308-250 **Notes on big-O Notation** **D. Avis**

Definition A function $f(n)$ is $O(g(n))$ if there exist two positive constants k and n_0 such that

$$|f(n)| \leq kg(n) \qquad \text{for } all\ n \geq n_0. \tag{1}$$

In working with big-O notation it is important to note the difference between proving that a function $f(n)$ is $O(g(n))$ and in proving that it is **not** $O(g(n))$. In the first case, you just have to give suitable constants and verify the inequality. In the second case, you have to show that **no** such constants exist.

1. Proving $f(n)$ is $O(g(n))$

1.1. Inequality Method.

In this method we guess suitable constants by inspection and try to verify the inequality. Note that the constants can be as big as you like, but they cannot depend on n.

Example 1: $f(n) = \dfrac{5n(n+12)}{3}$ is $O(n^2)$.

In this case we can see the high order term of f is $5n^2/3$, so let us try $k = 2$. We have to find n_0 such that for $n \geq n_0$ we have

$$\frac{5n^2}{3} + 20n \leq 2n^2$$

Multiplying by 3 and dividing by n we require

$$5n + 60 \leq 6n$$

or $n \geq 60$. So in this case $k = 2$ and $n_0 = 60$ satisfy the definition. You should check that $k = 5$ $n_0 = 6$ also work, as do an infinite number of other pairs of constants. □

For more complicated examples it is useful to know the inequalities below that are valid for $n \geq 16$ (why not for smaller n?).

$$\log_2 n < \sqrt{n} < n < n \log_2 n < n^2 < n^3 < 2^n < n! < n^n \tag{2}$$

It is also useful to know that

$$\log_k n < \log_2 n \qquad \text{for } k > 2 \text{ and } n \geq 2.$$

This applies when $k = e$, ie. when the log on the left hand side is the natural log, $\ln n$.

Example 2: $f(n) = 6n \ln n + 12n^{3/2} + n^2$ is $O(n^2)$.

Using (2) we see that for $n \geq 16$

$$6n \ln n + 12n^{3/2} + n^2 \leq 6n \log_2 n + 12n\sqrt{n} + n^2 \leq 6n^2 + 12n^2 + n^2 = 19n^2$$

so the values $k = 19$ and $n_0 = 16$ will work. □

1.2. Limit Method.

In this method we use the fact that we can prove (1) by showing

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < k \tag{3}$$

Example 1 (ctd): $f(n) = \dfrac{5n(n+12)}{3}$ is $O(n^2)$.

Now $g(n) = n^2$, so

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5n^2+60n}{3n^2} = \frac{5}{3} + \lim_{n\to\infty} \frac{20}{3n} = \frac{5}{3}.$$

This method shows that any $k > 5/3$ will suffice for this example. □

Example 2 (ctd): $f(n) = 6n\ln n + 12n^{3/2} + n^2$ is $O(n^2)$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{6n\ln n + 12n^{3/2} + n^2}{n^2} = \lim_{n\to\infty} \left(\frac{6\ln n}{n} + \frac{12}{\sqrt{n}} + 1\right) \tag{4}$$

In the last limit, clearly the second term goes to zero, so we need to investigate the first term. Both its numerator and denominator go to infinity, so we need L'Hôpital's rule:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

where $f'(n)$ is the derivative of $f(n)$ and $g'(n)$ is the derivative of $g(n)$. So

$$\lim_{n\to\infty} \frac{6\ln n}{n} = \lim_{n\to\infty} \frac{6/n}{1} = \lim_{n\to\infty} \frac{6}{n} = 0.$$

This shows that the limit in (4) is 1, so any $k > 1$ suffices for this example. □

2. Proving $f(n)$ is not $O(g(n))$

2.1. Inequality Method.

In this method we show by contradiction that no suitable constants exist. Note that is **not** sufficient to choose specific values for k and n_0, and then show that the inequality fails for these values.

Example 3: $f(n) = 3n\ln n + 12n$ is not $O(n)$.

Let us assume that constants k and n_0 exist that satisfy (1). Then for all $n \geq n_0$ we have $3n\ln n + 12n \leq kn$ and so $\ln n \leq (k-12)/3$. This is clearly a contradiction, since $\ln n$ gets arbitrarily large as n goes to infinity.

2.2. Limit Method.

In this method we use the fact that if (1) does not hold, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq k \tag{5}$$

for every constant k. As before, L'Hôpital's rule may sometimes be useful.

Example 4: $f(n) = 3^n$ is not $O(2^n)$.

$$\lim_{n\to\infty} \frac{3^n}{2^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$$