Algorithm Design Techniques - COMP 360

October 24, 2013 MIDTERM EXAM 8:35-9:55 am

NAME:_____ STUDENT ID:_____

No laptops, calculators, cell phones, books, notes or cheating.
This exam has 5 pages and 4 questions for a total of 50 points.
Write all answers on this exam, using the backs of pages if necessary.

1(a) (3 pts) What are the main features of a local improvement algorithm for an optimization problem?

1. Start at a feasible solution
2. Improve the solution by making local changes
3. Terminate when no improving solution is found

(b) (3 pts) What is meant by a "certificate of correctness" in relation to the output for a given problem. (The problem need not be a decision problem).

It is additional information (if necessary) that allows the output to be verified in polynomial time.

(c) (6 pts) Explain why the Ford-Fulkerson algorithm for finding the maximum flow in a network is a local improvement algorithm. Describe a certificate of correctness for the output.

1. Initially start with a zero flow. Source s, Sink t.
2. Use the flow augmenting path algorithm to increase the flow from s to t.
3. If none terminate. The certificate is a cut set $[S, \bar{S}]$ with capacity = the flow out of s.

S = {all nodes reachable from source s by a flow augmenting path}

2(a) (1 pt) Write down the standard form for a linear program.

$$\max\ z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, \dots, m$$

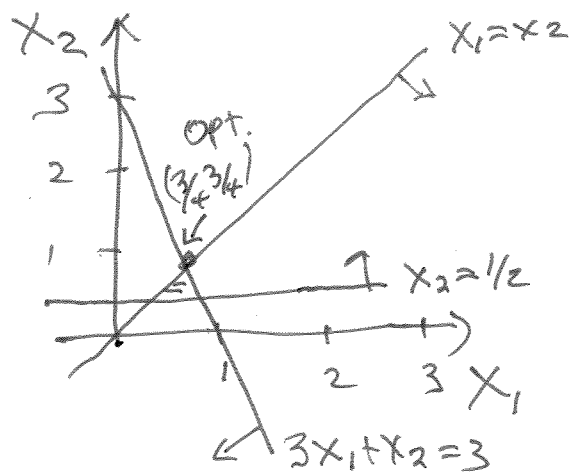
$$x_j \ge 0 \qquad j = 1, \dots, n$$

(b) (3 pts) What are the possible outcomes for a linear program?

1. Optimum solution
2. Infeasible solution
3. Unbounded solution

(c) (6 pts) For the following linear program determine its outcome (by any method) and give a certificate of correctness. Give a sketch to illustrate your answer.

$$z^* = \max z = x_1 + x_2$$
$$-x_1 + x_2 \le 0 \quad (1)$$
$$3x_1 + x_2 \le 3 \quad (2)$$
$$-2x_2 \le -1$$
$$x_1 \ge 0,\ x_2 \ge 0$$



The constraints define a triangle with vertices $(3/4, 3/4)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{5}{6}, \frac{1}{2})$

The maximum $z^* = \frac{3}{2}$ is obtained at $(3/4, 3/4)$.

Certificate: Adding rows (1) and (2) we get

$2x_1 + 2x_2 \le 3$ and so $x_1 + x_2 \le \frac{3}{2}$ for any feasible solution. So $z^* = \frac{3}{2}$ is optimum.

3. Consider the following LP dictionary for a maximization problem. You are about to apply the simplex method to it.

$$\begin{array}{rl} x_1 = 1 & + x_2 \quad - 2x_4 \; - x_6 \\ x_3 = 2 & - x_2 \qquad\qquad - x_6 \\ x_5 = 1 & - x_2 \\ z = 4 & + 2x_2 - x_4 - x_6 \end{array}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, 6$$

(a) (2 pts) What is the current basis and the current basic solution?

Basis $B = \{1, 3, 5\}$ $x_1 = 1$ $x_2 = 0$ $x_3 = 1$ $x_4 = x_6 = 0$ $x_5 = 1$

(b) (1 pt) What are the candidate(s) for entering the basis?

x_2

(c) (1 pt) Choose one of the candidate(s) in (b). Which variable will be chosen to leave the basis?

x_5

(d) (6 pts) Pivot using the variables in your answer for (c). Write down the resulting dictionary and the new basic solution. Is it optimal? If so, why? If not, why not?

$$\boxed{\begin{array}{l} x_1 = 2 - x_5 - 2x_4 - x_6 \\ x_3 = 1 + x_5 \qquad\qquad - x_6 \\ x_2 = 1 - x_5 \\ \hline z = 6 - 2x_5 - x_4 - x_6 \end{array}}$$

Basic Solution

$x_4 = x_5 = x_6 = 0$

$x_1 = 2$ $x_2 = 1$ $x_3 = 1$

$z = 6$

Yes it is optimum because all coefficients in the z-row are non-positive.

4. Let $G = (V, E)$ be an undirected graph with n vertices. Recall that an *independent* set in G is a subset S of vertices such that there is no edge in E between any two vertices in S. A subset T of vertices is a *vertex cover* if every edge in G is incident to at least one vertex in T.
Consider the problems:
IS: Input: Graph G, integer k.
Question: Is there an independent set of size at least k?

VC: Input: Graph G, integer m.
Question: Is there a vertex cover of size at most m?

(a) (3 pts) Prove that if S is an independent set in G then the complementary set of vertices $V - S$ is a vertex cover.

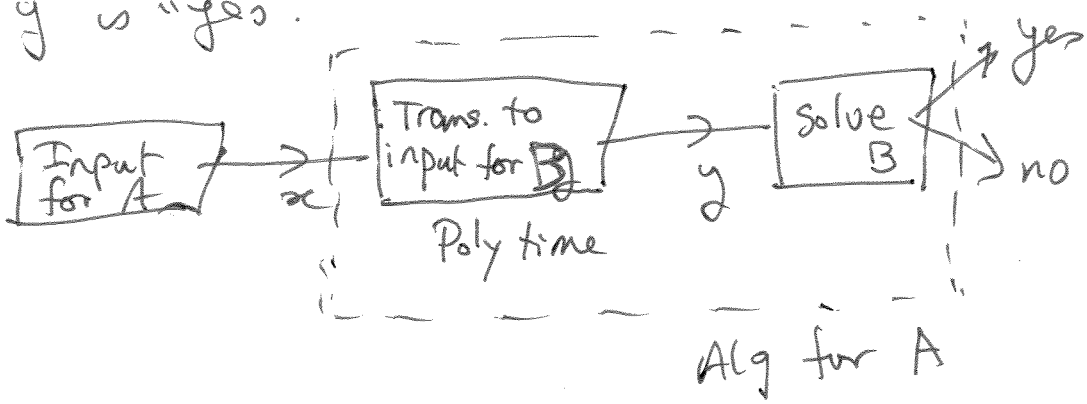
If S is an independent set then every edge $e \in E$ has at least one endpoint <u>not</u> in S, ie. in $V-S$. So $V-S$ has vertices incident to every edge in E and is a vertex cover. Note if $|S| = k$ then $|V-S| = n-k$, where $n = |V|$.

(b) (3 pts) Define the class *NP* and show that both IS and VC are in NP.

NP is the class of decision problems for which the "yes" answer can be verified in poly time using a certificate.
IS & VC are decision problems. The certificate for IS is an independent set S of size $\leq K$. We can check it has no edges in $O(|E|)$ time. For VC it is a subset T of vertices, $|T| \leq M$, and we can check each edge is covered in $O(E)$ time

(c) (3 pts) Define what is means for a decision problem A to be reducible in polynomial time to a decision problem B, ie. $A \leq_P B$. Use a diagram to illustrate the idea.

Every input x for A can be transformed in poly time to an input y for B, the answer to x is "yes" iff the answer to y is "yes".



(d) (3 pts) Define what it means for a problem A to be NP-complete and what it means for A to be NP-hard.

A is NP-complete if (a) it is in NP (b) for every $B \in NP$ we have $B \leq_p A$.

A is NP-hard if for every $B \in NP$ we have $B \leq_p A$.

(e) (6 pts) Show that $IS \leq_P VC$. If we know IS is NP-complete, what can we conclude about VC?

(Using an argument similar to (a) we see that if $V-S$ is a vertex cover then, S is an independent set. Therefore S is an ind. set $\Longleftrightarrow$ $V-S$ is a vertex cover. (*))

Take any input $\overbrace{G=(V,E), k}^{x}$ for IS and construct the input $\underbrace{G=(V,E)\ m=n-k}_{y}$ for VC, where $n=|V|$.

By (*) the answer to y is "yes" iff the answer to x is "yes" because G has VC of size $\leq n-k$ iff it has an independent set of size k.

Therefore $IS \leq_p VC$. Since IS is NP-complete for every $A \in NP$ $A \leq_p IS$. Therefore by transitivity $A \leq_p IS \leq_p VC$ and so VC is NP-hard. Since it is in NP (part (b)) we have proved it is NP-complete.