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Computational Geometry and Morphology

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Computational geometry is a relatively new and flourishing discipline in computer science that deals with the analysis and design of algorithms for solving geometric problems. Morphology is the study of form or structure as in the case of the measurement of biological shape and shape change or the automatic recognition of shape by machines. In the latter case, we may distinguish between two phases of the underlying process: (1) the analysis and resulting description of shapes and (2) their subsequent classification into categories. In this paper, we survey recent computational geometric approaches to the problem of shape description and recognition by machines. In particular, under (1) we consider the medial axis of a polygon, shape hulls of sets of points, decomposition of polygons into perceptually meaningful components, smoothing and approximating polygonal curves, and computing geodesic and visibility properties of polygons.

1. INTRODUCTION

The term *morphology* is used in several disciplines in a rather narrow sense. For example, in biology it is that branch of study that deals with the form and structure of animals and plants [1]. In linguistics, it is the study and description of word formation in a language. We use the term here in its broadest sense: *morphology* is the *study of form*. The term *computational geometry* has also been used in several different contexts. For example, it has been used to describe that aspect of *geometric modeling of solids* that deals with computational issues [2]. It has been used to describe the study of shape recognition by certain models of parallel machines [3]. In an entirely different context, it refers to the computational issues in *integral-geometry* or *geometric probability* [4]. In this paper we use the term *computational geometry* as in the work of Michael Shamos [5]. In this latter sense it forms a new discipline in computer science recently bearing much fruit. For a text book and collection of papers, see [6], [7]. Several surveys of this area have also recently appeared [8] - [11].

This paper is an incomplete but representative survey of recent results that lie in the interface between *computational geometry* and *morphology*. To avoid duplication we concentrate on results not covered in [6] - [11] and refer the reader to these references for earlier work as well as basics such as the models of computation used, etc..

2. THE MEDIAL AXIS OF A POLYGON

Let $P = (p_1, p_2, \ldots, p_n)$ be a simple polygon with *vertices* p_i, $i = 1,2,\ldots,n$ specified in terms of cartesian coordinates in order. The *medial* *axis* of P, denoted by MA(P), is the set of points {q} internal to P such that there are at least two points on the boundary of P that are equidistant from {q} and are closest to {q}. The medial axis of a figure was first proposed by Blum [12] as a descriptor of the shape or form of the figure. It has since evolved into a theory of biological shape and shape change [37] - [39].

Since the introduction of the notion of *medial axis* there has been considerable interest in computing it efficiently under different models of computation [13] - [23]. Most of these algorithms take time proportional to n^2. Recently, Lee and Drysdale [19] and Kirkpatrick [18] have presented a general algorithm for finding continuous medial axes (or *skeletons*) of a set of *disjoint* objects. Lee & Drysdale's algorithm takes $O(n \log^2 n)$ time whereas Kirkpatrick's algorithm runs in $O(n \log n)$ time. Even more recently, a simpler $O(n \log n)$ algorithm was proposed by D. T. Lee [24]. Lee [24] shows that the *medial axis* is a subgraph of a structure known as the *generalized Voronoi diagram* and his algorithm first computes this diagram and subsequently removes the edges of it that are incident on *concave* vertices of P.

3. THE SHAPE OF A SET OF POINTS

When points in the plane have a finite diameter so that they are visible, and when they are fairly densely and uniformly distributed in some region in the plane then a human observer is quick to perceive the "shape" of such a set. These sets are usually referred to as *dot patterns* or *dot figures*. A polygonal description of the boundary of the shape is referred to as the *shape hull* of a dot pattern, where the vertices are given in terms of the cartesian coordinates of the centers of the dots. There are two versions of the shape hull (SH) problem: in one there are no "holes" in the dot pattern and the dot pattern is "simply connected" and hence the shape hull is a simple polygon, whereas in the more difficult problem both "holes" and "disconnected" components may exist. To add to this difficulty, in some instances illusory contours are perceived between "disconnected" components as illustrated by Kennedy and Ware [25]. For more details on this problem and early approaches to solving it, see [8]. A more recent paper addressing this problem is that of Medek [26].

In addition to describing the shape or structure of a set of points by its *shape-hull* or *external shape* we may also use the *skeleton* or *internal shape*. An early step in this direction was taken by Zahn [27] with the minimal spanning tree. More recent approaches have used the *relative neighborhood graph* [28] - [33].

A very elegant definition of the external shape of a set of points was put forward recently by Edelsbrunner, Kirkpatrick, and Seidel [34]. They propose a natural generalization of convex hulls that they call α-hulls. The α-hull of a point set is based on the notion of generalized discs in the plane. For arbitrary real valued α, a *generalized disc of radius* 1/α is defined as follows[34]:

(i) if $\alpha>0$, it is a (standard) disc of radius $1/\alpha$;
(ii) if $\alpha<0$, it is the complement of a disc of radius $1/\alpha$; and
(iii) if $\alpha=0$, it is a half plane.

The α-*hull* of a point set S is defined to be the intersection of all closed generalized discs of radius $1/\alpha$ that contain all the points S. The convex hull of S is precisely the 0-hull of S. The family of α-hulls includes the smallest enclosing circle (when $\alpha=1/\text{radius}(S)$), the set S itself (for all α sufficiently small) and an essentially continuous set of enclosing regions in between these extremes.

Edelsbrunner *et al.* [34] also define a combinatorial variant of the α-hull, called the α-*shape* of a point set, which can be viewed without serious misunderstanding as the boundary of the α-hull with curved edges replaced by straight edges. Unlike the family of α-hulls, the family of distinct α-shapes has only finitely many members. These provide a spectrum of progressively more detailed descriptions of the external shape of a given point set. Figure 1 (copied from [34]) illustrates the α-shape of a point set for two different values of α. In [34] efficient algorithms are also presented for computing the α-shapes of dot patterns consisting of n points in O(n log n) time.

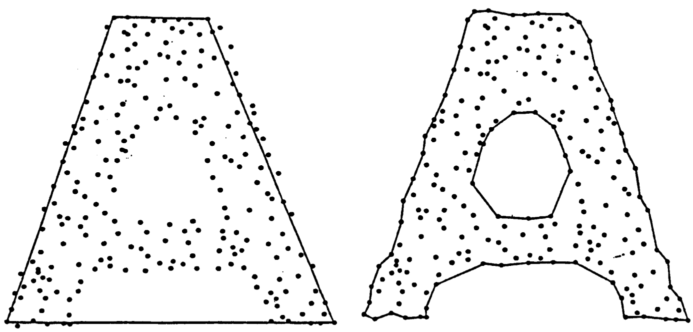

Fig. 1: Two different α-shapes of a common point set.

Subsequently, Kirkpatrick and Radke [69] outlined a new methodology for describing the *internal shape* of planar point sets. We should note that the ideas in [34] are closely related to the notions of *opening* and *closing* sets, found in mathematical morphology [35], [36].

We close this section by describing a new graph which I call the *sphere-of-influence* graph which I believe has some very attractive properties from the viewpoint of computer vision [40].

Let S be a finite set of points in the plane. For each point $x \in S$, let r_x be the closest distance to any other point in the set, and let C_x be the circle of radius r_x centered at x. The sphere of influence graph is a graph on S with an edge between points x and y if and only if the circles C_x and C_y intersect in at least two places. It is shown in [41] that:

(i) The sphere of influence graph has at most 29n edges ($n=|S|$).

(ii) Every decision tree algorithm for computing the sphere of influence graph requires at least $\Omega(n \log n)$ steps in the worst case.

As an application of (i), El Gindy observed that an algorithm of Bentley and Ottman [42] can be used to find the sphere of influence graph in O(n log n) time.

Motivated by the work in [34] we can also use the *sphere-of-influence* graph to define the *boundary* of a planar set S as either the *contour* of the union of the circles C_x, $x \in S$ (the *sphere-of-influence hull*) or the graph composed of those edges corresponding to pairs of points in S that have adjacent arcs on the *contour* defined above (the *sphere-of-influence shape*). These structures can be computed in O(n log n) time *without* computing the sphere of influence graph [40]. For the related problem of finding a perceptually meaningful *simple* polygon through S see [49]. Finally, the problem of drawing a simple polygon through a set of *line segments* is considered in [50].

4. DECOMPOSITION OF POLYGONS INTO PERCEPTUALLY MEANINGFUL COMPONENTS

Since [9] not much progress has been made on the morphological aspect of this problem. However, some new results are available on the geometrical side which are relevant. Chazelle & Dobkin [43] solve the problem of decomposing a non-convex simple polygon into a minimum number of convex polygons. They obtain an algorithm that runs in $O(n + c^3)$ time where n is the total number of vertices and c is the number of concave angles. For a recent survey of this area see [44].

One decomposition which appears to capture well the morphological aspect of the problem is the *relative-neighbour* decomposition [45]. In [45] an $O(n^3)$ algorithm is given for computing this decomposition. Recently, an $O(n^2)$ algorithm has been discovered [46].

5. APPROXIMATING POLYGONAL CURVES

Let $P = (p_1,p_2,\ldots,p_n)$ be a polygonal planar curve, i.e., P consists of a set of n distinct points (or nodes) $p_1,p_2,\ldots,p_n$ specified by their cartesian coordinates, along with a set of n-1 line segments joining pairs p_ip_{i+1}, $i=1,2,\ldots,n-1$. Note that in general P may intersect itself. Polygonal curves occur frequently in pattern recognition, image processing, and computer graphics. In order to reduce the complexity of processing polygonal curves it is often necessary to approximate P by a new curve which contains far fewer line segments and yet is a close-enough replica of P for the intended application. Many different approaches to this general problem exist and for a recent paper with 37 references the reader is referred to [47]. Different methods are more-or-less suited to different applications and yield solutions with different properties. In one instance of the problem it is required to determine a new curve $P' = (p'_1,p'_2,\ldots,p'_m)$ such that 1) m is less than n, 2) the p'_i are a subset of the p_i, and 3) any line segment $p'_jp'_{j+1}$ which substitutes the chain corresponding to $p_r,\ldots,p_s$ in P is such that the distance between each p_k, $r\le k\le s$, and the line segment $p'_jp'_{j+1}$ is less than some predetermined error tolerance w. An often used error criterion is the minimum distance between p_k and $p'_jp'_{j+1}$, i.e., $d(p_k,p'_jp'_{j+1}) = \min_x\{d(p_k,x)\,|\,x\varepsilon p'_jp'_{j+1}\}$, where $d(p_k,x)$ is the euclidean distance. No attempt has been made at minimizing m. Recently, Imai and Iri [48] proposed an $O(n^3)$ algorithm for determining the approximation that minimizes m subject to the two other constraints. Another criterion, often used, measures the distance between p_k and $p'_jp'_{j+1}$ as the minimum distance between p_k and a line $L(p'_j,p'_{j+1})$ colinear with p'_j and p'_{j+1}, i.e.,

$$d(p_k,p'_jp'_{j+1}) = \min_x\{d(p_k,x)\,|\,x \varepsilon L(p'_j,p'_{j+1})\}.$$

This is termed the "*parallel-strip*" criterion [47] since it is equivalent to finding a strip of width 2w such that p'_j and p'_{j+1} lie on the center line of the strip and all points p_k, $r\le k\le s$ lie in the strip. In [47] it is shown that if the *parallel-strip* criterion is used, the complexity of the algorithm of Imai and Iri can be reduced to $O(n^2 \log n)$. Furthermore, if the polygonal curves are *monotonic*, and a suitable error criterion is used, the complexity can be further reduced to $O(n^2)$.

6. COMPUTING GEODESIC PROPERTIES OF POLYGONS

Given a polygon P and two points a,b ε P, the shortest path (or *geodesic* path) between a and b is a polygonal path connecting a and b which lies entirely in P such that the sum of its euclidean edge-lengths is a minimum over all other internal paths. Intuitively, it is the shape an ideal elastic band would take if it were attached to a and b and the boundary of P consisted of barriers. We denote it by $GP(a,b|P)$ where the direction is from a to b. Geodesic paths find application in many areas such as image processing [51], operations research [52], visibility problems in graphics [53], and robotics [57] - [58]. Recently, Chazelle [54] and Lee & Preparata [52] independently discovered the same $O(n \log n)$ algorithm for computing $GP(a,b|P)$. Both of these algorithms first triangulate P and then find the shortest path in $O(n)$ time. An algorithm due to El Gindy [55] computes $GP(a,b|P)$ *without* first triangulating P, also in $O(n \log n)$ time. More recently, it has been discovered that a simple polygon can be triangulated in $O(n)$ time [59]. This result allows one to compute the geodesic path between two points in $O(n)$ time [60].

In the context of morphology, shortest paths are used for measuring shape properties of figures [38], [51], [61]. For example, the *length* of a biological object [51] is the length of the *longest* geodesic path (the *geodesic diameter*) between any pair of points in the object. Several algorithms have been proposed for computing the geodesic diameter of a simple polygon; for example, an $O(n^2)$ time and $O(n^2)$ space algorithm [54], an $O(c^2n)$ time and $O(n)$ space algorithm [61] where c is the number of convex vertices, and an $O(n^2)$ time and $O(n)$ space solution [62].

Another very useful geodesic property is the *geodesic center* of a polygon, i.e., that point in P that minimizes the length of the *longest* geodesic path to any point in P. Parallel algorithms for computing both the *geodesic diameter* and *center* of a pattern on a lattice are given in [51]. Asano and Toussaint [63] show that the geodesic center of a polyogn can be computed in $O(n^4 \log n)$ time.

7. COMPUTING VISIBILITY PROPERTIES OF POLYGONS

The notion of visibility is one that appears in many applications. In a morphological context visibility relations between vertices and edges of a polygon can be used as shape descriptors [9]. Much attention has been given to the problem of the visibility from a *point*.

A topic which has not been as much investigated as visibility from a *point* concerns the notion of *visibility from an edge*. A polygon P is *weakly visible from an edge* $[p_ip_{i+1}]$ if for every point x ε P there exists a y ε $[p_ip_{i+1}]$ such that [xy] lies inside P. Given a polygon P and a specified edge $[p_ip_{i+1}]$ of P, the *edge visibility polygon of P from an edge,* denoted by $EVP(P,[p_i,p_{i+1}])$ is that region of P that sees at least one point of $[p_ip_{i+1}]$. Intuitively, it is the region of P visible, at one time or another, by a guard patrolling edge $[p_ip_{i+1}]$. Recently, El Gindy [55], Lee & Lin [64], and Chazelle & Guibas [65] all independently proposed three different algorithms for computing $EVP(P,[p_i,p_{i+1}])$ in $O(n \log n)$ time. In the case where the polygon may have n "holes", Suri & O'Rourke [66] present an $O(n^4)$ algorithm for computing the boundary of the polygonal region *visible from an edge* and prove that it is optimal.

Toussaint [67] has shown that with the result of [59] the *edge visibility polygon* of P from an edge can be computed in $O(n)$ time. A similar algorithm was recently discovered independently by Guibas et al., [60].

A related problem concerns itself with answering queries of the type: are two specified edges in a polygon visible [53]? While this problem can be answered with the algorithms in [67] and [60] in O(n) time, the machinery used [59] is rather heavy. In [68] it is shown that even without all the heavy machinery of [59] such queries can still be answered in O(n) time.

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Q: Various geometrical concepts introduced by Toussaint in his interesting review appear to have alternative designations elsewhere especially in mathematical morphology (rf. Serra's 1982 book). Your example, 'medial axis' is alternatively 'skeleton', and 'α-hull' is alternatively 'opening or closing by a disk of radius α . It does seem as though stronger links should be forged between the relevant computational geometry and mathematical morphology groups. This present symposium is certainly a wonderful step in that direction! (R. Miles)

A: I agree whole heartedly. The main differnce between computational geometry and mathematical morphology lies not so much in the concepts used but in their approach. Mathematical morphology approaches problems on a digitized grid, matrix of pixels, or lattice whereas computational geometry works in a (continuous) vector graphics type mode. Therefore, the algorithms used and their measures of complexity are qualitatively quite different in both fields. A closer interaction between these two areas should answer the question of which approach is computationally more efficient in practice.