Mathematical Models for Binarization and Ternarization of Musical Rhythms

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April 28, 2009

Abstract

Musical cyclic rhythms with a cycle length (timespan) of 8 or 16 pulses are called *binary*; those with 6 or 12 pulses are called *ternary*. The process of mapping a ternary rhythm of, say 12 pulses, to a rhythm of 16 pulses, such that musicologically salient properties are preserved is termed binarization. By analogy, the converse process of mapping a binary rhythm to a ternary rhythm is referred to as *ternarization*. New algorithms are proposed and investigated for the binarization and ternarization of musical rhythms with the goal of understanding the historical evolution of traditional rhythms through inter-cultural contacts. The algorithms also have applications to automated rhythmic pattern generation, and may be incorporated in composition software tools.

1 Introduction

Research on the evolution of the structure of musical rhythms is difficult and risky for a variety of reasons. For one, musical phenomena must be well documented. Alas, most musical traditions are oral traditions, the Western one being a notable exception. Furthermore, analysis of evolution must take into account not only musical issues, but also historical and sociological issues, among others, as music itself is inherent to a particular human context. Fortunately, thanks to the recording and transcribing efforts made in several musical traditions by many musicologists over recent decades, the study of the transformation and evolution of musical phenomena is enjoying reinvigorating interest.

Prior to the study of evolution is the study of musical transformation. How do transformations in music take place? For example, in the musical style of the late 14th century known as *Ars Subtilior* [22], composers used elaborate rhythmic techniques that included *isorhythm* [27, 22] to perform rhythmic transformations. In Medieval theory, the term isorhythm referring to recurrent

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rhythmic configurations was not used. Instead, they had their own terminology: *color* and *talea*. Color designated the repetition of the same pitches to different rhythms, whereas talea designated the repetition of the same rhythms to different pitches. Coloration was actually indicated by writing the notes in red on the score. When a passage was written in red, its notes would undergo a reduction by one third of their value. This would produce syncopation effects such as hemiola and others. Thus we have here an example of systematic transformations in rhythm, many of which were later incorporated into Renaissance music.

Examples of the evolution of musical phenomena may be observed in those musical traditions created by the mixture of other existing traditions, such as in jazz [54] and Latin America music. Benzon [4] analyzes the development of ever more differentiated control over rhythmic patterns in the jazz music of the twentieth century, and argues that rhythmic elaboration in traditional jazz was followed by melodic progress in swing, and finally harmonic control in bop. In an award-winning book, Pérez Fernández [25] describes how African ternary rhythms that travelled to the Americas may have mutated to duple-metered forms as the more traditional music developed into more commercial popular music, a process he labelled *binarization*. For several critical discussions of the theory put forward in this book see [52], [39], and [8]. Manuel [42] describes a similar binarization transformation that occurred in Spain and Cuba, in which ternary 3/4 and 6/8 rhythms such as the flamenco *Guajira* mutated into the binary rhythm *Guajira-Son*. A more general discussion of the evolution of Cuban rhythms may be found in [1].

The approaches reviewed in [60] are different from the aforementioned methods, and mimic those used in bioinformatics, where an organism is represented by its DNA molecule which, in turn, is modelled as a sequence of symbols [34], [50]. Music can also be represented by its metric or timeline pattern (its "rhythmic DNA" [49]), which may, in turn, be modelled by a sequence of symbols [9]. The methods of sequence comparison used in bioinformatics have already been applied to musical sequences [44] and bird songs [6] as early as 1990. However, the phylogenetic analysis of families of rhythms is just beginning [57], [59], [20], [21].

This paper is concerned with mathematical models for two particular kinds of rhythmic transformations, namely, binarization and ternarization. In order to compare our models we will use the transformations contained in the work of Pérez Fernández [25, 26]. The algorithms proposed and investigated here are useful for a variety of purposes including their use as composition tools, as tools for studying the evolution of rhythms, and as tools for music theorists to study the mechanisms of rhythmic transformations.

2 Mechanisms of Rhythm Mutation

What is a mutation? This term is usually defined in the context of biology where it involves a modification of the DNA sequence of an organism. Here in the context of rhythm a mutation is broadly defined, by analogy, as an alteration of a sequence that represents a rhythm. By mechanism it is meant simply a rule for converting one rhythm to another. This section defines some terminology used in the fields of music theory and computer science. Although the two areas have evolved pretty much independently, they have developed similar tools. For a survey of this topic see [60].

2.1 Music Theory

In 1941 Joseph Schillinger published a massive work: *The Schillinger System of Musical Composition* consisting of many books [53]. Book I titled *Theory of Rhythm* contains mathematical algorithms for performing a large variety of mutations on rhythms including *contractions*, *expansions*, *circular* *permutations*, *general permutations*, and a special more technical class called *groupings*. Space limitations do not permit elaboration on these operations.

For the analysis of music from the rhythmic point of view, Pearsall [50] proposes what he calls the *duration-set theory* approach. First, he expresses a rhythm in terms of a sequence of numbers that indicates the ordered durations of the notes or onsets. For example, the *clave Son* timeline [x . . x . . x . . . x . x . . .] would be denoted by [33424]. This notation is identical to the *successive-interval-arrays* of Chrisman [11], [12]. Now the rhythm [66848] is the same rhythm at a slower tempo (at least from the mathematical point of view ignoring psychological perception issues). Pearsall expresses the durations as proportions, so that rhythms are independent of tempo. Therefore the latter rhythm would be written in the same way as the former, or both could be written as [(3/16),(3/16),(1/4),(1/8),(1/4)]. More relevant to the topic of this paper, Pearsall defines two mutation operations for changing one rhythm to another: *consolidation* and *proliferation*. A consolidation operation merges two adjacent durations into one duration of length equal to the sum of the two durations. Therefore, like an elision, one onset is not heard in the transformed rhythm, but unlike the elision, the overall sequence is not shortened. For example, merging the last two durations of the clave Son converts [33424] to [3346], yielding the rhythm [x . . x . . x . . . x . . . . .], used in much popular music. A proliferation operation is the inverse of a consolidation: it divides a duration into two durations such that the sum of their lengths is equal to the length of the original duration. Consider for example the clave Son again [33424]. We may divide the second duration [3] into [1,2] and the third duration [4] into [3,1] to obtain [3123124], yielding the rhythm [x . . x x . x . . x x . x . . .], the tom-tom drum ostinato pattern used by Buddy Holly in *Not Fade Away*. Mongeau and Sankoff [44] call these two operations *consolidation* and *fragmentation*, respectively.

These two operations, consolidation and proliferation (fragmentation), are called *fusion* and *fission*, respectively, by Jeff Pressing [51], a terminology that is adopted here. Note that the fusion and fission operations performed on a pair of adjacent duration intervals are equivalent to the *substitution* operation in the box representation of a rhythm. For example, merging the last two durations of the clave Son to obtain [3346] is equivalent to substituting the fifth onset [x] by a silent element [.]. Similarly, the fission operation may be viewed as the substitution of a silent element [.] by an onset [x]. Therefore, although the fusion and fission operations are attractive notions for pondering about the structural similarities among a set of rhythms, they are not by themselves very useful as mutation operations for phylogenetic analyses because the operation allows the insertion or deletion of any onset without taking into account the magnitude of the resulting change.

A fundamental mutation operation explored by David Lewin [37] which he calls a *flip* in the context of the pitch domain (scales, chords, and pitch-class-sets) interchanges two adjacent elements in the cyclic sequence. Transferring this idea into the rhythmic domain for example, we could transform the *clave Son* timeline given by [x . . x . . x . . . x . x . . .] into the *clave Rumba* timeline by performing a flip on the $7th$ and $8th$ pulses to obtain [x . . x . . . x . . x . x . . .]. If performing a flip operation on a cyclic rhythm R yields a new rhythm that is a rotation of the mirror image of R, then Lewin calls such an operation a *Cohn flip*.

In [59] the flip operation is called a *swap*, and the *swap distance* between two rhythms with the same number of onsets is defined as the minimum number of swaps needed to convert one rhythm into the other. In order to be able to compare two rhythms with different numbers of onsets the swap distance was generalized in [20], [21] as follows. Consider two rhythms A and B with n and m onsets respectively, where $n > m$. The *directed swap distance* is the minimum number of swaps needed to convert A to B with the following constraints: (1) every onset of A must move to the position of an onset of B, and (2) every onset of B must receive at least one onset from A. For example, if A and B are associated with the *Seguiriya* and *Fandango* meters, respectively, the directed swap distance is equal to four. These distance measures may be viewed as instances of the *assignment* problem in operations research [31], and may be generalized in several ways.

2.2 Computer Science

The first algorithms to compare two sequences in terms of the minimum number of a set of predefined operations necessary to convert one sequence to the other were designed for problems in coding theory by Vladimir Levenshtein [36]. His distance measure, which now goes by the name of *edit* distance (also Levenshtein distance) [40], allows three operations: insertions, deletions, and substitutions (also called reversals). Note that an insertion makes the sequence longer, a deletion makes it shorter, and a substitution does not change its length. At about the same time, researchers in text processing were discovering similar concepts for comparing strings of symbols. The basic string-to-string correction algorithm of Wagner and Fisher [61] permitted the same three operations: insertion, deletion, and substitution; it has since then been applied to measure music similarity [16], [48], [55]. Damerau [17] introduced the *transposition* operation which exchanges two symbols in arbitrary positions. The Damerau-Levenshtein distance is an extension of the Levenshtein distance that counts transposition as a single edit operation. Lowrance and Wagner [38] extended the edit (Levenshtein) distance to include the *swap*: the operation of interchanging any two *adjacent* characters. The *swap* operation is a good model of keyboard errors introduced into a text by typing, as well as *metathesis* in computational linguistics [32], and thus its addition improves on the performance of the edit-distance for such applications, albeit at an increase in computational complexity [62].

Mannila and Ronkainen [41] measure the distance between two sequences by the minimum cost needed to transform one sequence to the other using the operations: insertion, deletion and *move*. The move operation just moves an onset from one position to another, and thus consists of a deletion followed by an insertion. However, its cost is equal to the distance the onset is moved, multiplied by some predetermined constant.

Bookstein et al., [5] propose a distance measure they call the Fuzzy Hamming distance that uses the four operations: insertion, deletion, substitution and move. Since a substitution may be carried out by a deletion followed by an insertion, this measure is very similar to that of Mannila and Ronkainen [41].

Motivated by the problem of cursive script recognition, B. John Oomen [47] extended the edit distance of Lowrance and Wagner [38] to include the operations *squashing* and *expansion*. Recall that the Lowrance-Wagner distance uses insertion, deletion, substitution, and swap. The squashing operation transforms two or more contiguous characters of one string to one character of the other string, whereas the expansion operation is the inverse of squashing. Thus these operations change the overall length of the sequence. Kruskal [33] calls these operations *compression* and *expansion*, respectively. When used together the technique is also referred to as *time-warping*, and is popular in the application to speech recognition [35].

Adjeroh et al., [2] propose a string edit distance suitable for measuring dissimilarity in video sequences. Their measure extends the edit distance with four additional operations: *fusion*, *fission*, *swap*, and *break*. A fusion operation merges a consecutive stream of the same symbol into a single symbol, whereas a fission operation splits a single symbol into many symbols of the same type. Thus these fission and fusion operations are identical to time-warping, and different from the fission and fusion operations considered in music theory [51]. A swap operation interchanges two symbols (or blocks of symbols), and is thus more general than the swap operation used elsewhere (which does not interchange blocks of symbols). A break is a specialized operation related to the technicalities of handling video sequences.

A more general approach to the design of measures of string similarity is via the concept of an *assignment*, well developed in the operations research field. An assignment problem deals with the question of how to assign n items to m other items so as to minimize the overall cost [7]. If the two sets of n and m items are the corresponding two sets of onsets of two rhythms to be compared, and the cost of assigning an onset x of one rhythm to an onset y of the other rhythm is the minimum number of swaps needed to move x to the position of y, then the cost of the minimum-cost assignment is equal to the swap distance discussed in the preceding [20], [21]. The continuous version of the directed swap distance is called the *restriction scaffold assignment* problem in computational biology [3]. It turns out that the *restriction scaffold assignment* problem and the directed swap distance are special cases of the *surjection* distance between two sets S and T (where S has more elements than T) defined as follows:

$$\min_{\psi} \sum_{s \in S} \delta(s, \psi(s)), \tag{1}$$

where δ is a distance metric on the space, and ψ is a surjection between S and T. Eiter and Mannila [23] proposed an algorithm for computing the surjection distance in $O(n^3)$ time, where n is the number of elements in S, by reducing the problem to finding a minimum-weight perfect matching in an appropriate graph.

Ben-Dor et al. [3] presented an $O(n \log n)$ time algorithm to solve the *restriction scaffold assignment* problem. Their algorithm relies heavily on a result of Karp and Li [31] which provides a linear time algorithm (after sorting) for computing the *one-to-one* assignment problem in the special case where all the points lie on a line. In the one-to-one assignment problem between S and T some elements of S remain unassigned. However, their algorithm is not guaranteed to give the optimal solution, as is shown by a counter-example in [14], where an $O(n^2)$ time optimal algorithm is given. This complexity is reduced to $O(n \log n)$ in [13]. Other variants of distance measures between point sets are discussed in [15], where fusion and fission operations are performed implicitly via assignment constraints.

3 The Data

Most of the data used in our work have been taken from the book by Pérez Fernández [25], where a theory about how certain rhythms mutated from their original ternary form to their current binary form is laid out. The book is organized into three large chapters. The first chapter constitutes a historical, sociological and cultural study of the African influence in Iberia and colonial Latin America. It pays a great deal of attention to enculturation/deculturation processes among the slaves, and to mechanisms of cultural syncretism. Chapter two is devoted to the study of similarities and differences between the Hispanic and African musical traditions as well as the syncretism between both. Chapter three concerns the actual binarization processes; it explains binarization via the similarity between certain rhythmic structures (the metric feet), and provides detailed musical examples.

Pérez Fernández follows the work of the authoritative Ghanaian musicologist Nketia [45, 46] for some of his terminology. Nketia relates musical phrases to timespan, which is of fixed duration. The timespan, typically identified with a 12/8 bar when transcribing African music, is further divided into regulative beats, whose function is to serve as a reference for dancers. These regulative beats divide the timespan into two equal parts. By refining the timespan down to its smallest unit we find the basic pulse. The timespan is measured in terms of the number of basic pulses. Pérez Fernández then introduces the metric foot in between the regulative beat and the basic pulse as an intermediate level of rhythmic grouping. As will be seen in the following, metric feet will be helpful for rhythmic analysis. Metric feet consist of groupings of two or more basic pulses according to either their duration or accentuation patterns. Here we will consider metric feet only with regards to duration. The main metric feet considered in this paper appear in Figure 1 (L stands for a long duration and S for a short duration).

Rhythmic Foot	Rhythm	Duration Pattern
Trochee	[x . x]	L-S
Iamb	[x x .]	S-L
Molossus	[x . x . x .]	L-L-L
Tribrach	[x x x]	S-S-S
Choriamb	[x . x x x .]	L-S-S-L
Zamba foot	[x x x x x .]	Tribrach+Iamb

Figure 1: The relevant basic metric feet and their concatenation.

For example, the pattern [x . x x x . x x . x . x] has a timespan of 12 basic units, where its regulative beats are in positions 1 and 7. Its decomposition in terms of metric feet is trochee+iamb+ iamb+trochee.

We now describe the data used for testing our models. We first introduce a set of ternary rhythms having 6-pulse timespans ([25], pages 82, 83, 91 and 101). They are combinations of two metric feet. Figure 2 shows those rhythms and their binarizations. Names in the rightmost column correspond to one of the many possible names for the binarized rhythm.

Description of Rhythm	Ternary Rhythm	Binarized version	Description of the Binarized Version
Tribrach + Trochee	[x x x x . x]	[x x . x x . x .]	Argentinean milonga
Zamba Foot	[x x x x x .]	[x x . x x . x .]	Argentinean milonga
Choriamb	[x . x x x .]	[x . . x x . x .]	Habanera

Figure 2: Binarized rhythms having timespans of 6 basic units.

For the second set of rhythms, Pérez Fernández gathers rhythmic patterns with timespans of 12 pulses, and their corresponding binarized versions ([25], page 102 and following); see Figure 3. The rhythms are 6/8 clave son, also called the clave *fume-fume* [59], variations of it, and the ubiquitous bembé. Chernoff [10] was one of the first to study the binarization of the bembé.

Description of Rhythm	Notation of the Rhythm	Binarized version	Description of the Binarized Version
6/8 clave Son	[x . x . x . . x . x . .]	[x . . x . . x . . . x . x . . .]	clave Son
6/8 clave Son	[x . x . x . . x . x . .]	[x . . x . . x . . x . . x . . .]	Variation Son - 1
6/8 clave Son variation 1	[x . x . x . . x . x . x]	[x . . x . . x . . . x . x . x .]	Variation Son -2
6/8 clave Son variation 2	[x . x . x . . x . x x .]	[x . . x . . x . . . x . x . x .]	Variation Son -2
Bembé	[x . x . x x . x . x . x]	[x . . x . . x x . . x . x . . x]	Binarized bembé

Figure 3: Binarized rhythms with timespans of 12 basic units.

Names of the binarized rhythms on the last column are provided for reference in the following sections.

For a third set of rhythms, Pérez Fernández displays what he calls resources of rhythmic variation ([25], pages 73-74 and 112-122). These rhythms are formed by variations of the molossus [x . x . x .], the first part of the clave son. They were collected for his work [24] that classified cowbell, hand-clapping and clave patterns from traditional Cuban music, but they have also been documented in the African musical traditions belonging to enslavement areas. Figure 4 shows these rhythmic variations with their associated binarized counterparts. Note that some variations have more than one binarized version.

Description of Rhythm	Notation of the Rhythm	Binarized version	Name of the Rhythm
Variation 1(c)	[x x x . x .]	[x . x x . . x .]	Binarized var. 1(c)-1
Variation 1(c)	[x x x . x .]	[x x . x . . x .]	Binarized var. 1(c)-2
Variation 1(d)	[. x x . x .]	[. . x x . . x .]	Binarized var. 1(d)-1
Variation 1(d)	[. x x . x .]	[. x . x . . x .]	Binarized var. 1(d)-2
Variation 2(c)	[x x x . x x]	[x . x x . . x x]	Binarized var. 2(c)
Variation 4(c)	[x x x x x x]	[x x . x . x x x]	Binarized var. 4(c)
Variation 4(a)	[x . x x x x]	[x . . x x x x .]	Binarized var. 4(a)
Variation 5(a)	[x . x . . .]	[x . . x . . . .]	Binarized var. 5(a)
Variation 6(c)	[x x x . . x]	[x x . x . . . x]	Binarized var. 6(c)

Figure 4: Binarized rhythms derived from rhythmic variations.

Again, names of the binarized rhythms are mnemonics used for reference to them in the following sections.

4 Mapping Rules

The process of binarization proposed by Pérez Fernández uses the metric foot as the starting point. Indeed, the binarization of a ternary rhythm is broken down in terms of its metric feet. Afterwards, each foot is binarized according to a set of binarization rules, also called *mapping rules*. Finally, the binarized feet are put back together so that they constitute the new rhythm. For instance, let us consider the binarization of the zamba foot [x x x x x .]. It is formed by the concatenation of a tribrach [x x x] and an iamb [x x .]. For this case, the mapping rules are [x x x] $\longrightarrow$ [x x . x] and [x x .] $\longrightarrow$ [x . x .]. Finally, gluing these patterns together yields the binarized rhythm [x x . x x . x .]. All the mapping rules used by Pérez Fernández are shown in Figure 5.

Metric Feet	Binarized Patterns	Snapping Rules
[x x x]	[x x . x]	NN
	[x . x x]	CN
	[x x x .]	CCN
[x . x]	[x . . x]	NN/CN
	[x . x .]	FN/CCN
[. x . x . .]	[. . x . x . . .]	FN/CN
[x x .]	[x x . .]	NN/CCN
	[x . x .]	FN/CN
[x . x . x .]	[x . . x . . x .]	CN

Figure 5: The transformations used by Pérez Fernández, and their proposed geometric interpretations.

The rules described in the preceding are expressible in terms of *snapping rules*. Some transformations of a ternary metric foot (or rhythm) into a binary pattern can be interpreted geometrically as a snapping problem on a circle. Consider a three-hour clock with a four-hour clock superimposed on it, as depicted in Figure 16. The problem is reduced to finding a rule to snap onsets in the ternary clock to onsets in the binary clock. Since both clocks have a common onset at "noon" (the north pole), this onset is mapped to itself. For the remaining ternary onsets, several rules may be defined. One that arises naturally is snapping to the nearest onset. By doing so, the durational relationships among the onsets are perturbed as little as possible, and intuitively, one would expect that the perceptual structures of the two rhythms should remain similar. For instance, this rules takes a tribrach [x x x] to [x x . x]; it is called the *nearest neighbour* rule (NN). Other rules to be used in our study are the following: *furthest neighbour* rule (FN), where each onset is snapped to its furthest neighbour; *clockwise neighbour* rule (CN), which moves an onset to the next neighbour in a clockwise direction; and *counter-clockwise neighbour* rule (CCN), which is analogous to the clockwise rule, but travels in counter-clockwise direction. In Figure 5 the two rightmost columns identify the mapping rules used by Pérez Fernández in terms of the four snapping rules just introduced.

The reader may wonder what the rationale is for using the counter-intuitive FN rule. Two points are worth mentioning here. First, one would expect mapping rules that make musicological sense to use high-level musicologically relevant knowledge to select which onsets in one rhythm should be mapped to which onsets in the other. This is a difficult problem left for future research. In this study we have chosen to start our investigation with the simplest context-free rules possible, purely mathematical rules if you will, to determine how useful they can be. Therefore, from a combinatorial and logical point of view it makes sense to include the FN rule in our study. Second, and surprisingly, we observed that the musicological rules used by Pérez Fernández at the metric foot level were, in several cases, matched perfectly only by the FN snapping rule. Thus we were motivated to compare this rule with the others in order to better understand the entire snapping process.

Note that the nearest neighbour and furthest neighbour rules may snap two onsets onto the same onset. Consider a tribrach [x x x] and the furthest neighbour rule; one obtains the pattern [x . x .]. This creates the problem of breaking ties; we will deal with this problem later.

Snapping rules have been used previously in the study of rhythmic patterns. For example, Euclidean rhythms of k onsets and n pulses may be generated by snapping a set of k evenly spaced points on an n-hour clock; see [58, 18].

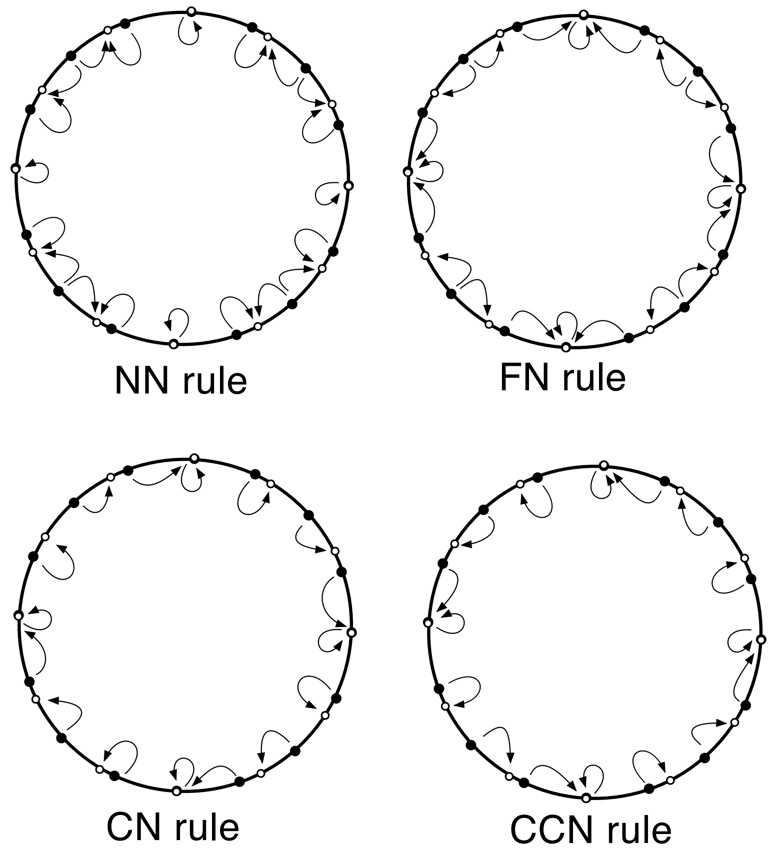

Figure 6: The snapping rules used.

5 Design of the Experiments

Since this paper is concerned with rhythmic transformations in general, we carry out the experiments in both directions, that is, from ternary rhythms to binary rhythms, and from binary rhythms to ternary rhythms. As a matter of fact, we would like to have at our disposal a set of purported ternarized rhythms, just as we have for binarization. In the absence of such a set, we will use Pérez Fernández's set of binarized rhythms; refer to the appropriate columns in Figures 2, 3 and 4.

The first experiment consists of the binarization of the ternary rhythms contained in Pérez Fernández's books [25, 26] (Figures 2, 3 and 4) by using the set of snapping rules defined in the preceding section. We use four snapping rules: NN, FN, CN and the CCN rule. The four rules not only yield a procedure for binarization, but also for ternarization, since the rules are applicable in both directions. The snapping rules will not be applied at the metric foot level, but on the whole rhythmic pattern, because it turns out that the same results are obtained with both methods. Indeed, we prove in the Appendix that the concatenation of the "local" snapping rules produces the "global" snapping rules.

The second experiment deals with centers of rhythm families. They were first used by Toussaint for analysing binary and ternary clave rhythms [57, 59], and proved to be good initial approximations to rhythmic similarity. Toussaint also computed phylogenetic graphs of families of rhythms. Due to lack of space, we will not carry out such an analysis here.

As pointed out in Section 4, when the NN and FN snapping rules are used, ties may arise when an onset has two equidistant nearest or furthest neighbouring pulses. Among the many ways to break ties, we have chosen a method based on rhythmic contours because of their importance in music perception [43, 29, 19, 56].

5.1 Rhythmic Contours

Rhythmic contours have been used for the analysis of non-beat-based rhythms, for the description of general stylistic features of music, for the design of algorithms for automatic classification of musical genres, and also for the study of perceptual discrimination of rhythms, among others. The rhythmic contour is defined as the pattern of successive relative changes of durations in a rhythm. Some authors represent the rhythmic contour as a sequence of integers reflecting these changes; others simply describe the changes in a qualitative manner, observing whether a duration becomes longer, shorter or remains the same. As an example, consider the rhythmic contour of the milonga [x x . x x . x .]. First, we determine its ordered set of durations 12122. The pattern of durations using integers is $\{1, -1, 1, 0, -1\}$, and if we are only concerned with the direction of these changes, we can write $\{+ - +0-\}$. We will use the latter definition of rhythmic contour. The length of the rhythmic contour depends only on the number of onsets in the rhythm. To break ties, we compare the rhythmic contours of the snapped rhythms with those of the original rhythms. Comparison of two rhythmic contours can be made by using the Hamming distance. The Hamming distance counts the number of places in which the rhythmic contours do not match. This distance, however, does not take into account where the mismatches occur. Finally, the contour that has the smallest Hamming distance to the contour of the original rhythm is chosen to break the tie. Some cases arise where one contour is shorter than the other, and hence both contours cannot be compared using the Hamming distance. Such cases appear when two onsets are snapped onto the same pulse. As a result, the total number of onsets in the snapped rhythm is smaller than that of the original rhythm. In these cases, a different measure has to be used for the comparison. The edit distance is a good alternative, and has already been used in the literature for similar purposes [28].

5.2 Centers of Rhythm Families

The second set of experiments comprises the computation of several types of centers. Given a family of rhythms with time-spans of n pulses, we define a center as a rhythm that optimizes some distance function either within the family or in the entire space of rhythms of time-span n. Here we consider centers that convey the idea of similarity. In order to do so, we select as optimization criteria the minimization of the maximum distance (min-max), and the minimization of the sum (min-sum) to all rhythms in either the family or the entire space. For distance (dissimilarity) functions, we select two common distances, the Hamming distance and the directed swap distance [20]. Thus, we have eight possible types of centers, given by the two possible distances, the two possible optimization criteria, the two possible sets of rhythms, and whether optimization is carried out within the given family or the entire space.

The swap distance between two rhythms of equal time-span is the minimum number of interchanges of adjacent elements required to convert one rhythm to the other. An interchange of two adjacent elements, either rests or onsets, is called a swap. If the condition of requiring that both rhythms have equal timespans is relaxed, then a more general distance, the directed swap distance, can be defined as follows. Let rhythm A have more onsets than rhythm B. Then, the directed swap distance is the minimum number of swaps required to convert A to B according to the following constraints: (1) Each onset in A must go to some onset in B; (2) Each and every onset in B must receive at least one onset from A; (3) No onset may travel across the boundary between the first and the last position in the rhythm.

6 Experimental Results

The families of the rhythms analysed were grouped according to the lengths of their timespans. Given the scope of this paper, we cannot describe all the details of all the experiments carried out for the four snapping rules. Due to the problem of breaking ties between rhythms, tables displaying the results are several pages long in some cases. We show the most relevant results and briefly comment on those remaining. For the centers of rhythm families we follow a similar approach. Centers computed on a given family of rhythms are discussed at length, whereas centers computed on the entire space of rhythms are not analysed in full; in some of these cases nearly one hundred instances of centers were obtained.

6.1 Snapped Rhythms

We begin with binarization using the NN rule. Figures 7 and 8 display the results of the calculations. The rhythms in boldface match the binarized rhythms in the books by Pérez Fernández. As can be seen, the NN rule yields almost no match. Furthermore, its binarizations are of little interest in the sense that they keep little perceptual resemblance to their ternary counterparts; see, for example, the binarization of the bembé and compare it to [x . . x . . x x . . x . x . . x].

Ternary rhythm	Name	Binarized rhythm
x x x x x . x . x . x .	Zamba+Molossus	x x . x x x . . x . . x . x . .
x . x . x . . x . x . .	6/8 clave Son	x . . x . x . . . x . . x . . .
x . x . x . . x . x . x	6/8 clave Son -var. 1	x . . x . x . . . x . . x . .x
x . x . x . . x . x x .	6/8 clave Son -var. 2	x . . x . x . . . x . . x x . .
x . x . x x . x . x . x	Bembé	x . . x . x . x . x . . x . . x

Figure 7: Binarization of 12-pulse rhythms using NN rules.

Ternary rhythm	Name	Binarized rhythm
x x x x . x	Tribrach+Trochee	x x . x x . . x
x x x x x .	Zamba foot	x x . x x x . .
x . x x x .	Choriamb	x x . x x x . .
x x x . x .	Var. 1(c)-1	x x . x . x . .
. x x . x .	Var. 1(d)-1	. x . x . x . .
x x x . x x	Var. 2(c)	x x . x . x . x
x x x x x x	Var. 4(c)	x x . x x x . x
x x x . . x	Var. 6(c)	**x x . x . . . x**
x . x . . .	Var. 5(a)	**x . . x . . . .**
x . x x x x	Var. 4(a)	x . . x x x . x

Figure 8: Binarization of 6-pulse rhythms using NN rules.

The table in Figure 9 shows the ternarization of the 8-pulse rhythms obtained with the NN rule. The meaning of the columns from left to right is the following: original rhythm; its name; the snapped rhythm; its name in case it is in our list of rhythms; the number of ties encountered; the list of rhythms with minimal Hamming distance in the tie breaking rule, or the list of rhythms generated by onset ties with both minimal and non-minimal Hamming distance, in case at least one occurrence of overlapping onsets happened in the ties; the contours of the ties and the contour of the original rhythm; the list of all rhythms generated by ties with overlapping onsets; the contours of the ties and the contour of the original rhythm. Several matches for the original binary rhythms are found. In this table the tie breaking procedure can be observed in detail. For instance, variation 6(c) was ternarized in a unique manner since no ties arose. However, variation 1(c)-1 produced two ties, [x x x . x .] and [x x x . . x]. In the latter case we compare their rhythmic contours to break the tie. The rhythmic contours are $\{0+0-\}$ and $\{0+-0\}$, respectively. For the original binary rhythm the rhythmic contour turns out to be $\{-+-0\}$; therefore, the rhythmic contour of [x x x . x .] is more similar, and [x x x . x .] is output as the ternarized rhythm. In the case of variation 1(d)-1, the rhythmic contour cannot break the tie between the two snapped rhythms; note that both contours are made up of the same symbols. Hence, the tie remains unresolved.

```
Bin. rhy | Name        |Tern. | Name ternary |Ties|Min.Hamm.|Rhyt. Contour |Ons. Ov.|Rhyt. Contour Ov. |
-----------------------------------------------------------------------------------------------------
xx.xx.x. | Milonga     |n/a   |n/a           |1   |xxxxx.   |000+- , +-+0- |n/a     |n/a                |
         |             |      |              |    |xxxx.x   |00+-0 , +-+0- |        |                   |
-----------------------------------------------------------------------------------------------------
x..xx.x. |Habanera     |x.xx.x|              |1   |n/a      |n/a           |n/a     |n/a                |
-----------------------------------------------------------------------------------------------------
x.xx..x. |Var1c1       |xxx..x|Tern. Var6c   |2   |xxx.x.   |0+0- , -+-0   |x.x.x.  |000 , -+-0         |
         |             |      |              |    |xxx..x   |0+-0 , -+-0   |x.x..x  |+-+ , -+-0         |
-----------------------------------------------------------------------------------------------------
xx.x..x. |Var1c2       |n/a   |n/a           |1   |xxx.x.   |0+0- , ++--   |n/a     |n/a                |
         |             |      |              |    |xxx..x   |0+-0 , ++--   |        |                   |
-----------------------------------------------------------------------------------------------------
..xx..x. |Var1d1       |n/a   |n/a           |2   |.xx.x.   |+0- , +--     |..x.x.  |00 , +--           |
         |             |      |              |    |.xx..x   |+-0 , +--     |..x..x  |-+ , +--           |
-----------------------------------------------------------------------------------------------------
.x.x..x. |Var1d2       |.xx..x|              |1   |n/a      |n/a           |n/a     |n/a                |
-----------------------------------------------------------------------------------------------------
x.xx..xx |Var2c        |xxx.xx|Tern. Var2c   |2   |xxx.xx   |0+-00 , -+-0+ |xxx..x  |0+-0 , -+-0+       |
         |             |      |              |    |         |              |x.x.xx  |0-0+ , -+-0+       |
         |             |      |              |    |         |              |x.x..x  |+-+ , -+-0+        |
-----------------------------------------------------------------------------------------------------
xx.x.xxx |Var4c        |n/a   |n/a           |1   |n/a      |n/a           |xxx.xx  |0+-00 , +0-000     |
         |             |      |              |    |         |              |xxx.xx  |0+-00 , +0-000     |
-----------------------------------------------------------------------------------------------------
xx.x...x |Var6c        |xxx..x|Tern. Var6c   |0   |n/a      |n/a           |n/a     |n/a                |
-----------------------------------------------------------------------------------------------------
x..x.... |Var5a        |x.x...|Tern. Var5a   |0   |n/a      |n/a           |n/a     |n/a                |
-----------------------------------------------------------------------------------------------------
x..xxxx. |Var4a        |x.xxxx|Tern. Var4a   |1   |x.xxxx   |-000+ , -00++ |x.xxx.  |-0+0 , -00++       |
-----------------------------------------------------------------------------------------------------
```

Figure 9: Ternarization of 8-pulse rhythms using NN rules.

Figures 10 and 11 display the binarizations given by the CN rule. For both 12-pulse and 6-pulse rhythms many matches are found. For example, the bembé is transformed to its commonly accepted binarized form [x . . x . . x x . . x . x . . x]. As mentioned in the preceding, this rule does not produce ties.

Ternary rhythm	Name	Binarized rhythm
x x x x x . x . x . x .	Zamba+Molossus	x . x x x . x . x . . x . . x .
x . x . x . . x . x . .	6/8 clave Son	**x . . x . . x . . . x . x . . .**
x . x . x . . x . x . x	6/8 clave Son -var. 1	x . . x . . x . . . x . x . . x
x . x . x . . x . x x .	6/8 clave Son -var. 2	**x . . x . . x . . . x . x . x .**
x . x . x x . x . x . x	Bembé	**x . . x . . x x . . x . x . . x**

Figure 10: Binarization of 12-pulse rhythms using CN rules.

Ternary rhythm	Name	Binarized rhythm
x x x x . x	Tribrach+Trochee	x . x x x . . x
x x x x x .	Zamba foot	x . x x x . x .
x . x x x .	Choriamb	**x . . x x . x .**
x x x . x .	Var. 1(c)-1	**x . x x . . x .**
. x x . x .	Var. 1(d)-1	**. . x x . . x .**
x x x . x x	Var. 2(c)	x . x x . . . x
x x x x x x	Var. 4(c)	x . x x x . x x
x x x . . x	Var. 6(c)	x . x x . . . x
x . x . . .	Var. 5(a)	**x . . x . . . .**
x . x x x x	Var. 4(a)	x . . x x . x x

Figure 11: Binarization of 6-pulse rhythms using CN rules.

The FN rule produced no matches in either the binarization or the ternarization. There were many with ternarization, in some cases as many as three. In numerous cases the ties are unresolved (in the case of 16-pulse rhythms no output was produced; all ties were unresolved). For the binarization, rhythms produced with the FN rule are somewhat monotonous, in many cases consisting of rhythms with many consecutive eighth notes, and again they do not reflect the perceptual structure of their counterparts. With respect to the CCN rule, the results are better than those obtained for the CN rule. Ternarization worked very well; for example, the ternarization of [x . . x . . x x . . x . x . . x] is the bembé. No match was obtained in the binarization, but the rhythms produced are still interesting inasmuch as they maintain a perceptual resemblance with their counterparts. CN and CCN rules do not give good results when transforming rhythms like [. x x . x .], since both rules produce rhythms with an onset on the first pulse. Obviously, that changes the essence of the rhythm since it takes an upbeat to a downbeat.

6.2 Centers of Rhythm Families

For the computation of centers of families of rhythms we used two distances, the Hamming distance and the directed swap distance; and two types of centers, the min-sum and the min-max functions. Both binary and ternary rhythms for all possible timespans of rhythms were included.

For binary rhythms of 8-pulses the results are summarized in Figure 12. We note that the rhythm [x x . x . . x .] is the center for all the distances and functions. The center for the directed swap distance with the min-max function contains four rhythms. Therefore, rhythm [x x . x . . x .] may be considered as the one most similar to the others.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	23	[x x . x . . x .]	Bin. var. 1(c)-2
Hamming	Min-Max	3	[x x . x . . x .]	Bin. var. 1(c)-2
Directed swap	Min-Sum	24	[x x . x . . x .]	Bin. var. 1(c)-2
Directed swap	Min-Max	4	[x . . x x . x .]	Habanera
			[x . x x . . x .]	Bin. var. 1(c)-1
			[x x . x . . x .]	Bin. var. 1(c)-2
			[. x . x . . x .]	Bin. var. 1(d)-2

Figure 12: Results for centers in the set of 8-pulse rhythms.

For the binary rhythms of 16 pulses we obtain the table shown in Figure 13. The clave son and its variation [x . . x . . x . . . x . x . x .] appear as centers in all cases. This is not surprising, since the set of binary rhythms considered are rhythms based mainly on the clave son.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	13	[x . . x . . x . . . x . x . . .]	Bin. clave Son
			[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1
Hamming	Min-Max	6	[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1
Directed swap	Min-Sum	12	[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1
Directed swap	Min-Max	4	[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1

Figure 13: Results for centers in the set of 16-pulse rhythms.

Figure 14 displays the results for the ternary rhythms of 6 pulses. As with the binary case, there is a rhythm that appears in all the centers, namely, variation 1(c)-1 [x x x . x .]. Again, this indicates that this rhythm is the one most similar to the others.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	19	[x x x . x .]	Var. 1(c)-1
Hamming	Min-Max	3	[x x x x x .]	Zamba foot
			[x x x . x .]	Var. 1(c)-1
			[x x x . x x]	Var. 2(c)
Directed swap	Min-Sum	18	[x x x . x .]	Var. 1(c)-1
Directed swap	Min-Max	3	[x . x x x .]	Choriamb
			[x x x . x .]	Var. 1(c)-1
			[. x x . x .]	Var. 1(d)

Figure 14: Results for centers in the set of 6-pulse rhythms.

Finally, the centers for ternary rhythms of length 12 are shown in Figure 15. The situation is very similar to that of binary rhythms. The 6/8 clave son and its variation [x . x . x . . x . x x .] determine the entire set of centers, the latter appearing in three centers out of four.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	11	[x . x . x . . x . x . .]	6/8 clave Son
Hamming	Min-Max	6	[x . x . x . . x . x x .]	6/8 clave Son var. 2
Directed swap	Min-Sum	9	[x . x . x . . x . x x .]	6/8 clave Son var. 2
Directed swap	Min-Max	4	[x . x . x . . x . x x .]	6/8 clave Son var. 2

Figure 15: Results for centers in the set of 12-pulse rhythms.

7 Concluding Remarks

In addition to some general conclusions, numerous specific conclusions may be drawn from the results of these experiments, concerning the data, the snapping rules, and the centers.

Data: It would be desirable to have more documented examples of binarized rhythms. The 12-pulse rhythms considered here are rather limited since they are almost all based on the 6/8 clave son. Figure 3 contains only five rhythms, four of which consist of the 6/8 clave son and its variants. Since we are using its binarizations as our set of binary rhythms, the situation repeats itself for the ternarization results.

Snapping rules: The snapping rules based on nearest (NN) and furthest (FN) neighbours appear not to work very well. In particular, the behaviour of the NN rule is surprizing. One would expect this rule to respect the perceptual structure of the original rhythm, but this is not the case, at least when it is applied to the entire rhythmic pattern. More experiments applying both rules at the metric foot level should be carried out. Rules based on snapping in a preferred direction, such as CN and CCN, work better than NN and FN. Curiously, CN works much better for binarization than for ternarization. On the other hand CCN performs better for ternarization.

Centers: As a consequence of the small size of the sets of 12- and 16-pulse rhythms, the results for the families of rhythms are poor in general. Centers computed on the families of 6- and 8-pulse rhythms are more meaningful. It would appear that perhaps a certain critical number of rhythms is necessary for the centers to make musical sense in the context of binarization and ternarization. Interestingly enough, on the whole, the computation of the centers yields large families of rhythms that are musicologically interesting on their own. Thus centers provide a nice tool for generating new rhythms that are similar to a given group, and can be used as composition tools as well as automatic rhythmic modulation rules. The entire collection is not listed here for lack of space, but it can be found on the web [30].

General conclusions: The results of these experiments are encouraging and suggest several avenues for further research towards our goals, which are the automatic generation of new rhythms as a composition tool, the possible testing of evolutionary theories of rhythm mutation via migration, the understanding of perceptual rhythm similarity judgements, and the development of a general computational theory of rhythm. The families of rhythms considered in this study are rather small since they consist of the documented examples of binarizations found in the literature. It would be very useful for comparison purposes to repeat these experiments using all known binary and ternary rhythms used in world music to discover which other binary-ternary pairs are identified by the snapping rules investigated here. Finally, mapping rules that use higher level musicological knowledge should be designed and compared to the context-free snapping rules used here, to determine how relevant such high level knowledge might be.

A Appendix

A.1 Preliminaries

We assume in this work that both rhythms are isochronous; this is also assumed in Perez-Fernandez's books. This implies that both rhythms have timespans of equal duration. Since the ternary rhythm has fewer pulses, it follows that these pulses are longer than those of the binary rhythm.

First we prove a lemma that will be used in the results to follow. Let K be a rhythm composed of p groups of k pulses each and M be a rhythm of p groups of m pulses each. Assume that $k, m > 1$. Let us consider the transformation of K into M.

Lemma 1 *There are two consecutive pulses in M equidistant to a pulse in K if and only if k is even and m is odd.*

Proof:

- For the "if" part: Since 2 divides k, point $\frac{k/2}{k} = 1/2$ is a pulse of K. Since m is odd, $m = 2\lfloor m/2 \rfloor + 1$. Let us consider the consecutive pulses $\lfloor m/2 \rfloor / m$ and $\lceil m/2 \rceil / m$ in M. Then, $\frac{k/2}{k}$ is equidistant to $\lfloor m/2 \rfloor / m$ and $\lceil m/2 \rceil / m$. Indeed:

$$\frac{k/2}{k} - \frac{\lfloor m/2 \rfloor}{m} = \frac{1}{2} - \frac{\lfloor m/2 \rfloor}{m} = \frac{m - 2\lfloor m/2 \rfloor}{2m} = \frac{1}{2m},$$

$$\frac{\lceil m/2 \rceil}{m} - \frac{k/2}{k} = \frac{\lfloor m/2 \rfloor + 1}{m} - \frac{1}{2} = \frac{2\lfloor m/2 \rfloor + 2 - m}{2m} = \frac{2-1}{2m} = \frac{1}{2m}.$$

- For the "only if" part: Let i, j be two indices such that $\frac{i}{m}$ and $\frac{i+1}{m}$ are equidistant to $\frac{j}{k}$. Then, the following equality holds:

$$\frac{i+1}{m} - \frac{j}{k} = \frac{j}{k} - \frac{i}{m} \Longrightarrow k(i+1) - mj = mj - ki \Longrightarrow k \cdot (2i+1) = 2j \cdot m.$$

  The last equality implies that k must be even and m must be odd.

■

A.2 Snapping the Concatenation of Metric Feet - Binarization

Let T be a ternary rhythm composed of p groups of 3 pulses each. Let B be binary rhythm composed of p groups of 4 pulses each. In this paper p takes on the values 2 and 4. Consider the transformation of T into B under the preceding snapping rules. The pulses in T and B (for $p \geq 1$) are given by the following sequences:

$$T = \left\{0, \frac{1}{3p}, \frac{2}{3p}, \ldots, \frac{3p-1}{3p}\right\},$$
$$B = \left\{0, \frac{1}{4p}, \frac{2}{4p}, \ldots\ldots, \frac{4p-1}{4p}\right\}.$$

By the definition of the snapping rules, pulses in T of the form $\frac{3i}{3p}$, for $i = 0, \ldots, p-1$, are assigned to $\frac{4i}{4p}$ in B. Consider the remaining pulses, namely: those of the form $\frac{3i+1}{3p}$ and $\frac{3i+2}{3p}$, for $i = 0, \ldots, p-1$. Given a fixed pulse t_i of T, when T and B are superimposed, it follows from the lemma that the distance function of t_i to the pulses of B possesses only one minimum value attained by only one pulse of B. Note that this statement is no longer true if we interchange T and B.

Observation:

1. the two nearest neighbours of pulses $\frac{3i+1}{3p}$ in B are $\frac{4i+1}{4p}$ and $\frac{4i+2}{4p}$, and the minimum is attained at $\frac{4i+1}{4p}$.
2. the two nearest neighbours of pulses $\frac{3i+2}{3p}$ in B are $\frac{4i+2}{4p}$ and $\frac{4i+3}{4p}$, and the minimum is attained at $\frac{4i+3}{4p}$.

For (1), the following three computations establish the result:

$$\text{distance from } \tfrac{3i+1}{3p} \text{ to } \tfrac{4i}{4p} : \frac{3i+1}{3p} - \frac{4i}{4p} = \frac{12pi + 4p - 12pi}{12p^2} = \frac{4}{12p}$$

$$\text{distance from } \tfrac{3i+1}{3p} \text{ to } \tfrac{4i+1}{4p} : \frac{3i+1}{3p} - \frac{4i+1}{4p} = \frac{12pi + 4p - 12pi - 3p}{12p^2} = \frac{1}{12p}$$

$$\text{distance from } \tfrac{4i+2}{4p} \text{ to } \tfrac{3i+1}{3p} : \frac{4i+2}{4p} - \frac{3i+1}{3p} = \frac{12pi + 6p - 12pi - 4p}{12p^2} = \frac{2}{12p}$$

From these computations it follows that the nearest and counter-clockwise neighbour of $\frac{3i+1}{3p}$ is $\frac{4i+1}{4p}$, and its furthest and clockwise neighbour is $\frac{4i+2}{4p}$.

For (2), analogous computations establish the result:

$$\text{distance from } \tfrac{3i+1}{3p} \text{ to } \tfrac{4i+2}{4p} : \frac{3i+2}{3p} - \frac{4i+2}{4p} = \frac{12pi + 8p - 12pi - 6p}{12p^2} = \frac{2}{12p}$$

$$\text{distance from } \tfrac{4i+3}{4p} \text{ to } \tfrac{3i+1}{3p} : \frac{4i+3}{4p} - \frac{3i+2}{3p} = \frac{12pi + 9p - 12pi - 8p}{12p^2} = \frac{1}{12p}$$

$$\text{distance from } \tfrac{4i+4}{4p} \text{ to } \tfrac{3i+1}{3p} : \frac{4i+4}{4p} - \frac{3i+2}{3p} = \frac{12pi + 12p - 12pi - 8p}{12p^2} = \frac{4}{12p}$$

The nearest and clockwise neighbour of $\frac{3i+2}{3p}$ is $\frac{4i+3}{4p}$, and its furthest and counter-clockwise neighbour is $\frac{4i+2}{4p}$.

A.3 Concatenating the Snapped Metric Feet - Binarization

Now we will snap 3-pulse ternary rhythms into 4-pulse binary rhythms, and concatenate p of them. We want to prove that the concatenation of the"local" snapping rules produces the "global" snapping rules described in the preceding. A ternary rhythm has 3 pulses, and its sequence is $\{0, \frac{1}{3}, \frac{2}{3}\}$. A binary rhythm has 4 pulses, and its sequence is $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$. Recall the rules from Figure 16, where distances between pulses have been added.

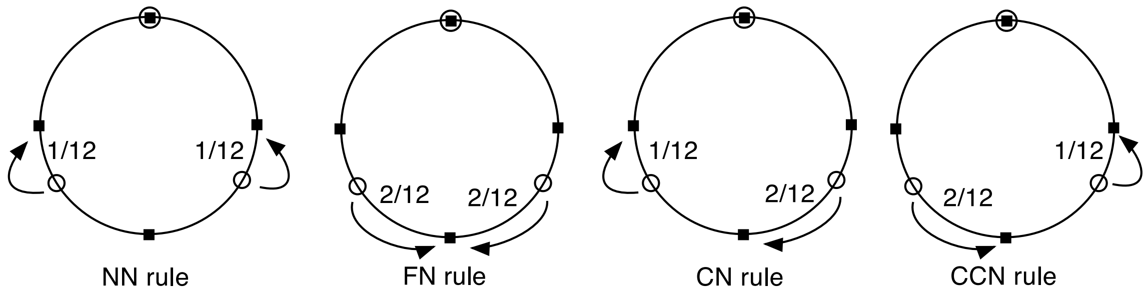

Figure 16: Snapping rules for binarization of metric feet.

Summarizing the information in Figure 16:

- The nearest, furthest, clockwise and counter-clockwise neighbour of 0 is 0.
- The nearest and counter-clockwise neighbour of $\frac{1}{3}$ is $\frac{1}{4}$ at distance $\frac{1}{12}$, and its furthest and clockwise neighbour is $\frac{2}{4}$ at distance $\frac{2}{12}$.
- The nearest and clockwise neighbour of $\frac{2}{3}$ is $\frac{3}{4}$ at distance $\frac{1}{12}$, and its furthest and counter-clockwise neighbour is $\frac{2}{4}$ at distance $\frac{2}{12}$.

When the ternary rhythms are concatenated, they must fit into a common time span. In order to do so, the ternary rhythm has to be scaled down by a factor of $1/p$. The relative distances between pulses are not affected by this scaling, and therefore, neither are the neighbours of a pulse. For example, if $p = 4$, the sequence $\{0, \frac{1}{3}, \frac{2}{3}\}$ is transformed into the sequence $\{0, \frac{1}{12}, \frac{2}{12}\}$, which in turn is transformed into $\{0, \frac{1}{12}, \frac{2}{12}, \ldots, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}\}$ when the 4 ternary rhythms are concatenated. The nearest neighbour of $1/12$ is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

When the division by p is performed and the rhythms concatenated we obtain the following:

- Pulses of the form $\frac{3i}{3p}, i = 0, \ldots, p-1$ are snapped to $\frac{4i}{4p}$ under all the rules.
- For pulses of the form $\frac{3i+1}{3p}$, for $i = 0, \ldots, p-1$, we have: $\frac{4i+1}{4p} < \frac{3i+1}{3p} < \frac{4i+2}{4p}$, where $\frac{4i+1}{4p}$ is the nearest neighbour of $\frac{3i+1}{3p}$.
- For pulses of the form $\frac{3i+2}{3p}$, for $i = 0, \ldots, p-1$, we have: $\frac{4i+2}{4p} < \frac{3i+2}{3p} < \frac{4i+3}{4p}$, where $\frac{4i+3}{4p}$ is the nearest neighbour of $\frac{3i+2}{3p}$.

Hence, we find the snapping rules are the same as the ones deduced above.

A.4 Snapping the Concatenation of Metric Feet - Ternarization

Let us consider ternarization, that is, the transformation from B to T. Computations similar to the case for binarization, yield the following table:

Element of B	First NN	Distance	Second NN	Distance	Third NN	Distance
$\frac{4i}{4p}$	$\frac{3i}{3p}$	0	NA	NA	NA	NA
$\frac{4i+1}{4p}$	$\frac{3i+1}{3p}$	$\frac{1}{12p}$	$\frac{3i}{3p}$	$\frac{3}{12p}$	$\frac{3i+2}{3p}$	$\frac{5}{12p}$
$\frac{4i+2}{4p}$	$\frac{3i+1}{3p}$, $\frac{3i+2}{3p}$	$\frac{2}{12p}$				
$\frac{4i+3}{4p}$	$\frac{3i+2}{3p}$	$\frac{1}{12p}$	$\frac{3i+3}{3p}$	$\frac{3}{12p}$	$\frac{3i+1}{3p}$	$\frac{5}{12p}$

Table 1: Table of distances for ternarization.

Legend: First NN refers to the nearest neighbour of the element of B with respect to the elements of T; the next column shows the distance between the pulse of B and its nearest neighbour. Second NN refers to the second nearest neighbour followed by its distance. Third NN denotes the third nearest neighbour followed by its distance. NA means not applicable, as $\frac{4i}{4p}$ is always mapped to $\frac{3i}{3p}$ according to all the rules.

Note that for pulse $\frac{4i+2}{4p}$ in B there are two pulses, $\frac{3i+1}{3p}$ and $\frac{3i+2}{3p}$, at an equal distance. The distance function still takes a unique minimum value, but this is attained at two values. This situation happens because B is a binary rhythm and has a pulse in the middle of each interval $[0, \frac{4}{4p}], \ldots, [\frac{4p-1}{4p}, \frac{4p}{4p}]$. Therefore, in this case, the nearest neighbour is not unique, and neither is the furthest neighbour.

A.5 Concatenating the Snapped Metric Feet - Ternarization

Figure 17 shows the snapping rules for ternarization of metric feet. The presence of ties is evident. The nearest- and furthest-neighbours are not unique.

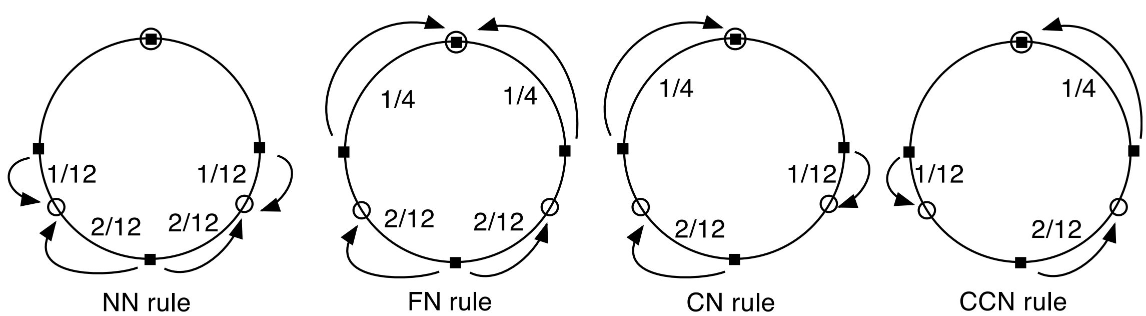

Figure 17: Snapping rules for the ternarization of metric feet.

Again, a reasoning similar to that for the case of binarization shows that the concatenation of the snapped feet results in equivalent snapping rules.

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