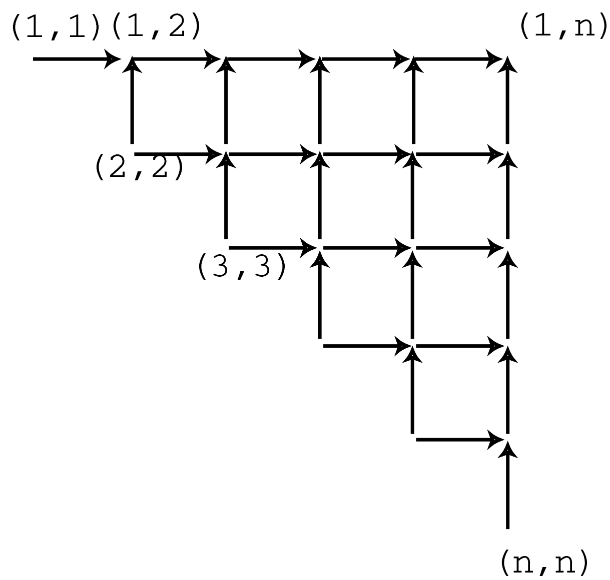
$MCR(i,j)$.
3. **if** $m[i,j] \ge 0$ **then**
4. **return** $m[i,j]$
5. **else**
6. **if** $i = j$ **then**
7. $m[i,j] = 0$.
8. **else**
9. $m[i,j] \leftarrow \min_{i \le k < j} \{MCR[i,k] + MCR[k+1,j] + p_{i-1}p_k p_j\}$
10. **return** $m[i,j]$.

This takes $O(n^3)$ time.

Recursion-free version

The subproblem for (i, j) depends on subproblem (i, k) and also on subproblem $(k + 1, j)$ for all $i \leq k \leq j$.

Dependency graph looks like:



So a loop like:

for $i \leftarrow 1$ to n
  for $j \leftarrow 1$ to n
    Compute $m[i, j]$

won't work. Instead we need something like:

for $l \leftarrow 1$ to n
  for all i, j such that $j = i + l - 1$
    Compute $m[i, j]$

Recursion free dynamic programming solution

$MatrixChainOrder(p_1, p_2, \ldots, p_n)$.

for $l \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to $n - l + 1$ **do**
    $j \leftarrow i + l - 1$ [so $[i, j]$ contains l elements]
    if $i = j$ **then**
      $m[i, j] \leftarrow 0$
    else
      $m[i, j] \leftarrow \min_{i \leq k < j} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\}$

Multiplying the matrices

We have computed the minimum number of multiplications required. How do we go about multiplying the matrices.

We assume that there is a library function $MatrixMultiply(A, B)$ that multiplies A and B.

The following returns the product of $A_i A_{i+1} \ldots A_j$.

$Multiply(i, j)$
if $i = j$ **then**
    return A_i.
else
    find k such that $m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$.
    $A_L \leftarrow Multiply(i, k)$
    $A_R \leftarrow Multiply(k + 1, j)$
    return $MatrixMultiply(A_L, A_R)$.