History based pivot rules for acyclic USOs on hypercubes

David Avis (Kyoto, McGill)
joint work with
Yoshikazu Aoshima, Theresa Deering, Yoshitake Matsumoto
and Sonoko Moriyama

January 13, 2011
Warning!

This problem may be addictive ...
Works for low dimensions ...
Let’s try $d = 10$
Starting point

(Courtesy: G. Ziegler)

Dear Victor,

Please post this offer of $1000 to the first person who can find a counterexample to the least entered rule or prove it to be polynomial. The least entered rule enters the improving variable which has been entered least often.

Sincerely,

Norman Zadeh
Thursday, 10:30am
Subexponential Lower Bounds for the Simplex Algorithm
Oliver Friedmann
Ludwig-Maximilians-Universität München
"... We also give a subexponential lower bound for Zadehs pivoting rule which among all improving pivoting steps enters the variable that has been entered least often. "

Reward claimed!
Unpublished gem

Unpublished gem


Victor Klee (1925-2007)

Vic Klee at Oberwolfach in 1981
(photo: L. Danzer)
Klee-Minty paper (1970)

How Good Is the Simplex Algorithm?

Victor Klee*
Department of Mathematics, University of Washington, Seattle, Washington

AND

George J. Minty†
Department of Mathematics, Indiana University, Bloomington, Indiana

1. Introduction

By constructing long “increasing” paths on appropriate convex polytopes, we show that the simplex algorithm for linear programs (at least with its most commonly used pivot rule, Dantzig [1]) is not a “good algorithm” in the sense of Jack Edmonds. That is, the number of pivots or iterations that may be required is not majorized by any polynomial function of the two parameters that specify the size of the program. In particular, $2^d - 1$ iterations may be required in solving a linear program whose feasible region, defined by $d$ linear inequality constraints in $d$ nonnegative variables or by $d$ linear equality constraints in $2d$ nonnegative variables, is projectively equivalent to a $d$-dimensional cube. Further, for each $d$ there are positive constants $\alpha_d$ and

The start of Polyhedral Computation?
Norm Zadeh

Norm Zadeh creator of Perfect Ten Magazine at his Beverly Hills Mansion November 2001 with his perfect 10 models (photo: Jonas Mohr)
For Sale!

Hot property: Norm Zada, publisher of Perfect 10 magazine has listed his house in Beverly Park at $24.5 million

Hot property: Norm Zada - Los Angeles Times

http://www.latimes.com/classified/realestate/printedition/hm-hotpropzad...
Sold!

72 BEVERLY PARK Dr, Beverly Hills, CA 90210 | MLS# 09-352603

Sold on 11/16/2010
$16,500,000

72 BEVERLY PARK Dr
Beverly Hills, CA 90210

BEDS: 11
BATHS: 18
SQ. FT.: 20,000
$/SQ. FT.: $825
LOT SIZE: 6.79 Acres
PROPERTY TYPE: Residential, Single Family
STYLE: Architectural
VIEW: Canyon, City Lights, Mountain, Yes
YEAR BUILT: 2000
COMMUNITY: Beverly Hills Post Office
COUNTY: Los Angeles
MLS#: 09-352603
SOURCE: TheMLS
STATUS: Closed

The absolute best opportunity to purchase a pristine almost new Beverly Park compound in years. Trophy contemporary estate by the Landry Design Group sited on the highest elevation in Beverly Park. The free-flowing approx 20,000 sq. ft. estate includes a new 6,100 sq. ft. guest house + 1,700 sq. ft. bonus room at basement level, which is connected to the main house by a glass and stainless steel-columned bridge. There is a total of 11bdrm/18bath on 6.79 acres (over 2 acres level) w/ mtn views and city lights. This private hilltop setting's outdoor amenities include vast pristine lawns, large pool and paddle tennis ct with pavilions.

http://www.redfin.com/CA/Beverly-Hills/72-Beverly-Park-90210...
LP-digraphs

The Simplex Method and LP digraphs

Linear Programming (LP) $A$:

$$\min f := x + 2y + 4z$$

$$\begin{cases} 
-1 \leq x \leq 1 \\
-1 \leq y \leq 1 \\
-1 \leq z \leq 1
\end{cases}$$

The Simplex Method:

Algorithm of searching a sink of LP digraphs by some pivotting rules.

Strongly polynomial-time algorithms for LP?

Good characterizations for LP digraphs?
Basic problem

Can we efficiently find the sink of an LP-digraph by following a directed path from any given vertex, using a given edge selection rule (pivoting)?
Necessary conditions for LP digraphs

- Unique Sink Orientation (USO) ['01 Szabo, Welzl]
- Acyclicity
- Holt Klee Property ['99 Holt, Klee]
- Shelling Property ['09 Avis, Moriyama]
Necessary conditions for LP digraphs

Unique Sink Orientation (USO)
[‘01 Szabo, Welzl]

Each subgraph $G(P,H)$ of $G(P)$ induced by a face $H$ of $P$ has a unique sink (and then a unique source).
Necessary conditions for LP digraphs

Acyclicity

G(P) has no directed cycle.

Acyclic

Not acyclic
Necessary conditions for LP digraphs

Holt Klee property [‘99 Holt, Klee]

G(P) has a USO, and for every k-dimensional face H of P there are k disjoint paths from the unique source to the unique sink in $G(P,H)$. 
Klee-Minty Examples

• 3-cube (Chvátal, P.47)

\[
\begin{align*}
\text{maximize} & \quad 100x_1 + 10x_2 + x_3 \\
\text{s.t.} & \quad x_1 \leq 1 \\
& \quad 20x_1 + x_2 \leq 100 \\
& \quad 200x_1 + 20x_2 + x_3 \leq 10000 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
Klee-Minty Examples

• 3-cube (Chvátal, P.47)

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\begin{align*}
\text{maximize} & \quad 100x_1 + 10x_2 + x_3 \\
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& \quad 200x_1 + 20x_2 + x_3 \leq 10000 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Vertices:

\[
\begin{align*}
0 & 0 & 0 & 0 & 100 & 8000 \\
1 & 0 & 0 & 1 & 80 & 8200 \\
1 & 80 & 0 & 1 & 0 & 9800 \\
0 & 100 & 0 & 0 & 0 & 10000
\end{align*}
\]
Pivot Sequence (Dantzig’s rule)

\[
\begin{array}{c}
\begin{array}{ccc}
  x_1 & x_2 & x_3 \\
  0 & 0 & 0 \\
  1 & 0 & 0 \\
  1 & 80 & 0 \\
  0 & 100 & 0 \\
  0 & 100 & 8000 \\
  1 & 0 & 9800 \\
  1 & 80 & 8200 \\
  0 & 0 & 10000 \\
\end{array}
\end{array}
\]

• \(x_1\) stays out of basis for \(2^n - 1\) iterations.
• \(x_1\) pivots \(2^n - 1\) times.
Pivot Sequence (Dantzig’s rule)

$x_1$ $x_2$ $x_3$
0 0 0
1 0 0
1 80 0

- 0 100 0
- 0 100 8000
- 0 100 8000
- 1 0 9800
- 1 80 8200
- 0 0 10000

- $x_n$ stays out of basis for $2^{n-1}$ iterations.
Pivot Sequence (Dantzig’s rule)

\[
\begin{align*}
&x_1 \ x_2 \ x_3 \\
&0 \ 0 \ 0 \\
&1 \ 0 \ 0 \\
&1 \ 80 \ 0 \\
&\bullet \ 0 \ 100 \ 0 \\
&0 \ 100 \ 8000 \\
&0 \ 100 \ 8000 \\
&1 \ 0 \ 9800 \\
&1 \ 80 \ 8200 \\
&0 \ 0 \ 10000 \\
&\bullet \ x_n \text{ stays out of basis for } 2^{n-1} \text{ iterations.} \\
&\bullet \ x_1 \text{ pivots } 2^{n-1} \text{ times.}
\end{align*}
\]
Klee-Minty construction

1D  2D  3D  4D
Klee-Minty path
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
- Facets $F_1, F_2, \ldots, F_{2n}$. For $i = 1, \ldots, n$,

  \[ F_i = \{ (x_1, x_2, \ldots, x_n) | x_i = 0 \}, \quad F_{n+i} = \{ (x_1, x_2, \ldots, x_n) | x_i = 1 \}. \]
n-cube USOs

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- Cobasis $C(v) = \{i : v \in F_i, i = 1, \ldots, 2n\}, v \in V$
n-cube USOs

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- Facets \( F_1, F_2, \ldots, F_{2n} \). For \( i = 1, \ldots, n \),

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\]
- Cobasis \( C(v) = \{i : v \in F_i, i = 1, \ldots, 2n\}, \quad v \in V \)
- Basis \( B(v) = \{i : v \notin F_i, i = 1, \ldots, 2n\}, \quad v \in V \)
n-cube USOs

- Vertices $V = \{0, 1, \ldots, 2^n - 1\} = \{00..00, 00..01, \ldots, 11..11\}$
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- Basis $B(v) = \{i : v \notin F_i, i = 1, \ldots, 2n\}, \; v \in V$
- Note $i \in B(v)$ iff $n + i \in C(v)$. 

n-cube USOs

- Vertices $V = \{0, 1, ..., 2^n - 1\} = \{00..00, 00..01, ..., 11..11\}$
- Facets $F_1, F_2, ..., F_{2n}$. For $i = 1, ..., n$,
  
  $$F_i = \{(x_1, x_2, ..., x_n) | x_i = 0\}, \quad F_{n+i} = \{(x_1, x_2, ..., x_n) | x_i = 1\}.$$

- Cobasis $C(v) = \{i : v \in F_i, i = 1, ..., 2n\}, \quad v \in V$
- Basis $B(v) = \{i : v \notin F_i, i = 1, ..., 2n\}, \quad v \in V$
- Note $i \in B(v)$ iff $n + i \in C(v)$.
- A pivot interchanges a pair of indices $i$ and $n + i$ between $B(v)$ and $C(v)$. (flips bit $i$ of $v$)
3-cube acyclic USO

- Vertices $V = \{0, 1, \ldots, 7\} = \{000, 001, \ldots, 111\}$
3-cube acyclic USO

- Vertices $V = \{0, 1, \ldots, 7\} = \{000, 001, \ldots, 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, , x_3)|x_i = 1\}$
3-cube acyclic USO

- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$
3-cube acyclic USO

- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, , x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$
- $v = 6$ pivots to vertices 2,4,7 by flipping bits 1,2,3
3-cube acyclic USO

- Vertices $V = \{0, 1, \ldots, 7\} = \{000, 001, \ldots, 111\}$
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, , x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$
- $\nu = 6$ pivots to vertices 2,4,7 by flipping bits 1,2,3
- Pivots correspond to moves in the 4,2,1 directions
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
- Least recently considered (Cunningham)
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
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- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least number of times to enter basis (Zadeh)
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
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- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least number of times to enter basis (Zadeh)
- All of the above break Klee-Minty type constructions
History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)
- Least recently considered (Cunningham)
- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least number of times to enter basis (Zadeh)
- All of the above break Klee-Minty type constructions
- We try to find an acyclic USO for which a given rule follows a Hamiltonian path
Least recently basic (Johnson)
Least recently considered (Cunningham)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Sequence</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0</td>
<td>+ 2, - 2, + 1, - 1, + 3, - 3, + 4, - 4</td>
<td>+ 2</td>
</tr>
<tr>
<td>2 0 0 0</td>
<td>- 2, + 1, - 1, + 3, - 3, + 4, - 4</td>
<td>+ 3</td>
</tr>
<tr>
<td>6 0 1 0</td>
<td>- 3, + 4, - 4, + 2, - 2, + 1, - 1</td>
<td>+ 4</td>
</tr>
<tr>
<td>14 1 1 0</td>
<td>- 4, + 2, - 2, + 1, - 1, + 3, - 3</td>
<td>- 3</td>
</tr>
<tr>
<td>10 1 0 0</td>
<td>+ 4, - 4, + 2, - 2, + 1, - 1, + 3</td>
<td>- 2</td>
</tr>
<tr>
<td>8 1 0 0</td>
<td>+ 1, - 1, + 3, - 3, + 4, - 4, + 2</td>
<td>- 2</td>
</tr>
</tbody>
</table>
Least recently entered (Fathi-Tovey)
Least number of iterations in basis (A-M-M)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>(orientation, direction)-pair</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0</td>
<td>0 1 0 1 0 1 0 1</td>
<td>+1, +2, +3, +4</td>
</tr>
<tr>
<td>1 0 0 1</td>
<td>0 2 0 2 0 2 1 1</td>
<td>+2, +3, +4</td>
</tr>
<tr>
<td>5 0 1 0</td>
<td>0 3 1 2 0 3 2 1</td>
<td>+2, +4</td>
</tr>
<tr>
<td>13 1 1 0</td>
<td>1 3 2 2 0 4 3 1</td>
<td>+2</td>
</tr>
<tr>
<td>15 1 1 1</td>
<td>2 3 2 2 1 4 4 1</td>
<td>-1</td>
</tr>
<tr>
<td>14 1 1 0</td>
<td>3 3 4 2 2 4 4 2</td>
<td>-3</td>
</tr>
<tr>
<td>10 1 0 1</td>
<td>4 3 4 3 3 4 4 3</td>
<td>+1, -2</td>
</tr>
<tr>
<td>11 1 0 1</td>
<td>5 3 4 4 4 4 5 3</td>
<td>-2</td>
</tr>
<tr>
<td>9 1 0 0</td>
<td>6 3 4 5 4 5 6 3</td>
<td>-1</td>
</tr>
<tr>
<td>8 1 0 0</td>
<td>7 3 4 6 4 6 6 4</td>
<td></td>
</tr>
</tbody>
</table>
Least used direction (A-M-M)
Least used direction

For n-cube $H_n$, directions $i = 1, ..., n$

- $nv(i)$ = the number of times that direction $i$ has been taken.
Least used direction

For n-cube $H_n$, directions $i = 1, ..., n$

- $nv(i)$ = the number of times that direction $i$ has been taken.
- Initialize: $nv(i) = 0$ for $i = 1, ..., n$
Least used direction

For n-cube $H_n$, directions $i = 1, ..., n$

- $nv(i) =$ the number of times that direction $i$ has been taken.
- Initialize: $nv(i) = 0$ for $i = 1, ..., n$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
Least used direction

For n-cube $H_n$, directions $i = 1, \ldots, n$

- $nv(i) =$ the number of times that direction $i$ has been taken.
- Initialize: $nv(i) = 0$ for $i = 1, \ldots, n$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
- Set $nv(j) = nv(j) + 1$.
- Special case of Zadeh’s rule.
Unique $H_3$

Hamilton path using least used direction rule
It satisfies the Holt-Klee condition

Least visited rule

$nv(1) = nv(2) = nv(4) = 0$
$4: \ nv(1) = \ nv(2) = 0, \ nv(4) = 1$
$2: \ nv(1) = 0, \ nv(2) = 1, \ nv(4) = 1$
$1: \ nv(1) = 1, \ nv(2) = 1, \ nv(4) = 1$
$2: \ nv(1) = 1, \ nv(2) = 2, \ nv(4) = 1$
$4: \ nv(1) = 1, \ nv(2) = 2, \ nv(4) = 2$
$2: \ nv(1) = 1, \ nv(2) = 3, \ nv(4) = 2$
$1$
Unique $H_4$

Hamilton path using least used direction rule
It satisfies the Holt-Klee condition

Least visited rule

$[3, 4, 3, 2, 1, 2, 1, 3, 1, 0, 0, 2, 1, 3, 2, 2]$
$[9, 8, 12, 4, 6, 14, 10, 11, 15, 13, 5, 7, 3, 2, 0, 1]$

nv(1) = nv(2) = nv(4) = nv(8) = 0
1: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 0
4: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 0
8: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 1
2: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 1
8: nv(1) = 1, nv(2) = 2, nv(4) = 2, nv(8) = 2
1: nv(1) = 2, nv(2) = 1, nv(4) = 2, nv(8) = 2
4: nv(1) = 2, nv(2) = 1, nv(4) = 1, nv(8) = 2
2: nv(1) = 2, nv(2) = 2, nv(4) = 3, nv(8) = 2
8: nv(1) = 2, nv(2) = 2, nv(4) = 3, nv(8) = 3
2: nv(1) = 2, nv(2) = 3, nv(4) = 3, nv(8) = 3
4: nv(1) = 2, nv(2) = 2, nv(4) = 4, nv(8) = 2
1: nv(1) = 3, nv(2) = 2, nv(4) = 4, nv(8) = 2
2: nv(1) = 3, nv(2) = 3, nv(4) = 4, nv(8) = 2
1
$H_5$

David’s example
Acyclic, USO, non-HK
One H_4 cube

Least visited rule

Acyclic, USO, non-HK
One H_4 cube
Another candidate for $H_5$

Yoshitake’s example
Acyclic, USO, non-HK
No H_4 cubes

Least visited rule
Computational results: least used direction

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of Hamilton paths</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
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Computational results: least used direction

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- For $n \leq 4$, each example extends to the next dimension
Computational results: least used direction

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<td>1</td>
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</tr>
</tbody>
</table>

- For $n \leq 4$, each example extends to the next dimension
- Since HK fails for $n = 5$, these examples are not LP-digraphs
Computational results: least used direction

But things do not go well for $n \geq 6$ ...

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
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<tbody>
<tr>
<td>number of Hamilton paths</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>

We did a computer search of all acyclic USOs that contain Hamiltonian paths.
Least times to enter basis (Zadeh’s rule)

Facets \( F_i, i = 1, ..., 2n \)

- \( n_v(i) \) = the number of times that \( F_i \) has been visited.
Least times to enter basis (Zadeh’s rule)

Facets $F_i, i = 1, ..., 2n$

- $nv(i) =$ the number of times that $F_i$ has been visited.
- Initialize: $nv(i) = 0$ for all $i$
Least times to enter basis (Zadeh’s rule)

Facets $F_i, i = 1, \ldots, 2n$

- $nv(i) =$ the number of times that $F_i$ has been visited.
- Initialize: $nv(i) = 0$ for all $i$
- Update: From current vertex $y$ choose an outgoing edge to a facet $F_j$ minimizing $nv(j)$
Least times to enter basis (Zadeh’s rule)

Facets $F_i, i = 1, \ldots, 2n$

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- Set $nv(j) = nv(j) + 1$. 
## Computational results: least times to enter basis

The deluge!

<table>
<thead>
<tr>
<th>dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ham. paths</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td>1,072</td>
<td>3,262,342</td>
<td>$\geq 42,500,000,000$</td>
</tr>
<tr>
<td>Holt-Klee</td>
<td>1</td>
<td>2</td>
<td>12</td>
<td>79</td>
<td>360</td>
<td>none yet</td>
</tr>
</tbody>
</table>
## Computational results: all rules

<table>
<thead>
<tr>
<th>Dimension</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least-entered(Zadeh)</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td>1,072</td>
<td>3,262,342</td>
<td>$&gt;10^{10}$</td>
</tr>
<tr>
<td>Least-used-direction</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-recently-entered</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-recently-considered</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-recently-basic</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Least-iterations-in-basis</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table:** Hamiltonian paths produced by history based pivot rules
How do we get the results?

Williamson Hoke’s theorem (1988)

- Given an oriented $n$-cube $H$, let $d_k = \text{number of vertices with in-degree } k$
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- Eg. Exactly $n$ vertices have in-degree one.
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- Hopeless trying to generate all AUSOs directly
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- A H.P. defines the orientation of all edges of an acyclic orientation
- Williamson-Hoke’s theorem characterizes which orientations are AUSOs
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**Canonical paths lemma**

For any history based pivot rule generating a HP there is a labelling of the cube s.t.
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- HP starts at vertex 0
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  Eg. $+1, +2, -1, +3, ....$ is valid.
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In a canonical HP the indegree of a vertex is 1 if and only it is reached by a signed direction that is being used for the first time.
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- By WH there are no others.
- Eg: $+1,+2,+3,-1,...$ does give a HP by Zadeh.
Non-existence of Hamiltonian paths

Theorem

- For $d \geq 3$ the following rules do not generate any Hamiltonian paths on AUSO cubes: least-iterations-in-basis, least-recently-basic, least-recently considered.
Non-existence of Hamiltonian paths

Theorem

- For $d \geq 3$ the following rules do not generate any Hamiltonian paths on AUSO cubes: least-iterations-in-basis, least-recently-basic, least-recently considered.

- For $d \geq 5$ the least-recently-entered rule does not generate any Hamiltonian paths on AUSO cubes.
Proof of non-existence of Hamiltonian paths

- The least-iterations-in-basis, least-recently-basic, least-recently considered, rules start with $+1, +2, \ldots, +d, -1, \ldots$
Proof of non-existence of Hamiltonian paths

- The least-iterations-in-basis, least-recently-basic, least-recently considered, rules start with $+1, +2, \ldots, +d, -1, \ldots$
- These $d + 1$ vertices have indegree one violating Williamson-Hoke
Non-existence: least-recently-entered, \( d \geq 5 \)

**Theorem 1.1.** The least-recently entered rule does not have any Hamiltonian paths on a \( d \)-cube for \( d \geq 5 \).

**Proof.** Suppose \( P \) is a Hamiltonian path produced by Algorithm 1 for the least-recently entered rule when \( d \geq 5 \). We will show that \( P \) must begin with the sequence of vertices \( Q = Q_1, Q_2, \ldots, Q_{d+1} \) where \( Q_1 = \{0, 1, 3, \ldots, 2^d - 1\} \), \( Q_2 = \{2, 3, \ldots, 2^d - 2\} \), \( Q_3 = \{2^d - 1, 2^d - 2, \ldots, 2^{d-1}\} \), \( Q_4 = \{2^{d-1} + 2, 2^d, 6, 14, \ldots, 2^{d-2} - 2\} \) and \( Q_5 = \{2^d - 2 - 2^{d-2}, 2^d - 2 - 2^{d-2} - 2^{d-3}, \ldots, 2^{d-1} + 2 + 4 + 8, 2^{d-1} + 2 + 4\} \).

\( Q \) includes the vertices \( \{2^{d-1} + 2, 2^{d-1} + 2 + 4 + 8, 2^{d-1} + 4\} \) as a subsequence and does not contain the vertex \( 2^{d-1} + 2 + 8 \). These four vertices lie on a 2-face which has two sources, \( 2^{d-1} + 2 \) and \( 2^{d-1} + 2 + 4 + 8 \), a contradiction. It remains to show that \( P \) begins as specified.

- \( Q_1 = 0, 1, 3, \ldots, 2^d - 1 \). This follows from Lemma 2.

- \( Q_2 = 2^d - 1, 2^d - 2 - 1, 2^d - 2 - 2 - 2^d - 3 - 1, \ldots, 2^d - 1 \).

We prove this by mathematical induction. For the basic step, we will show only \( 2^d - 2 - k \), \( k = 1 \). When we visited the vertex \( 2^d - 1 \), all of the bits are 1. It means the next vertex can be represented as \( 2^d - 2 - 1 = \sum_{i=0}^{d-1} 2^i \) \( (d - 1 > k > 0) \). By Corollary 3, vertex \( \sum_{i=0}^{d-1} 2^i \) \( - 2^i \) should have two visited neighbours, one of which is obviously the vertex \( 2^d - 1 \). In other words, there exists \( j \neq k \) such that \( \sum_{i=0}^{d-1} 2^i - 2^i \in \{0, 1, 3, \ldots, 2^d - 1\} \) \( \{v|\exists x \text{ s.t. } v = \sum_{i=0}^{d-1} 2^i\} \cup \{0\} \).

Since \( d \geq 3 \) forces \( \sum_{i=0}^{d-1} 2^i - 2^i \) not to be equal to 0, \( \sum_{i=0}^{d-1} 2^i - 2^i \) should be represented as \( \sum_{i=0}^{d-1} 2^i = \sum_{i=0}^{d-1} 2^i - \sum_{j=0}^{d-1} 2^j \) for certain \( j \). Therefore, the \( (k, j) \) equal \( (d - 1, 1) \) or \( (d - 2, d - 1) \), and \( d - 1 > k \) requires \( k = d - 2 \).

We can prove the inductive step similarly. If the path is continued by \( 2^d - 1, 2^d - 1 - 2^d - 2, \ldots, 2^d - 1 - \sum_{i=0}^{d-1} 2^i \), the next vertex should be equal to \( \sum_{i=0}^{d-1} 2^i - \sum_{i=0}^{d-2} 2^i + 2^i \) \( (d - 2 - k \leq j \leq d - 2) \) or \( \sum_{i=0}^{d-1} 2^i - \sum_{i=0}^{d-2} 2^i - 2^i \) \( (j = d - 1 \text{ or } j < d - 2 - k) \). By Corollary 3, two neighbours of it are in \( \{0, 1, 3, \ldots, 2^d - 1, 2^d - 1 - 2^d - 2, \ldots, 2^d - 1 - \sum_{i=0}^{d-2} 2^i \} \).

Using binary numbers, \( 2^d - \{\sum_{k=0}^{d-1} 2^k}\} \) \( - 1 \) can be denoted \( 100 \ldots 0011 \ldots 11 \), where we have \( k \geq 0 \) 0s.

- \( Q_3 = \{2^{d-1} + 2, 2^d, 6, 14, \ldots, 2^{d-2} - 2\} \)

At the vertex \( 2^{d-1} \), the history information function becomes

\[
f(x) = \begin{cases} 
  d + x & (\text{if } x > 0) \\
  1 & (\text{if } x = -d) \\
  2d + 1 - x & (\text{if } x < 0) 
\end{cases}
\]

Although its minimum value is 1, when \( x = -d \), and the second smallest value is 2, when \( x = +1 \), we can not use either the direction \( -d \) or \( +1 \),
How few times can a variable enter the basis?

- In Klee-Minty examples, one variable never enters the basis...
How few times can a variable enter the basis?

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- ...and one variable enter $2^{n-1}$ times.
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- ...and one variable enter $2^{n-1}$ times.
- Zadeh’s rule tries to balance this.
- Theorem
  Let $H$ be a AUSO $n$-cube with a H.P. followed by Zadeh’s rule. Each variable enters the basis at least $\frac{2^{n-2}}{n} - 1$ times.
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
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- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} nv(i) = 2^n - 1$
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} n v(i) = 2^n - 1$
- Pivot $\pm d$ is blocked at $v$ if $v$’s twin already visited
Proof of the lower bound

- Variable $-d$ enters the basis min number $k$ times
- At the sink $\sum_{i=1}^{2^n} n\nu(i) = 2^n - 1$
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- Any pivot with $n\nu(i) \geq k + 2$ must be blocked
Proof of the lower bound

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- $\sum_{i=1}^{2^n} nv(i) \leq 2n(k + 2) - 1 + 2^{n-1}$
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- Pivot $\pm d$ is *blocked* at $v$ if $v$'s twin already visited
- Any pivot with $n v(i) \geq k + 2$ must be blocked
- Number of blocked pivots is at most $2^{n-1}$
- $\sum_{i=1}^{2n} n v(i) \leq 2n(k + 2) - 1 + 2^{n-1}$
- Combining, $k \geq \frac{2^{n-2}}{n} - 2$
Hamiltonian paths are special

- The theorem does not generalize to arbitrary exponential length Zadeh paths
Hamiltonian paths are special

- The theorem does not generalize to arbitrary exponential length Zadeh paths
- Let $C_1$ and $C_2$ be copies of an AUSO with an exponential length Zadeh path.
And the open problems are ...
And the open problems are ...

- Show Hamiltonian paths exist in all dimensions for Zadeh’s rule
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- Show Hamiltonian paths exist in all dimensions for Zadeh’s rule
- Are there exponential lower bounds for all rules?
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- Show Hamiltonian paths exist in all dimensions for Zadeh’s rule
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- Are any of these rules subexponential for LPs?