Question #1: Snacks and foods that will be part of my low cost diet:

Table 1: Nutritional information

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Description	Milk 1% fat with calcium added (250 ml)	30 g cereal serving (Rice Krispies)	Apple-tropical fruit snack	2 slices of multi-grain bread	Orange juice (250 ml)	Almonds (12)	Peanut butter (2 Tbsp)	Raw carrot (1)	Boiled egg	Cheddar cheese (25 g)
Vitamin A (RE)	120	0	200	0	50	0	0	3000	0	80
Vitamin D (mcg)	2.25	0	0	0	0	0	0	0	1	0.1
Vitamin E (mg)	0	0	0.1	0.2	0	3.9	3.15	0.5	1	0.2
Vitamin C (mg)	0	0	18	0	60	0	0	2	0	0
Calcium (mg)	312	10	10	48	12	50	10	50	30	200
Iron (mg)	0.14	3.92	0.1	2.24	0.4	0.8	0.5	1	1.5	0.1
Protein (g)	9.2	2.1	0.2	7.1	2	3	6.2	1.1	6	7.8
Energy (Cal)	116	110	80	161	112	70	170	40	71	128
Price ($)	0.35	0.34	0.39	0.33	0.21	0.23	0.02	0.15	0.30	0.31
Max I want to eat per day	2	2	1	1.5	3	2	1.5	1	2	4

a) Primal problem

Objective function:
max $-0.35x_1 - 0.34x_2 - 0.39x_3 - 0.33x_4 - 0.21x_5 - 0.23x_6 - 0.02x_7 - 0.15x_8 - 0.30x_9 - 0.31x_{10}$;
Which is equivalent to
min $0.35x_1 + 0.34x_2 + 0.39x_3 + 0.33x_4 + 0.21x_5 + 0.23x_6 + 0.02x_7 + 0.15x_8 + 0.30x_9 + 0.31x_{10}$;

Constraints:
VitA (y1): $120x_1 + 200x_3 + 50x_5 + 3000x_8 + 80x_{10} >= 1000$;
VitD (y2): $2.25x_1 + 1.0x_9 + 0.1x_{10} >= 5$;
VitE (y3): $0.1x_3 + 0.2x_4 + 3.9x_6 + 3.15x_7 + 0.5x_8 + x_9 + 0.2x_{10} >= 10$;
VitC (y4): $18x_3 + 60x_5 + 2x_8 >= 60$;
Cal (y5): $312x_1 + 10x_2 + 10x_3 + 48x_4 + 12x_5 + 50x_6 + 10x_7 + 50x_8 + 30x_9 + 200x_{10} >= 800$;
Iron (y6): $0.14x_1 + 3.92x_2 + 0.1x_3 + 2.24x_4 + 0.4x_5 + 0.8x_6 + 0.5x_7 + x_8 + 1.5x_9 + 0.1x_{10} >= 14$;
Prot (y7): $9.2x_1 + 2.1x_2 + 0.2x_3 + 7.1x_4 + 2x_5 + 3x_6 + 6.2x_7 + 1.1x_8 + 6x_9 + 7.8x_{10} >= 55$;
Ener (y8): $116x_1 + 110x_2 + 80x_3 + 161x_4 + 112x_5 + 70x_6 + 170x_7 + 40x_8 + 71x_9 + 128x_{10} <= 1900$;
bound1 (y9): $x_1 <= 2$;
bound2 (y10): $x_2 <= 2$;
bound3 (y11): $x_3 <= 1$;
bound4 (y12): $x_4 <= 1.5$;
bound5 (y13): $x_5 <= 3$;
bound6 (y14): $x_6 <= 2$;
bound7 (y15): $x_7 <= 1.5$;
bound8 (y16): $x_8 <= 1$;
bound9 (y17): $x_9 <= 2$;
bound10 (y18): $x_{10} <= 4$;

Dual problem

Objective function:
max $1000y_1 + 5y_2 + 10y_3 + 60y_4 + 800y_5 + 14y_6 + 55y_7 – 1900y_8 - 2y_9 - 2y_{10} - y_{11} - 1.5y_{12} - 3y_{13} - 2y_{14} - 1.5y_{15} - y_{16} - 2y_{17} - 4y_{18}$;

Constraints:
x1: $120y_1 + 2.25y_2 + 312y_5 + 0.14y_6 + 9.2y_7 - 116y_8 – y_9 <= 0.35$;
x2: $10y_5 + 3.92y_6 + 2.1y_7 - 110y_8 –y_{10} <= 0.34$;
x3: $200y_1 + 0.1y_3 + 18y_4 + 10y_5 + 0.1y_6 + 0.2y_7 - 80y_8 –y_{11} <= 0.39$;
x4: $0.2y_3 + 48y_5 + 2.24y_6 + 7.1y_7 - 161y_8 –y_{12} <= 0.33$;
x5: $50y_1 + 60y_4 + 12y_5 + 0.4y_6 + 2y_7 – 112y_8 –y_{13} <= 0.21$;
x6: $3.9y_3 + 50y_5 + 0.8y_6 + 3y_7 - 70y_8 –y_{14} <= 0.23$;
x7: $3.15y_3 + 10y_5 + 0.5y_6 + 6.2y_7 - 170y_8 –y_{15} <= 0.02$
x8: $3000y_1 + 0.5y_3 + 2y_4 + 50y_5 + 1y_6 + 1.1y_7 - 40y_8 –y_{16} <= 0.15$
x9: $1y_2 + 1y_3 + 30y_5 + 1.5y_6 + 6y_7 - 71y_8 –y_{17} <= 0.30$
x10: $80y_1 + 0.1y_2 + 0.2y_3 + 200y_5 + 0.1y_6 + 7.8y_7 - 128y_8 –y_{18} <= 0.31$;

b) Solution as given by lp_solve:

For the primal problem:

Value of objective function:	-2.618667
x10	0
x1	1.8691
x2	1.4868
x3	0
x4	1.5
x5	0.99193
x6	0.81579
x7	1.5
x8	0.24204
x9	1.6724

Dual values:

vitA (y1)	-6.9156e-06
vitD (y2)	0
vitE (y3)	-0.019646
vitC (y4)	-0.0020463
cal (y5)	-0.00027545
iron (y6)	-0.071285
prot (y7)	-0.027527
ener (y8)	0

For the dual problem:

Value of objective function:	2.618667
y10	0
y11	0
y12	0.042273
y13	0
y14	0
y15	0.25095
y16	0
y17	0
y18	0
y1	6.9156e-06
y2	0
y3	0.019646
y4	0.0020463
y5	0.00027545

y6	0.071285
y7	0.027527
y8	0
y9	0

Dual values:

x1	1.8691
x2	1.4868
x3	0
x4	1.5
x5	0.99193
x6	0.81579
x7	1.5
x8	0.24204
x9	1.6724
x10	0

These results show that the solution given for the primal and the dual problems are the same. The values for the duals given by lp_solve when solving for the primal problem are negative because the minimization problem was written as a maximization problem as necessary in lp_solve. However, the magnitude of all the values is the same for both the primal and the dual problems. Therefore, the optimal cost of my diet is approximately 2.62$.

c) Verification of optimality

Our optimal solution for the primal problem is 2.618667. Let's verify its optimality.

y1 * ($120x_1 + 200x_3 + 50x_5 + 3000x_8 + 80x_{10} >= 1000$)
y2 * ($2.25x_1 + 1.0x_9 + 0.1x_{10} >= 5$)
y3 * ($0.1x_3 + 0.2x_4 + 3.9x_6 + 3.15x_7 + 0.5x_8 + x_9 + 0.2x_{10} >= 10$)
y4 * ($18x_3 + 60x_5 + 2x_8 >= 60$)
y5 * ($312x_1 + 10x_2 + 10x_3 + 48x_4 + 12x_5 + 50x_6 + 10x_7 + 50x_8 + 30x_9 + 200x_{10} >= 800$)
y6 * ($0.14x_1 + 3.92x_2 + 0.1x_3 + 2.24x_4 + 0.4x_5 + 0.8x_6 + 0.5x_7 + x_8 + 1.5x_9 + 0.1x_{10} >= 14$)
y7 * ($9.2x_1 + 2.1x_2 + 0.2x_3 + 7.1x_4 + 2x_5 + 3x_6 + 6.2x_7 + 1.1x_8 + 6x_9 + 7.8x_{10} >= 55$)
y8 * ($116x_1 + 110x_2 + 80x_3 + 161x_4 + 112x_5 + 70x_6 + 170x_7 + 40x_8 + 71x_9 + 128x_{10} <= 1900$)
y9 * ($x_1 <= 2$)
y10 * ($x_2 <= 2$)
y11 * ($x_3 <= 1$)
y12 * ($x_4 <= 1.5$)
y13 * ($x_5 <= 3$)
y14 * ($x_6 <= 2$)
y15 * ($x_7 <= 1.5$)
y16 * ($x_8 <= 1$)
y17 * ($x_9 <= 2$)
y18 * ($x_{10} <= 4$)

We know the values for y1 … y18 for the optimal solution (see section b).

6.9156e-06 * ($120x_1 + 200x_3 + 50x_5 + 3000x_8 + 80x_{10} >= 1000$)
0 * ($2.25x_1 + 1.0x_9 + 0.1x_{10} >= 5$)
0.019646 * ($0.1x_3 + 0.2x_4 + 3.9x_6 + 3.15x_7 + 0.5x_8 + x_9 + 0.2x_{10} >= 10$)
0.0020463 * ($18x_3 + 60x_5 + 2x_8 >= 60$)
0.00027545 * ($312x_1 + 10x_2 + 10x_3 + 48x_4 + 12x_5 + 50x_6 + 10x_7 + 50x_8 + 30x_9 +200x_{10} >= 800$)
0.071285 * ($0.14x_1 + 3.92x_2 + 0.1x_3 + 2.24x_4 + 0.4x_5 + 0.8x_6 + 0.5x_7 + x_8 +1.5x_9 +0.1x_{10} >= 14$)
0.027527 * ($9.2x_1 + 2.1x_2 + 0.2x_3 + 7.1x_4 + 2x_5 + 3x_6 + 6.2x_7 + 1.1x_8 + 6x_9 + 7.8x_{10} >= 55$)
0 * ($116x_1 + 110x_2 + 80x_3 + 161x_4 + 112x_5 + 70x_6 + 170x_7 + 40x_8 + 71x_9 + 128x_{10} <= 1900$)
0 * ($x_1 <= 2$)
0 * ($x_2 <= 2$)
0 * ($x_3 <= 1$)
0.042273 * ($x_4 <= 1.5$)
0 * ($x_5 <= 3$)
0 * ($x_6 <= 2$)
0.25095 * ($x_7 <= 1.5$)
0 * ($x_8 <= 1$)
0 * ($x_9 <= 2$)
0 * ($x_{10} <= 4$)

This gives us:

$0.000829872x_1 + 0.00138312x_3 + 0.00034578x_5 + 0.0207468x_8 + 0.000553248x_{10} >=$ 0.0069156

$0.0019646x_3 + 0.0039292x_4 + 0.0766194x_6 + 0.0618849x_7 + 0.009823x_8 + 0.019646x_9 + 0.0039292x_{10} >= 0.19646$)
$0.0368334x_3 + 0.122778x_5 + 0.0040926x_8 >= 0.122778$)
$0.0859404x_1 + 0.0027545x_2 + 0.0027545x_3 + 0.0132216x_4 + 0.0033054x_5 + 0.0137725x_6 + 0.0027545x_7 + 0.0137725x_8 + 0.0082635x_9 + 0.05509x_{10} >= 0.22036$)
$0.0099799x_1 + 0.2794372x_2 + 0.0071285x_3 + 0.1596784x_4 + 0.028514x_5 + 0.057028x_6 + 0.0356425x_7 + 0.071285x_8 + 0.1069275x_9 + 0.0071285x_{10} >= 0.99799$)
$0.2532484x_1 + 0.0578067x_2 + 0.0055054x_3 + 0.1954417x_4 + 0.055054x_5 + 0.082581x_6 + 0.1706674x_7 + 0.0302797x_8 + 0.165162x_9 + 0.2147106x_{10} >= 1.513985$
$-0.042273\ x_4 >= -0.0634095$
$-0.25095\ x_7 >= -0.376425$

When summed up together:

0.349998572x1 + 0.3399984x2 + 0.10511812x3 + 0.329979x4 + 0.20999718x5 + 0.2300009x6 + 0.0199993x7 + 0.1401766x8 + 0.299999x9 + 0.281411548x10 >= 2.6186541

The function we wanted to minimize in the first place was:
$0.35x_1 + 0.34x_2 + 0.39x_3 + 0.33x_4 + 0.21x_5 + 0.23x_6 + 0.02x_7 + 0.15x_8 + 0.30x_9 + 0.31x_{10}$

Therefore, by comparing the 2 functions,
$0.35x_1 + 0.34x_2 + 0.39x_3 + 0.33x_4 + 0.21x_5 + 0.23x_6 + 0.02x_7 + 0.15x_8 + 0.30x_9 + 0.31x_{10} >= 0.349998572x_1 + 0.3399984x_2 + 0.10511812x_3 + 0.329979x_4 + 0.20999718x_5 + 0.2300009x6 + 0.0199993x_7 + 0.1401766x_8 + 0.299999x_9 + 0.281411548x_{10}$
$>= 2.6186541$

And thus
$0.35x_1 + 0.34x_2 + 0.39x_3 + 0.33x_4 + 0.21x_5 + 0.23x_6 + 0.02x_7 + 0.15x_8 + 0.30x_9 + 0.31x_{10} >= 2.6186541$

Therefore, the cost of my nutritional diet should be at least 2.6186541, which is extremely close to the value lp_solve finds for the optimal solution of 2.618667. And thus, the solution lp_solve found must be the actual optimal solution.

As well, by the weak duality theorem, since I find a feasible solution to the primal problem that is equal to a feasible solution to the dual problem, it must be that both these solutions are optimal for their respective problem. Therefore the cost of approximately 2.62$ is the lowest I will get given that I chose the 10 above foods.

The dual variables represent the cost for one unit of each of the vitamins and minerals they represent. In the dual problem, we are trying to minimize the cost of the diet by choosing the appropriate amount of each food (each different food has a different amount of vitamins and minerals for a certain price), given the price and nutrition of each food and the minimum nutritional requirements.

<u>Remarks and common mistake:</u>
1) Each bound constraint in primal creates a dual variable. Hence, most of you had 18 primal inequalities and thus needed 18 dual variables.
2) You cannot impose the primal variables to be INTEGERS!!! because you then have an Integer Program which is NP-complete and where you don't have a strong duality theorem. This means the solution of the dual problem will not be the same as the primal one.
3) I accepted the calorie constraint to be an upper bound or a lower bound on the calories. People did both.
4) Look at the solution carefully... (borrowed from one student in the class)

<u>Marks:</u>
(a) primal 5 pts and dual 5 pts
(b) primal solution 5 pts and dual solution 5 pts
(c) interpretation 3 pts
proof 7 pts

Question #2

Everybody did very well on this question except for the certificates in some cases. Especially in (a), even if you get a contradiction in the constraints which shows that the problem is infeasible, it is NOT the certificate. You need to use Farkas Lemma. Really re-read the hand-out and chap. 2 before the final!

<u>Marks:</u>
(a) certificate 3 pts
primal-argument-LP_solve 7 pts
(b) certificate 3 pts
dual-argument-LP_solve 7 pts