On Reconfiguring Tree Linkages: Trees can Lock

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Abstract

It is an open problem to determine whether a polygonal chain can be "straightened" in the plane if its links are not allowed to cross. In this paper we propose a related question: whether a *tree linkage* can always be "straightened" in the plane, without allowing its links to cross. We prove that this is not always possible. Indeed, we prove that an N-link tree linkage can have $2^{\Omega(N)}$ equivalence classes of configurations.

Key words: Motion Planning, Linkage Reconfiguration, Graph Embedding, Tree Embedding, Distance Geometry.

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1 Introduction

Consider a graph, each edge labelled with a positive number. Such a graph may be thought of as a collection of distance constraints between pairs of points in a Euclidean space. A *realization* of such a graph maps each vertex to a point, also called a *joint*, and maps each edge to the closed line segment, called a *link*, connecting its incident joints. The link length must equal the label of the underlying graph edge. If a graph has one or more such realizations, we call it a *linkage*.

An embedding of a linkage in space is called a *configuration* of the linkage. In a *simple configuration*, any pair of links whose underlying edges are incident on a common vertex intersect only at the common joint; all other pairs of links are disjoint. A *motion* of a linkage is a continuous movement of its joints in which the link lengths are respected at all times. In a *free motion*, the configuration of the linkage must remain simple throughout the motion. In a *semifree* motion, the joint of one link can touch another link, provided it does not properly cross over it. This notion can be elaborated technically in a variety of ways.

The *linkage reconfiguration* problem is the following: given two configurations, can the linkage move from one to the other? To complete the problem definition, one must specify the type of linkage considered, and what types of motion and configurations are allowed. This paper defines and explores a reconfiguration problem for linkages in the plane, where the underlying graph of the linkage is a tree. We call such a linkage a *tree linkage*. The initial configuration for the linkage is simple, and the linkage moves under free motion.

Many of the previous linkage motion results in the computational geometry literature (e.g. [9–13, 15–20, 22–24]) allow the links to cross or to pass through or over one another. In other words, the links represent distance constraints between joints, not physical obstacles that must avoid each other. Recently, other results for free motion have been obtained; see [1, 3, 4, 6, 8]. There is also the very general algebraic approach to motion planning of [5] and [21], since the constraint that the links not cross can be specified algebraically. For related work from a topological point of view, see [14] and references therein; again, this work allows links to cross.

Even the most basic questions concerning the reconfiguration of linkages have proved difficult to solve. For example, consider a linkage whose graph is a path. The question of whether a given configuration of such a linkage can always be reconfigured by a free motion to a stretched out configuration in which adjacent links form angles of measure π, has been open for several years. This question has been posed independently by several researchers, including W.

Lenhart and S. Whitesides [16], and J. Mitchell. For certain classes of starting configurations, positive results have been obtained [4]. Similarly, the question of whether a linkage whose graph is a single cycle can be reconfigured to form a convex polygon is only known for special cases. In the case that the initial configuration forms a star-shaped simple polygon, i.e., some point inside the polygon sees the entire boundary of the polygon, a free motion that convexifies always exists [8]. The same is true for the case that the initial configuration is a monotone simple polygon, i.e., each vertical line intersects the closed region bounded by the polygon in at most a single line segment [3]. In three dimensions, while a complete characterization isn't known, there are linkage configurations that can be straightened or convexified, and other configurations that cannot be [1]. In four or more dimensions, any path linkage can be straightened, and any cycle linkage can be convexified [6].

An equivalence relation on the set of simple configurations can be defined as follows. One simple configuration is *equivalent* to another means that the one can be moved to the other by a free motion. The basic problem we consider in this paper is whether the simple configurations of a tree linkage are reachable, one from another, under free motion. In other words, we ask whether a tree linkage can exhibit more than one equivalence class of simple configurations. Our main result is that this is indeed possible. This paper establishes in Theorem 5 that a suitably-constructed tree linkage has two simple configurations (pictured in Figures 1 and 2) but no motion between them is possible. As a corollary, we obtain the result that an N-link tree linkage can exhibit $2^{\Omega(N)}$ equivalence classes of simple configurations. Theorem 5 also answers a question posed in [7], arising from a paper folding problem.

The rest of this paper is organized as follows. Section 2 gives definitions, Section 3 explains the basic idea for constructing a tree configuration that cannot be straightened, Section 4 gives the construction itself, and Sections 5 and 6 give the correctness proof. Section 7 concludes with some open problems.

2 Preliminaries

In this section, we introduce the technical definitions used. From now on, we often do not distinguish between vertices and edges in the underlying tree, the corresponding joints and links in the tree linkage, and the points and line segments occupied by the joints in links in a particular configuration. The context should make the meaning clear.

Definition: Choose a vertex r to be the root of a given tree. For any vertex v, an incident link that joins v with its parent is called a *parent link*; otherwise

it is called a *child link*. An *ε-straightened configuration* of a tree linkage is one in which, for each joint, the angle between any two child links is $< \varepsilon$, and the angle between a parent link and a child link is $> \pi - \varepsilon$. To *ε-straighten* a linkage configuration means to move it to a ε-straightened configuration. We say a linkage configuration can be straightened, if it can be ε-straightened for any $\varepsilon > 0$. We say a linkage configuration is *locked* if there exists some $\varepsilon > 0$ for which it cannot be ε-straightened.

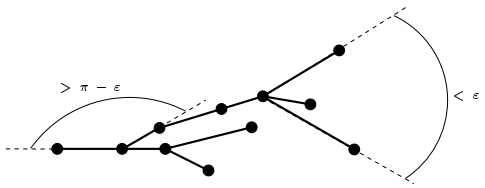

Fig. 1. Illustrating an ε-straightened configuration.

One can show that if a tree linkage configuration can be straightened, then it can be straightened using any joint as the root.

Next, we define more precisely what motion we allow.

Definition: A *motion* M is a map taking the set of vertices, V, into the plane, as a continuous function of time. We require the motion to begin at time 0 and end at time 1; thus $M : V \times [0, 1] \to E^2$. Furthermore, the distance constraint between adjacent joints given by the edge label must be respected over the entire time interval.

Definition: A motion M is *free* provided all configurations of the linkage are simple throughout the motion.

The trees considered in this paper consist of n *petals*, each comprised of three links. The initial configuration is sketched in Figure 2 and detailed as the paper unfolds.

The petals all meet at joint O; the other joints of petal i are labelled A_i, B_i, and C_i. Designate the *petal angle* as $\theta_i = \angle A_{i-1}OA_i$; let $\bar{\theta} = 2\pi/n$. All index arithmetic is taken mod n; all angles are measured in the interval $[0, 2\pi)$. The link lengths are as follows: $\|OA_i\| = 1$, $\|A_iB_i\| = l_1$, and $\|B_iC_i\| = l_2$, where for two points in the plane, X and Y, we use XY to designate the closed line segment between X and Y, and $\|XY\|$ to denote its length. The values of l_1 and l_2 will be discussed in Section 4.

We often focus on a single petal at a time, so the notation is simplified by suppressing the petal index. Joints of petal i are referred to as A, B, and C, joint A_{i-1} is denoted A', and the petal angle, i.e., $\angle A'OA$, as just θ. Let

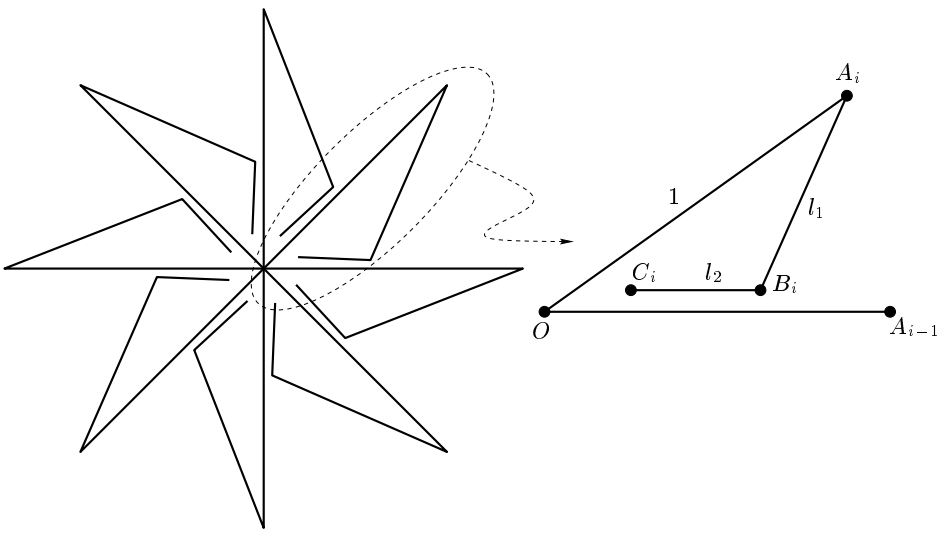

Fig. 2. An example linkage with enlarged view of petal i, showing the joint labels and the link lengths.

L be the line through OA', and choose a reference frame with L oriented horizontally, O to the left of A'.

3 The Intuition

We first give a brief outline of the main argument, before delving into the details.

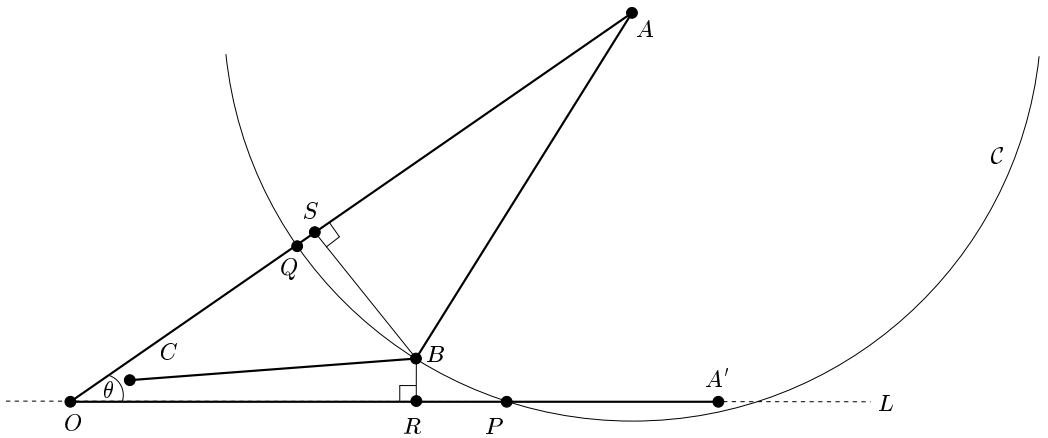

Fig. 3. Illustrating the definition of *restricted configuration*.

In the initial configuration, all petals are in a congruent configuration, pictured in Figure 3, with petal angle $\bar{\theta} = 2\pi/n$. By choosing l_1 long enough, we ensure that link AB cannot swing out so as to straighten joint A. Thus, the petal angle must be increased before joint A can straighten. Because the petals all join at O, opening up one petal necessitates squeezing the other petals.

Consider now how small we can squeeze a petal angle. By choosing l_2 long enough, we ensure that link BC cannot swing over to fold up against link AB. In fact, joint C is trapped inside quadrilateral $\square ORBS$. We show later that the smallest petal angle is obtained by moving joint C to O, pictured in Figure 4. Lengthening l_2 increases this minimum petal angle, so by choosing l_2 long enough, we can ensure that squeezing $n-1$ petals to their minimum angle is still not enough to let the last petal open. This, in essence, is why the tree is locked.

From the above discussion, we note that l_1 and l_2 must be chosen "long enough". On the other hand, they must also be "short enough" that the simple configuration shown in Figure 3 is achievable. Section 4 details all the constraints needed to satisfy these two requirements. Section 5 defines a "restricted" class of petal configurations (to which the configuration in Figure 3 belongs) and proves that there is a non-zero minimum petal angle for this class of configurations. Finally, Section 6 proves that a linkage with parameters n, l_1, and l_2, satisfying all the constraints of Section 4, can be put into a simple initial configuration that is locked.

4 Constructing a Locked Tree

This section details the constraints on the parameters n, l_1, and l_2 that are necessary to construct our locked linkage. Lemma 1, at the end of this section, demonstrates that the constraints are simultaneously satisfiable. Refer to Figure 3 throughout this section.

A three-link petal cannot be locked if the petal angle is greater than or equal to $\pi/2$. Thus we must have the initial petal angle ($\bar{\theta} = 2\pi/n$) strictly less than $\pi/2$, or

$$n > 4. \tag{1}$$

Henceforth, assume $\theta < \pi/2$. We want to have joint B to the left of the vertical line through A. In order that link AB fits, we must have

$$l_1 < 1. \tag{2}$$

For any configuration of a petal, let $\mathcal{C}$ be the circle centred at A with radius l_1. Let β be the petal angle at which $\mathcal{C}$ is tangent to L,

$$\beta = \arcsin l_1. \tag{3}$$

The range for the principal value of arcsine is $[-\pi/2, \pi/2]$. However, we know that l_1 must be positive, and less than 1 (Inequality 2), so β is actually in the range $(0, \pi/2)$.

When $\theta < \beta$, circle $\mathcal{C}$ will properly intersect L. We want this to be true for the initial configuration, in which all petal angles are $\bar{\theta}$, so we require

$$\bar{\theta} < \beta. \tag{4}$$

Suppose $\mathcal{C}$ properly intersects L and let P be the leftmost intersection point. Applying the cosine rule to $\triangle OAP$ and noting that $\|OA\| = 1$ and $\|AP\| = l_1$, yields $l_1^2 = \|OP\|^2 + 1 - 2\|OP\| \cos\theta$. Since l_1, the radius of $\mathcal{C}$, is strictly less than $\|OA\| = 1$ by Inequality 2, joint O is outside the circle $\mathcal{C}$. From this, and $\theta < \pi/2$, it follows that P is to the right of O, and hence $\|OP\|$ is the smaller root of the quadratic equation. We define the function

$$l_P(\theta) = \|OP\| = \cos\theta - \sqrt{l_1^2 - \sin^2\theta}. \tag{5}$$

This function is only defined on values of θ for which $\mathcal{C}$ intersects L; i.e., for θ in $[0, \beta]$. Differentiating 5 shows that l_P is a strictly increasing function on this interval.

Furthermore, P is to the left of the vertical line through A, and hence $P \in OA'$, but P is not at A'. We know that A is inside the circle $\mathcal{C}$, while O is outside, so $\mathcal{C}$ intersects OA; let Q be this intersection point. Note that the small arc PQ of $\mathcal{C}$ lies inside $\triangle A'OA$.

Suppose $\theta < \beta$, and B is on the small arc PQ of $\mathcal{C}$. Then B is inside $\triangle A'OA$. Since the triangle is isosceles, and the angle θ is $< \pi/2$, $\triangle A'OA$ is acute. This ensures that there is a line passing through B perpendicular to each edge of the triangle. Let $R \in OA'$ be such that $BR \perp OA'$ and $S \in OA$ be such that $BS \perp OA$. We want to have joint C inside quadrilateral $\square ORBS$. This is feasible for the initial configuration if we choose

$$l_2 < l_P(\bar{\theta}), \tag{6}$$

for then we may obtain a simple configuration by placing B near P, and C along OB. At the same time, we wish to have l_2 long enough that C remains trapped in $\square ORBS$. This is ensured by choosing

$$l_2 > \sin\beta \cos\beta, \tag{7}$$

as we show later in Section 5.

Define

$$\alpha = (2\pi - \beta)/(n-1). \tag{8}$$

Later, we will see that if a petal angle is opened past β, then some other petal angle must be smaller than α. The proof that the tree is locked then hinges on showing that there is a minimal petal angle, which is greater than α.

To obtain a non-zero minimal petal angle, we require

$$l_1 + l_2 > 1, \tag{9}$$

for otherwise, links AB and BC could fold flat against link OA, and the petal angle could squeeze to zero. Indeed, we show later, in Lemma 3, that the minimum possible petal angle is bounded from below by the petal angle obtained in the non-simple configuration with C at O, and $B \in OA'$, pictured in Figure 4. Define θ_m to be the resulting petal angle. With $\triangle OAB$, the cosine rule yields $l_1^2 = l_2^2 + 1 - 2l_2 \cos\theta_m$, or

$$\theta_m = \arccos\left(\frac{1 - l_1^2 + l_2^2}{2l_2}\right). \tag{10}$$

We are using the principal value of arccosine, so $\theta_m \in [0, \pi]$.

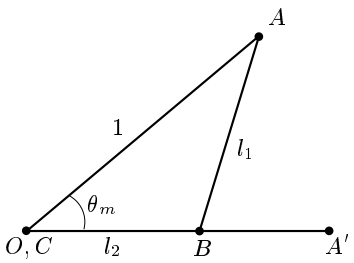

Fig. 4. For calculating θ_m.

Finally, in order to prove Theorem 5, we assume that l_1 and l_2 are such that

$$\alpha < \theta_m. \tag{11}$$

We prove, in Appendix A, the following lemma which states that all the constraints of this section may be simultaneously satisfied by an appropriate choice of n, l_1, and l_2.

Lemma 1 *For each integer $n > 4$, there exists two real numbers l_1 and l_2 satisfying simultaneously all the constraints of Section 4 (i.e., Inequalities 2, 4, 6, 7, 9, and 11).*

In the sequel, we assume some choice of parameters has been made such that all the constraints of this section hold.

5 Restricted Configurations

Referring to Figure 3, recall from the definition of β (Equation 3) that $\theta < \beta$ implies that circle $\mathcal{C}$ properly intersects link OA'. Thus, points P and Q are well-defined. When joint B is on the small arc PQ, recall that points R and S are also well-defined. And finally, joint C also fits inside $\square ORBS$, due to Inequality 6.

Definition: A petal configuration is said to be *restricted* if the following conditions hold:

(i) $\theta < \beta$,
(ii) B is on the open small arc PQ of $\mathcal{C}$, and
(iii) C is in the open region bounded by the quadrilateral $\square ORBS$.

Note in Figure 3 that $\|BR\|$ and $\|BS\|$ are both smaller than $\|BC\|$. We show now that this is always the case in a restricted configuration.

Lemma 2 *In a restricted configuration, both $\|BR\|$ and $\|BS\|$ are strictly less than l_2.*

Proof: For a point X, let $d(X, L)$ denote the distance from X to the line L through O and A'. We have that $\|BR\| = d(B, L)$. Because B is on the small arc PQ, $d(B, L) \leq d(Q, L) = \|OQ\| \sin\theta$. Hence,

$$\|BR\| \leq \|OQ\| \sin\theta. \tag{12}$$

By similar reasoning,

$$\|BS\| \leq \|OP\| \sin\theta. \tag{13}$$

Note that $\|OQ\| = 1 - l_1 = 1 - \sin\beta$, by Equation 3. For $x \in [0, \pi/2]$, $\sin x + \cos x \geq 1$, so $\|OQ\| \leq \cos\beta$. Because l_P is an increasing function, $\|OP\| = l_P(\theta) \leq l_P(\beta)$, and $l_P(\beta) \leq \cos\beta$, by Equation 5.

Both $\|OP\|$ and $\|OQ\|$ are $\leq \cos\beta$. Hence, by Inequalities 12 and 13, both $\|BR\|$ and $\|BS\|$ are bounded from above by $\cos\beta \sin\theta < \cos\beta \sin\beta$, as $\theta < \beta \leq \pi/2$ by assumption. But this is strictly less than l_2 by Inequality 7. $\square$

Lemma 3 *In a restricted configuration, $\theta \geq \theta_m$.*

Proof: Suppose, for contradiction, the petal is in a restricted configuration with $\theta < \theta_m$.

Consider the two triangles $\triangle OAB$ and $\triangle OAP$. These triangles share the common side OA, and $\|AB\| = \|AP\| = l_1$. Two sides of $\triangle OAB$ are equal length with two sides of $\triangle OAP$. Moreover the included angles satisfy $\angle OAB < \angle OAP$, since by definition of restricted configuration (condition (ii)), joint B lies on the small arc PQ of $\mathcal{C}$. Applying the cosine law to the remaining side in each triangle (or using Euclid's Proposition 24, Book I) we see $\angle OAB < \angle OAP$ implies $\|OB\| < \|OP\|$. By definition (Equation 5) $\|OP\| = l_P(\theta)$, which is less than $l_P(\theta_m)$ since l_P is an increasing function. A direct computation shows that $l_P(\theta_m) = l_2$, so we have $\|OB\| < l_2$.

By Lemma 2, we have also $\|BR\| < l_2$ and $\|BS\| < l_2$.

All four points O, R, B and S are strictly inside the circle of radius l_2 centred at B. Joint C is of course on this circle, so it cannot be inside $\square ORBS$. This contradicts condition (iii) of a restricted configuration. □

Lemma 4 *Consider a petal in a restricted configuration. Throughout any motion during which θ is strictly less than β, the petal remains in a restricted configuration.*

Proof: Note that the points P, Q, R, S, and the circle $\mathcal{C}$, are defined by the positions of the joints A and B; as the joints move, so do the points P, Q, etc. For simplicity, we omit displaying this dependence on time.

Consider, in turn, the three conditions required for a restricted configuration. Condition (i) holds throughout the motion by assumption.

In any configuration, B must be on $\mathcal{C}$. Since B starts the motion on the small arc PQ, for condition (ii) to be violated, B must pass through point P or through point Q. P and Q are on the interior of links OA' and OA, respectively, and since $\theta < \beta$, $\mathcal{C}$ always properly intersects these links. Thus B may not move through P or Q, and hence condition (ii) holds throughout the motion.

Given that B remains on the small arc PQ, points R and S are well defined. As C starts the motion inside $\square ORBS$, condition (iii) is violated only if C passes through one of the sides of this quadrilateral. Sides OR and OS are portions of links, so C may not pass through them. By Lemma 2, $\|BR\|$ and $\|BS\|$ are both strictly less than $\|BC\|$, so C cannot pass through side BR or side BS. Thus, condition (iii) holds throughout the motion. □

6 Trees Can Lock

This section describes our main result: a tree linkage in a simple configuration that cannot be straightened.

Recall that θ_m is defined by a triangle with sides 1, l_1, and l_2, pictured in Figure 4. With $\triangle OAB$, the cosine rule yields

$$l_1^2 = l_2^2 + 1 - 2l_2 \cos\theta_m,$$

while from Inequality 6 we can obtain

$$l_1^2 < l_2^2 + 1 - 2l_2 \cos\bar{\theta},$$

which implies $\theta_m < \bar{\theta}$. Putting this together with Inequalities 11 and 4, we have

$$\alpha < \theta_m < \bar{\theta} < \beta, \tag{14}$$

hence (α, β) is a non-empty interval.

Consider a tree in a configuration in which the petal configurations are all congruent, with petal angles all equal to $\bar{\theta} \in (\alpha, \beta)$. We place joint B on the small arc PQ. Because $l_2 < l_P(\bar{\theta}) = \|OP\|$ (Inequality 6) we can place B near, but not at, P to ensure $\|OB\| > \|BC\|$. This ensures that joint C may be placed along OB, which is inside $\square ORBS$. The configuration is simple. Furthermore, it satisfies all the conditions of a restricted configuration. Yet it cannot be straightened, as the next theorem shows.

Theorem 5 *Consider a tree of n petals in a configuration such that $\theta_i \in (\alpha, \beta)$ for $0 \leq i < n$, and with each petal in a restricted configuration. Such a configuration cannot be straightened.*

Proof: Suppose, to the contrary, that the configuration can be straightened. If so, we can choose to ε-straighten the tree with $\varepsilon < \alpha$, and joint O as the root of the tree. Since all the joints A_i are children of O, they will all lie in a small cone with vertex O. In particular, all but one of the petal angles will be less than $\varepsilon < \alpha$ when straightened (the last petal angle will be greater than $2\pi - \varepsilon$).

For the putative motion that straightens the configuration, let t_α be the first instant that some petal angle, say for petal k, reaches α. Let t_β be the first instant that some petal angle reaches β.

If $t_\beta < t_\alpha$, then at time t_β all angles are strictly greater than α and at least one is equal to β. This means

$$2\pi = \sum_{i=0}^{n-1} \theta_i > (n-1)\alpha + \beta = 2\pi,$$

a contradiction. Hence $t_\alpha \leq t_\beta$.

During the straightening motion, the joint angles change continuously in time. Since $\alpha < \theta_m$ by Inequality 11, and θ_k approaches α from above as t approaches t_α from below, we may choose $t_0 < t_\alpha$ such that $\theta_k \in (\alpha, \theta_m)$ at time t_0.

Note that during the motion up to time t_0, all petal angles are strictly less than β, as $t_0 < t_\alpha \leq t_\beta$. By Lemma 4 all petals remain in a restricted configuration throughout the interval $[0, t_0]$, so Lemma 3 applies to petal k. This means $\theta_k \geq \theta_m$, contradicting the choice of t_0.

Thus no motion to straighten this configuration exists. □

Recall that two simple configurations of a tree linkage are equivalent if one can be moved to the other by a free motion. Theorem 5 yields the following corollary.

Corollary 6 *There exist N-link tree linkages such that the linkages have $2^{\Omega(N)}$ equivalence classes of simple configurations.*

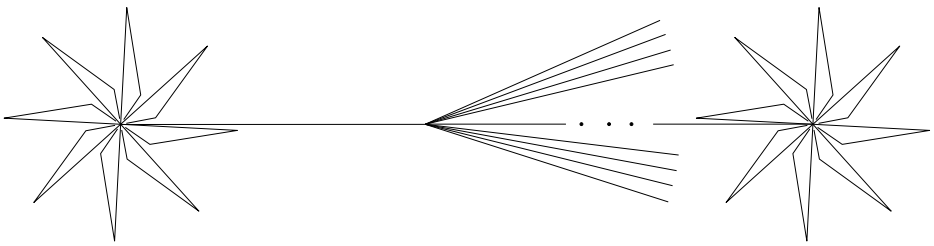

Fig. 5. A tree linkage formed by joining k copies of a lockable tree. Each subtree may be in either an open (as is the middle subtree) or a closed (the first and last subtrees) configuration. This linkage has at least 2^k configuration classes.

Proof: Consider the linkage in Figure 5, in which there are k copies of an eight-petal lockable tree connected by long links joining the O joints of the subtrees. The connecting links are long enough that when they are stretched out to form a straight chain, each subtree can be in either an open or a closed configuration without crossing links.

Consider simple configurations in which the long links form a straight chain, and each subtree is in either an open or a closed configuration. Label such

configurations by a k-bit vector, specifying for each subtree, whether its configuration is open or closed. Configurations with different labels are clearly not equivalent, as a free motion of the entire linkage that would take some subtree in a closed configuration to an open configuration would imply, by removing links outside the subtree, the existence of a free motion that would straighten a locked tree. Hence the number of inequivalent configurations is at least $2^k \in 2^{\Omega(N)}$, as $N = 24k + (k-1)$. □

7 Conclusion

The straightenability question for chain linkages (i.e. those whose underlying graph is a path) is still open. In this paper, we showed that for a tree linkage, there can be more than one equivalence class of simple configurations. Indeed, some N-link trees have $2^{\Omega(N)}$ equivalence classes.

The tree construction of Section 4 constrains the link lengths to be non-equal (it appears difficult to even get them nearly equal). This prompts the following question: can a tree linkage with equal-length links have a locked configuration? One "nearly equilateral" tree linkage is shown in Figure 6. We conjecture that if the link lengths are very nearly equal, this configuration is locked; the intuition is as follows. Each of the six triangular petals cannot collapse, so each remains nearly an equilateral triangle. If that is the case, it seems that each petal can only move by pivoting about its degree-3 joint, which it cannot do without crossing a link of the adjacent petal.

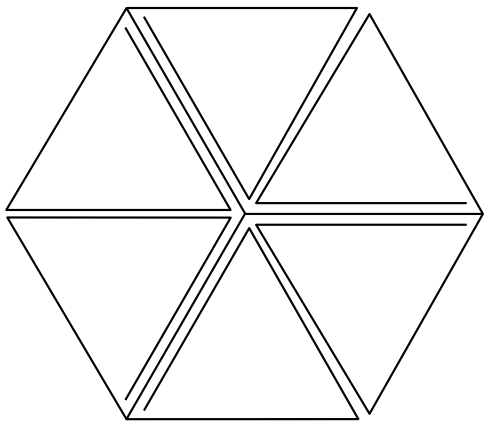

Fig. 6. A nearly equilateral tree linkage configuration.

Notice, however, that if the links truly are of equal length, the configuration pictured cannot be simple. Moreover, it is reasonable to suppose the configuration is locked (under a suitable notion of semifree motion). Whether weakly-simple configurations of chains can always be straightened with semifree motion is an open problem.

The nearly equilateral example has the feature that the graph has maximum degree three. It is easy to replace a high-degree joint with a number of degree-three joints joined by a chain of tiny links. Do equilateral locked tree linkages with maximum degree three exist? With maximum degree two?

Finally, many interesting questions can be posed for linkages moving in higher dimensions. See [1, 6] for recent work on chain and cycle linkages moving under simple motion in three and more dimensions.

Acknowledgements

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Our locked tree was inspired by a polygonal chain designed by Joe Mitchell, kindly shared with one of the coauthors. In particular, we borrowed the circular structure from his example.

A preliminary version of this work appeared in [2].

A Proof of Lemma 1

This appendix proves the following lemma.

> For each integer $n > 4$, there exists two real numbers l_1 and l_2 satisfying simultaneously all the constraints of Section 4 (i.e., Inequalities 2, 4, 6, 7, 9, and 11).

Proof:

We show that $l_1 = \sin(\bar{\theta} + \epsilon)$ and $l_2 = l_P(\bar{\theta} - \epsilon/n)$ are feasible link lengths, where $\epsilon = 0.01°$ is feasible for $n = 5, n = 6$ and any $0 < \epsilon < 0.4°$ is feasible for $n \geq 7$. The proof consists of checking, in turn, each of the constraints mentioned above.

Constraint 2: $l_1 < 1$

Given our choice of l_1, this is satisfied as long as $\bar{\theta} + \epsilon < \pi/2$. Recall that $\bar{\theta} = 2\pi/n$ and $n \geq 5$ (Inequality 1), so $\bar{\theta} \leq \frac{2\pi}{5}$. Since $\epsilon < 0.4°$, the constraint is satisfied.

Constraint 4: $\bar{\theta} < \beta$

By the definition of β (Equation 3), and our choice of l_1, we have $\beta = \bar{\theta} + \epsilon$. Since $\epsilon > 0$, the constraint is satisfied.

Constraint 6: $l_2 < l_P(\bar{\theta})$

Since $\epsilon > 0$, $\bar{\theta} - \epsilon/n < \bar{\theta}$. Using the definition $\bar{\theta} = 2\pi/n$, we see that $\bar{\theta} - \epsilon/n = (2\pi - \epsilon)/n$. Since $\epsilon < 0.4°$, $\bar{\theta} - \epsilon/n > 0$. Now, $l_P(\theta)$ is an increasing function on $[0, \beta] \supset (0, \bar{\theta})$, and $l_2 = l_P(\bar{\theta} - \epsilon/n)$, so the constraint is satisfied.

We summarize, for future reference, some inequalities derived so far,

$$0 < \bar{\theta} - \epsilon/n < \bar{\theta} < \beta = \bar{\theta} + \epsilon < \pi/2. \tag{A.1}$$

Constraint 7: $l_2 > \sin\beta\cos\beta$

This constraint is the only one for which we distinguish cases based on n. For $n = 5$ and $n = 6$, a direct calculation shows that the given link lengths (with $\epsilon = 0.01°$) satisfy the constraint.

For the case $n \geq 7$, we show that any $0 < \epsilon < 0.4°$ yields $l_2 > \frac{1}{2}$, whence the constraint follows since $\sin\beta\cos\beta = \frac{1}{2}\sin(2\beta) \leq \frac{1}{2}$ (the equality is an identity, the inequality uses $\sin x \leq 1$).

Plugging our choice of l_2 into the definition of l_P (5),

$$l_2 = l_P(\bar{\theta} - \epsilon/n) = \cos(\bar{\theta} - \epsilon/n) - \sqrt{l_1^2 - \sin^2(\bar{\theta} - \epsilon/n)}. \tag{A.2}$$

Cosine is a decreasing function on $[0, \pi]$, so $\cos(\bar{\theta} - \epsilon/n) > \cos\bar{\theta} > \cos(\frac{2\pi}{7})$ given $0 < \bar{\theta} - \epsilon/n < \bar{\theta} < 2\pi/7$ (A.1 and $n \geq 7$).

Under the radical of A.2, substituting $l_1 = \sin(\bar{\theta} + \epsilon)$ gives the expression $\sin^2(\bar{\theta}+\epsilon) - \sin^2(\bar{\theta}-\epsilon/n)$. Using the identity $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$, this expression becomes $\sin(2\bar{\theta}+\epsilon-\epsilon/n)\sin(\epsilon+\epsilon/n) \leq \sin(\epsilon+\epsilon/n) < \sin(2\epsilon) < \sin(0.8°)$. The last two inequalities follow by noting that $0 < \epsilon + \epsilon/n < 2\epsilon < 0.8° < \pi/2$, and sine is increasing on the interval $[0, \pi/2]$.

Using these bounds for the two terms in Equation A.2,

$$l_2 > \cos\left(\frac{2\pi}{7}\right) - \sqrt{\sin(0.8°)} \approx 0.505 > \frac{1}{2}.$$

Constraint 9: $l_1 + l_2 > 1$

Define function $f(\theta) = l_1 + l_P(\theta)$ on $[0, \beta]$. We know l_P is a strictly increasing function on $[0, \beta]$, hence so is f. Furthermore, $f(0) = 1$, so $f(\theta) > 1$ for $\theta \in (0, \beta]$.

From A.1, we see $\bar{\theta} - \epsilon/n \in (0, \beta)$, so $f(\bar{\theta} - \epsilon/n) > 1$. By definition of f, this becomes $l_1 + l_P(\bar{\theta} - \epsilon/n) > 1$. Since $l_2 = l_P(\bar{\theta} - \epsilon/n)$, we obtain that $l_1 + l_2 > 1$ as desired.

Constraint 11: $\alpha < \theta_m$

By definition, $\alpha = (2\pi - \beta)/(n-1)$ (Equation 8). This shows $\alpha \geq 0$. Rewriting 2π as $n\bar{\theta}$, and β as $\bar{\theta} + \epsilon$ (A.1), the equation for α becomes $\alpha = (n\bar{\theta} - \bar{\theta} - \epsilon)/(n-1) = \bar{\theta} - \epsilon/(n-1) < \bar{\theta} - \epsilon/n$. Combining all this with (A.1), we find

$$0 < \alpha < \bar{\theta} - \epsilon/n < \bar{\theta} < \beta < \pi/2. \tag{A.3}$$

From (A.3) we see that $\alpha \in (0, \pi)$. By definition (Equation 10), θ_m is also in $(0, \pi)$. Since cosine is decreasing on this interval, the constraint $\alpha < \theta_m$ holds if, and only if, $\cos\alpha > \cos\theta_m$. Using the definition of θ_m (Equation 10), this latter inequality becomes $2l_2\cos\alpha > 1 - l_1^2 + l_2^2$. We collect the terms in l_2 to one side, $l_2^2 - 2l_2\cos\alpha < l_1^2 - 1$, and complete the square to get

$$(l_2 - \cos\alpha)^2 < l_1^2 - \sin^2\alpha.$$

Since $l_1 = \sin\beta$ (by definition of β, Equation 3), the right hand side can be written as $\sin^2\beta - \sin^2\alpha$, which is positive since $\alpha < \beta$ by (A.3). Since both sides of the inequality are positive, we can take square roots which leads to

$$|l_2 - \cos\alpha| < \sqrt{l_1^2 - \sin^2\alpha},$$

so

$$-\sqrt{l_1^2 - \sin^2\alpha} < l_2 - \cos\alpha < \sqrt{l_1^2 - \sin^2\alpha},$$

and we deduce that l_2 (which equals $l_P(\bar{\theta} - \epsilon/n)$) must satisfy

$$\cos\alpha - \sqrt{l_1^2 - \sin^2\alpha} < l_P(\bar{\theta} - \epsilon/n) < \cos\alpha + \sqrt{l_1^2 - \sin^2\alpha}. \tag{A.4}$$

Comparing the left inequality of this with the definition of l_P (5), we see that the former can be written $l_P(\alpha) < l_P(\bar{\theta} - \epsilon/n)$. From (A.3) we note that α is less than $\bar{\theta} - \epsilon/n$ and both quantities are in the range $[0, \beta]$. The increasing property of l_P ensures that the left inequality of (A.4) is satisfied.

For the upper bound, note that by (A.1), $\bar{\theta} - \epsilon/n < \bar{\theta}$ and both quantities are in the range $[0, \beta]$. The increasing property of l_P ensures that $l_P(\bar{\theta} - \epsilon/n) < l_P(\bar{\theta})$. By the definition of l_P (5), $l_P(\bar{\theta}) < \cos\bar{\theta}$. Given $0 < \alpha < \bar{\theta} < \pi/2$ (A.3), the decreasing nature of cosine on this interval ensures $\cos\bar{\theta} < \cos\alpha$. Putting this together, we find $l_P(\bar{\theta} - \epsilon/n) < \cos\alpha$, so the upper bound of Inequality A.4 is satisfied.

□

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