Computational Intractability	2013/7/10
Lecture 10	
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In this lecture we consider how to generate classes of cutting planes for specific integer programming problems, especially those with binary variables. We use the knapsack problem as an illustration. First we generate an ideal formulation for a small problem and then investigate and generalize the output obtained so that it can be applied to larger problems. The material on cover inequalities is derived from Ch. 9 of Wolsey [1], to where the reader is referred for a more detailed description.

1 Computing ideal formulations for binary integer programs

The ideal formulation for a binary integer program can be found (for small problems) by using software for transforming an H-representation to a V-representation and vice versa, using programs such as cdd, PORTA, or lrs. We use lrs [2] in these notes. Suppose we have an LP-relaxation $P = \{x \in R^n \mid Ax \le b,\ 0 \le x \le 1\}$. We denote this H-representation of P as $H(P)$.

- Compute the V-representation $V(P)$ of $H(P)$
- Delete any fractional vertices from $V(P)$ getting $IV(P)$
- Compute the H-representation $IH(P)$ from $IV(P)$

$IH(P)$ is the ideal formulation for the integer solutions in P.

A knapsack problem is a binary ILP with a single constraint consisting of only positive integers. The constraint region is given by

$$KP = \{x \in R^n \mid \sum_{j=1}^{n} a_j x_j \le b, x_j \in \{0,1\}, j = 1, ..., n\}$$

We assume without loss of generality that $a_1 \ge a_2 \ge ... \ge a_n$.

An LP relaxation $H(P)$ of KP is formed by replacing the binary constraints with upper and lower bounds on the variables. Consider, for example,

$$H(P) = \{x \in R^5 \mid 9x_1 + 8x_2 + 6x_3 + 6x_4 + 5x_5 \le 14,\ 0 \le x \le 1\}$$

The input format for H-representations in lrs is $[b - A]$, and the lrs input file *3i.ine* for $H(P)$ is:

```
H-representation
begin
11 6 rational
14 -9 -8 -6 -6 -5
1 -1 0 0 0 0
1 0 -1 0 0 0
1 0 0 -1 0 0
1 0 0 0 -1 0
1 0 0 0 0 -1
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 1 0 0
0 0 0 0 1 0
0 0 0 0 0 1
end
```

The output V-representation, $V(P)$, consists of 31 vertices, 13 of which are binary. We construct a V-representation, $IV(P)$ (*3i_int.ext*) from these 13 binary variables, where column one is always 1 and indicates a vertex:

```
V-representation
begin
13 6 rational
 1  0  0  0  0  0
 1  1  0  0  0  0
 1  0  1  0  0  0
 1  0  0  1  0  0
 1  0  1  1  0  0
 1  0  0  0  1  0
 1  0  1  0  1  0
 1  0  0  1  1  0
 1  0  0  0  0  1
 1  1  0  0  0  1
 1  0  1  0  0  1
 1  0  0  1  0  1
 1  0  0  0  1  1
end
```

Running lrs on this input we obtain the ideal formulation, $IH(P)$ (*3i_int.ine*), for KP where we have added line numbers for later reference:

```
H-representation
begin
11 6 rational
1:   0  1  0  0  0  0
2:   0  0  1  0  0  0
3:   2 -1 -1 -1 -1 -1
4:   1  0  0  0  0 -1
5:   0  0  0  1  0  0
6:   1 -1  0  0 -1  0
7:   0  0  0  0  0  1
8:   2 -2 -1 -1 -1  0
9:   1 -1 -1  0  0  0
10:  0  0  0  0  1  0
11:  1 -1  0 -1  0  0
end
```

2 Analyzing the output

We first observe that only lines 1,2,4,5,7,10 of $IH(P)$ appear in $H(P)$. These correspond to the 5 constraints $x_j \geq 0$ and the single upper bound $x_5 \leq 1$.

2.1 Cover inequalities

Lines 6,9,11 of $IH(P)$ correspond to the inequalities

$$x_1 + x_4 \leq 1 \qquad x_1 + x_2 \leq 1 \qquad x_1 + x_3 \leq 1.$$

These are stronger than the upper bound inequalities $x_2 \leq 1,\ x_3 \leq 1,\ x_4 \leq 1$ which is why these do not appear in $IH(P)$. They indicate that at most one of a subset of two items can be chosen, and are examples of *cover inequalities.*

Consider a general knapsack constraint $\sum_{i=1}^{n} a_j x_j \leq b$, where all $a_j > 0$. Let $S \subseteq \{1, 2, ..., n\}$ be a subset such that

$$\sum_{j \in S} a_j > b$$

Then we obtain the *cover inequality*

$$\sum_{j \in S} x_j \leq |S| - 1.$$

2.2 Extended cover inequalities

The examples in the last subsection have $|S| = 2$. Consider the case $S = \{3, 4, 5\}$. Since $a_3 + a_4 + a_5 > 14$ we obtain the cover inequality $x_3 + x_4 + x_5 \leq 2$. However this inequality does not appear in $IH(P)$. Instead in line 3 we find the stronger inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2.$$

This is an extended cover inequality, which we now define in general.

Consider a general knapsack constraint $\sum_{i=1}^{n} a_j x_j \leq b$ and cover inequality defined by S. Let k be the minimum index in S. If $k > 1$ we define the *extended cover inequality* from S as

$$\sum_{j=1}^{k-1} x_j + \sum_{j \in S} x_j \leq |S| - 1. \tag{1}$$

Since we assumed that $a_1 \geq a_2 \geq ... \geq a_n$ this is clearly valid and stronger than the original cover inequality. Inequality 3 of $IH(P)$ displayed above is the extended cover inequality derived from $S = \{3, 4, 5\}$.

2.3 Strengthened extended cover inequalities

We have now analyzed all constraints of $IH(P)$ except the constraint on line 8:

$$2x_1 + x_2 + x_3 + x_4 \leq 2. \tag{2}$$

Since the coefficient of x_1 is 2 it cannot be an extended cover inequality. Nevertheless we can observe that $x_2 + x_3 + x_4 \leq 2$ is a cover inequality and hence $x_1 + x_2 + x_3 + x_4 \leq 2$ is an extended cover inequality. However (2) is stronger than this.

If we consider the coefficients $a_1 = 11, a_2 = 8, a_3 = a_4 = 6$ we can observe that if $x_1 = 1$ then we must have $x_2 = x_3 = x_4 = 0$. However if $x_1 = 0$ then we just have the original cover inequality. This is what (2) and non-negativity of the variables implies. It is called a strengthened extended cover inequality.

Again consider a general knapsack constraint $\sum_{i=1}^{n} a_j x_j \leq b$ and extended cover inequality (1). We attempt to strengthen it by finding the largest integer $c_1 \geq 1$ such that

$$c_1 x_1 + \sum_{j=2}^{k-1} x_j + \sum_{j \in S} x_j \leq |S| - 1. \tag{3}$$

is a valid inequality, where the first summation is empty if $k = 2$. Validity is immediate for the case when $x_1 = 0$. For the case $x_1 = 1$ we need to find the largest integer c_1 such that

$$\sum_{j=2}^{k-1} x_j + \sum_{j \in S} x_j \leq |S| - 1 - c_1 \tag{4}$$

is valid. This is achieved by solving the knapsack problem

$$\begin{aligned} z^* = max \sum_{j=2}^{k-1} x_j &+ \sum_{j \in S} x_j \\ s.t. \quad \sum_{i=2}^{n} a_j x_j &\leq b - a_1 \\ x_j &\in \{0, 1\} \end{aligned}$$

and setting $c_1 = |S| - 1 - z^*$. In our example we would obtain $z^* = 0$ and hence $c_1 = 2$. Observe that finding c_1 involves solving an NP-hard problem! However in practice these problems may be rather small and easy to solve.

The procedure described in this section can be iterated for each variable added to extend a cover inequality. We illustrate on the larger example:

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 14$$

We set $S = \{3, 4, 5, 6\}$ giving the cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$. We extend this by first considering x_2 and finding its maximum coefficient c_2 as described above. We solve

$$\begin{aligned} z^* = max \quad & x_3 + x_4 + x_5 + x_6 \\ s.t. \quad & 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 14 - 6 = 8 \\ & x_j \in \{0, 1\} \end{aligned}$$

by inspection obtaining $z^* = 1$. This gives coefficient $c_2 = 3 - z^* = 2$. We obtain the strengthened extended cover inequality $2x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$.

Next we consider x_1 and find its coefficient c_1. For this we solve the knapsack problem

$$\begin{aligned} z^* = max \;\; & 2x_2 + x_3 + x_4 + x_5 + x_6 \\ s.t. \;\; & 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \le 14 - 11 = 3 \\ & x_j \in \{0, 1\} \end{aligned}$$

by inspection obtaining $z^* = 0$, and hence $c_1 = 3 - 0 = 3$. We obtain the strengthened extended cover inequality

$$3x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 \le 3.$$

Note that the order that the variables are considered in computing the strenghtened inequalities may influence the final inequality obtained (see Exercise 2). However it can be shown (see [1], Ch. 9) that the procedure described here applied to any cover inequality always results in a facet of the knapsack polytope.

3 Exercises

1. Take the knapsack inequality $11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \le 14$ with all variables binary valued and compute its ideal formulation using the method of Section 1. For each inequality in the ideal formulation that is not in the original formulation, show how it can be derived using the methods described in Section 2.

2. Find an example where different strengthened extended cover inequalities are obtained depending on the order the the variables are considered for strengthening.

References

[1] L. Wolsey, *Integer Programming* (Wiley) 1998.

[2] D. Avis, *lrs home page*, http://cgm.cs.mcgill.ca/~avis/C/lrs.html