Computational Aspects of Musical Rhythms:

COMP 251 Course Notes
Godfried Toussaint

1. Necklaces and Bracelets,
2. Homometric Rhythms,
3. The Hexachordal Theorem,
4. Patterson's Theorems, and
5. Flat Rhythms and Deep Rhythms

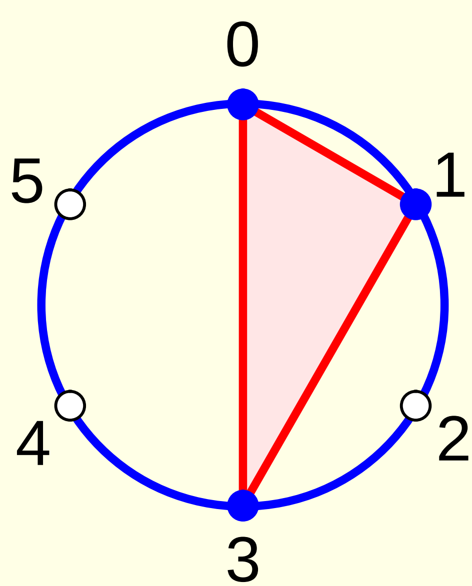
All-Interval Flat Rhythms

Consider a rhythm with k onsets in a time-span (clock) of n (even) units.

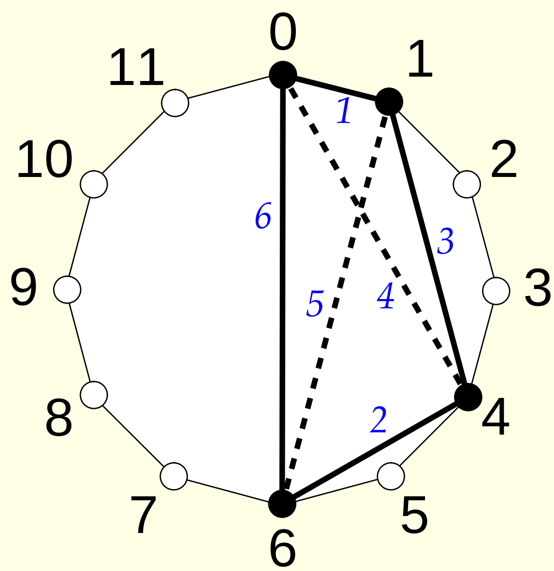
The rhythm clock determines n/2 different possible durations between pairs of pulse points.

A rhythm is called all-interval flat if it contains all the n/2 duration intervals, and each of the intervals is used precisely once.

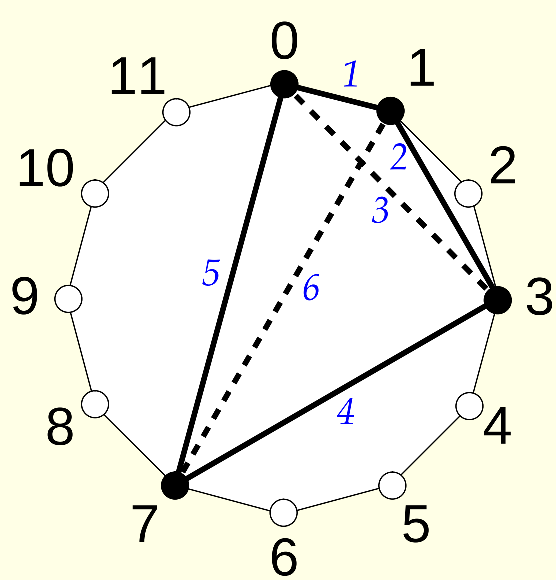
Example of all-interval flat rhythm for n=6, k=3



An (k = 4, n = 12) All-Interval flat Rhythm Bracelet



Another (k = 4, n = 12) All-Interval flat Rhythm Bracelet



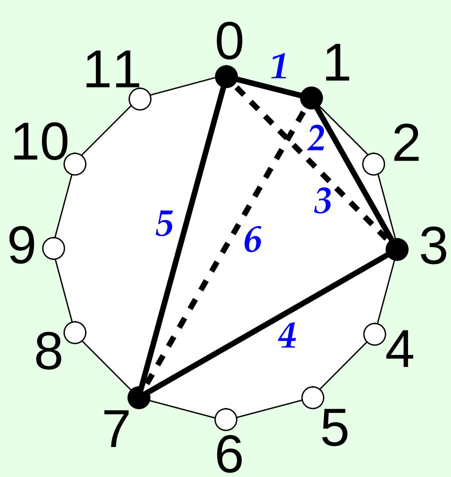
The *Two-Bracelets* Theorem

For n = 12 there exist only two all-interval flat bracelets yielding 16 all-interval rhythms.

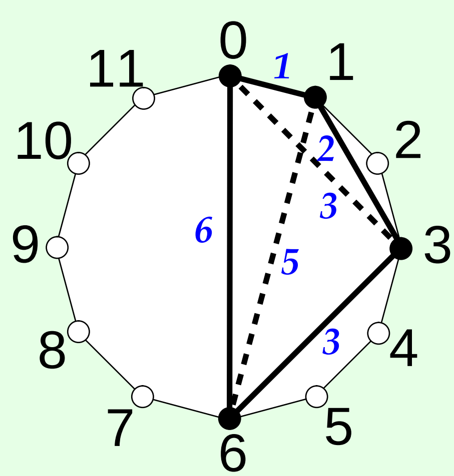
The equation k(k-1)/2 = 6 has only one solution: k = 4. Therefore only rhythms with 4 onsets are candidates.

There are two cases: the longest interval is either: (1) a diagonal or (2) an edge of the resulting quadrilateral.

case 1

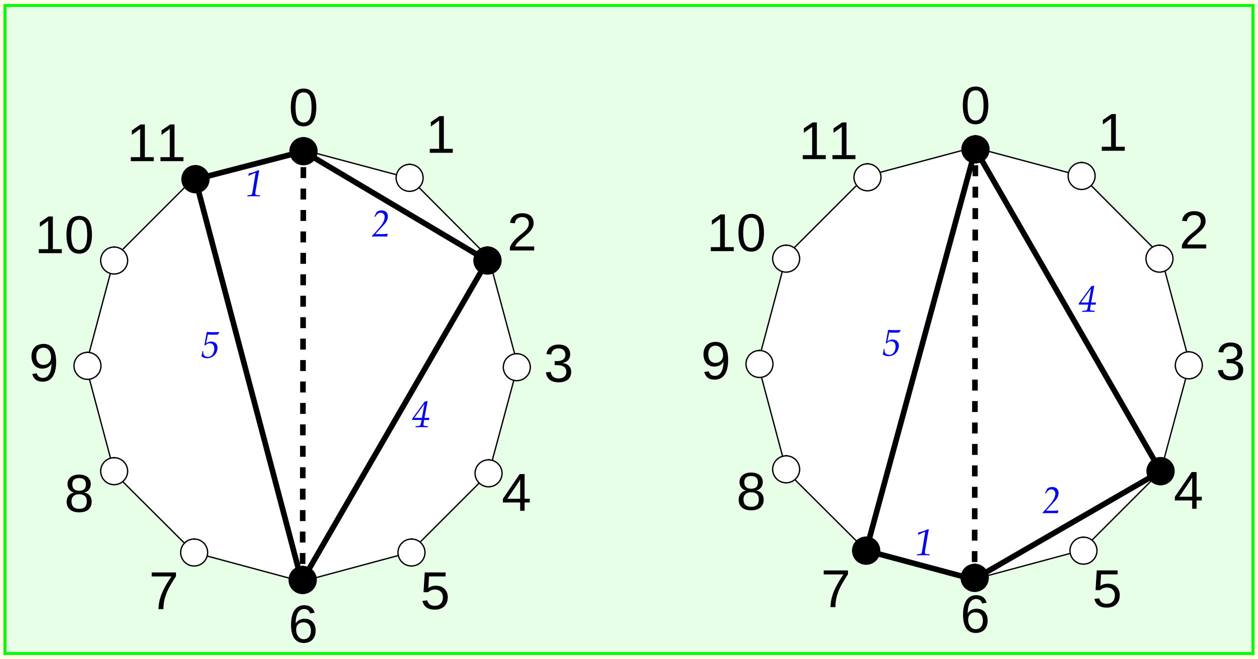


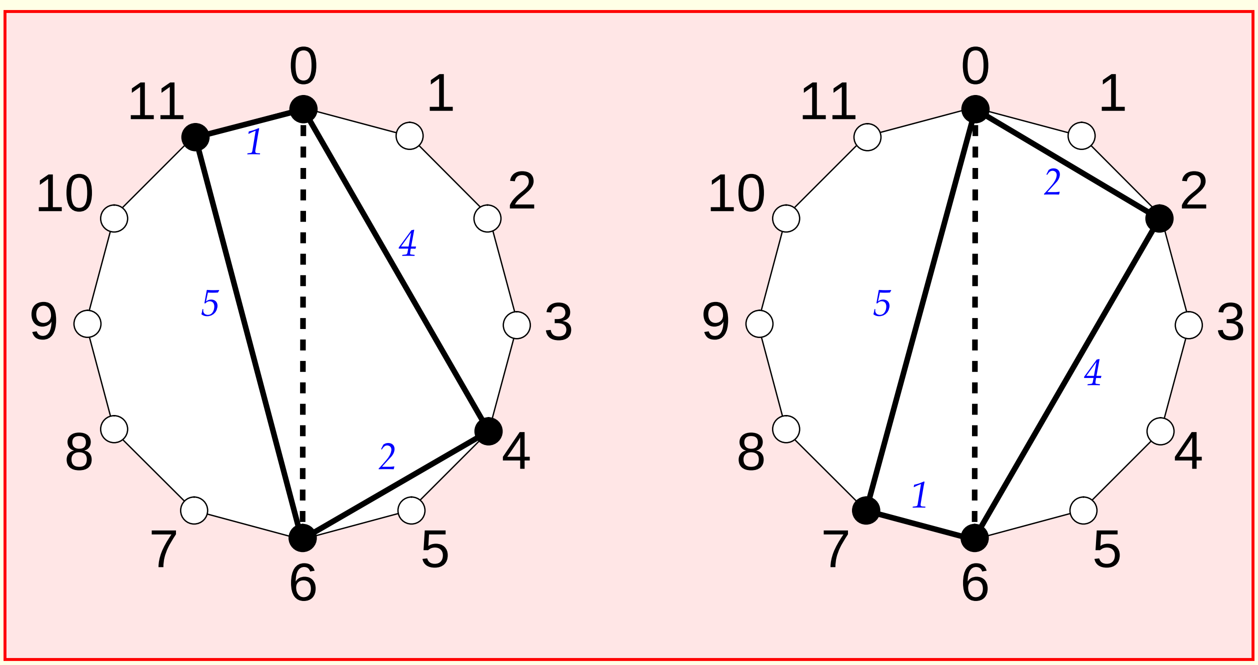
case 2



The *Two-Bracelets* Theorem - *cont.*

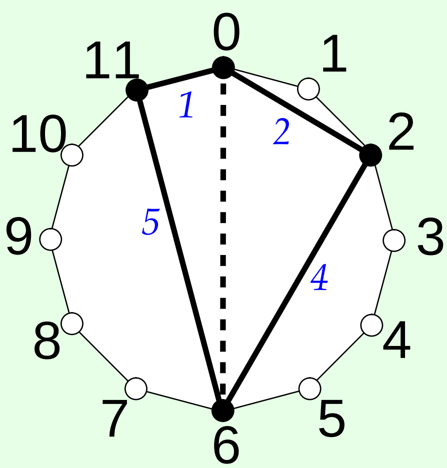
case 1: *diameter determined by diagonal*

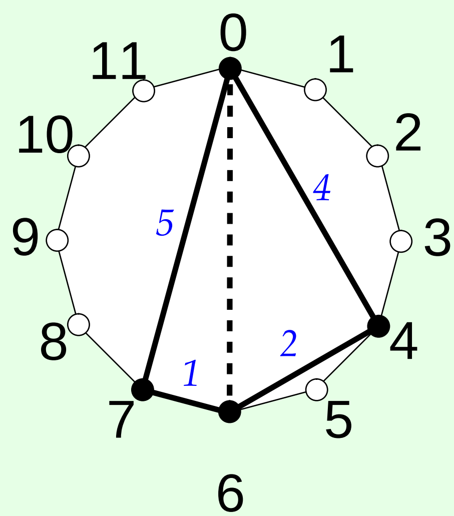


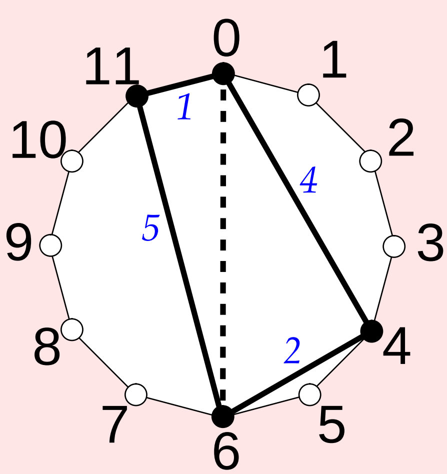


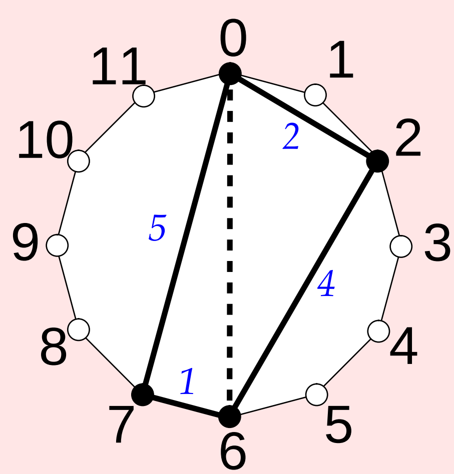
The *Two-Bracelets* Theorem - *cont.*

case 1: *diameter determined by diagonal*

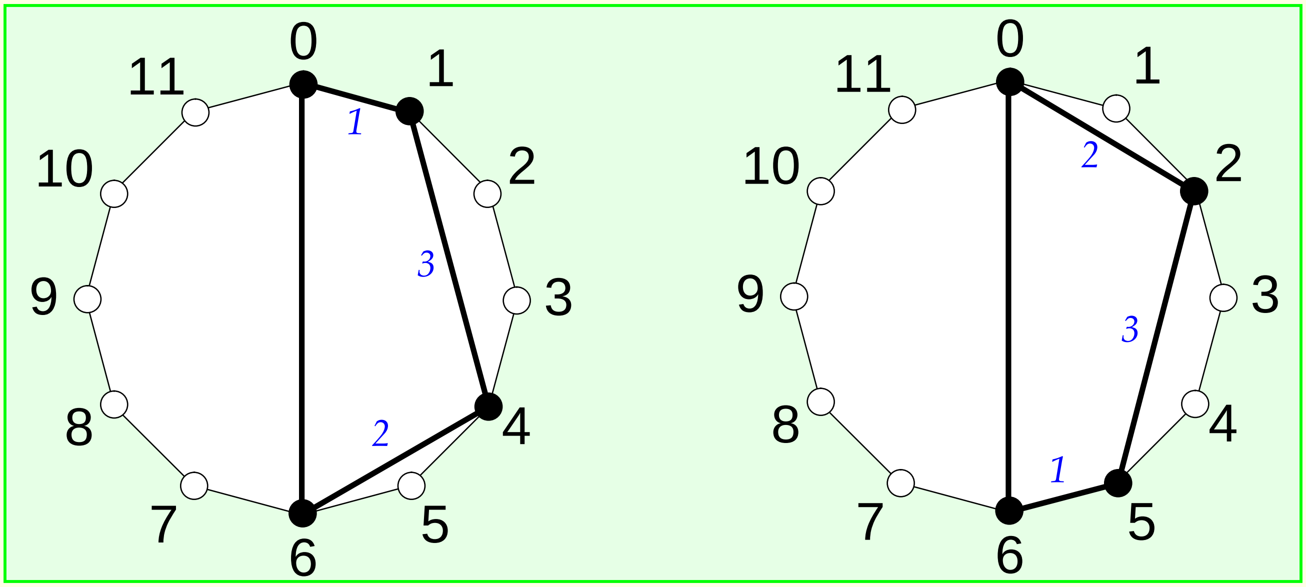


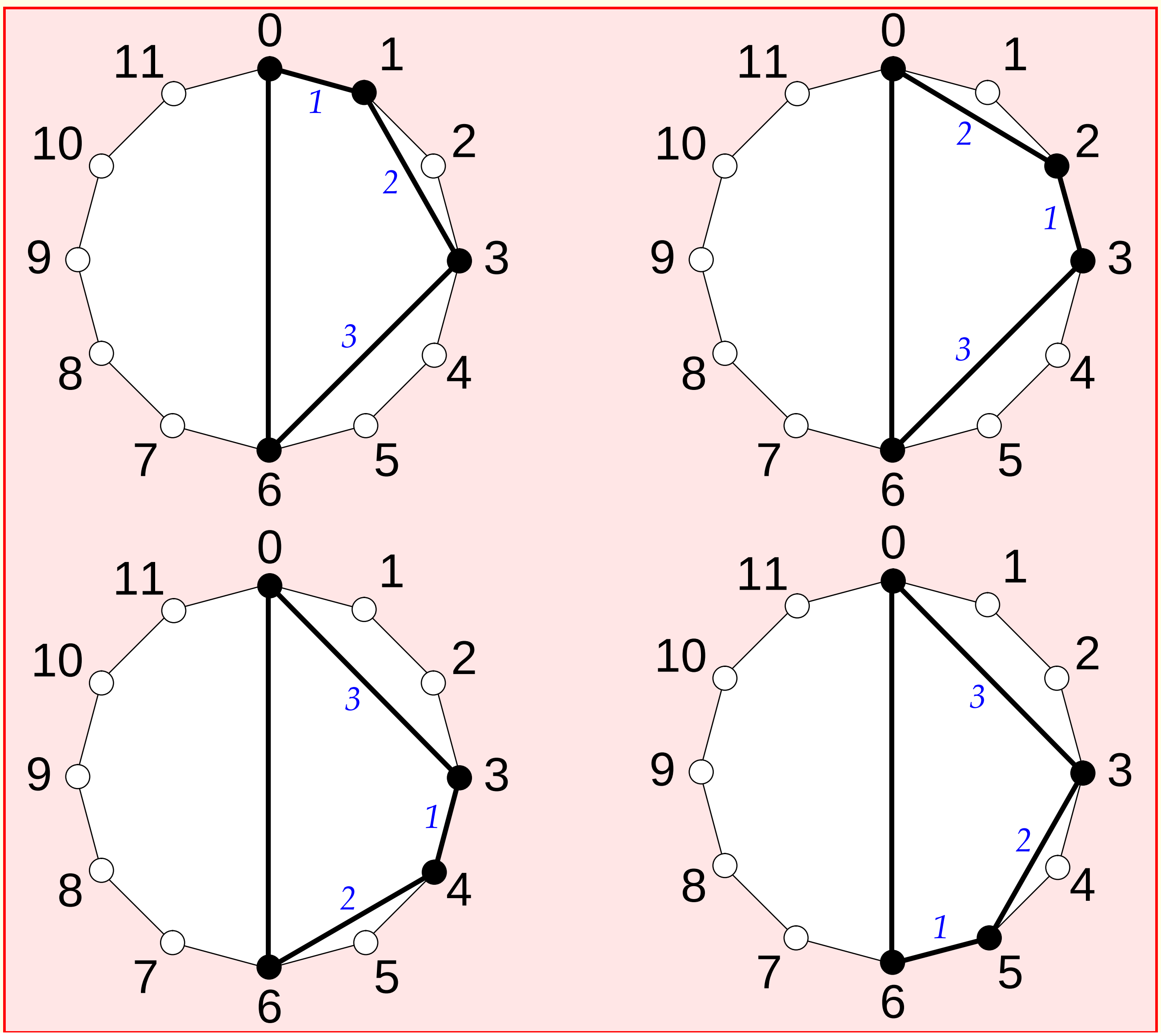




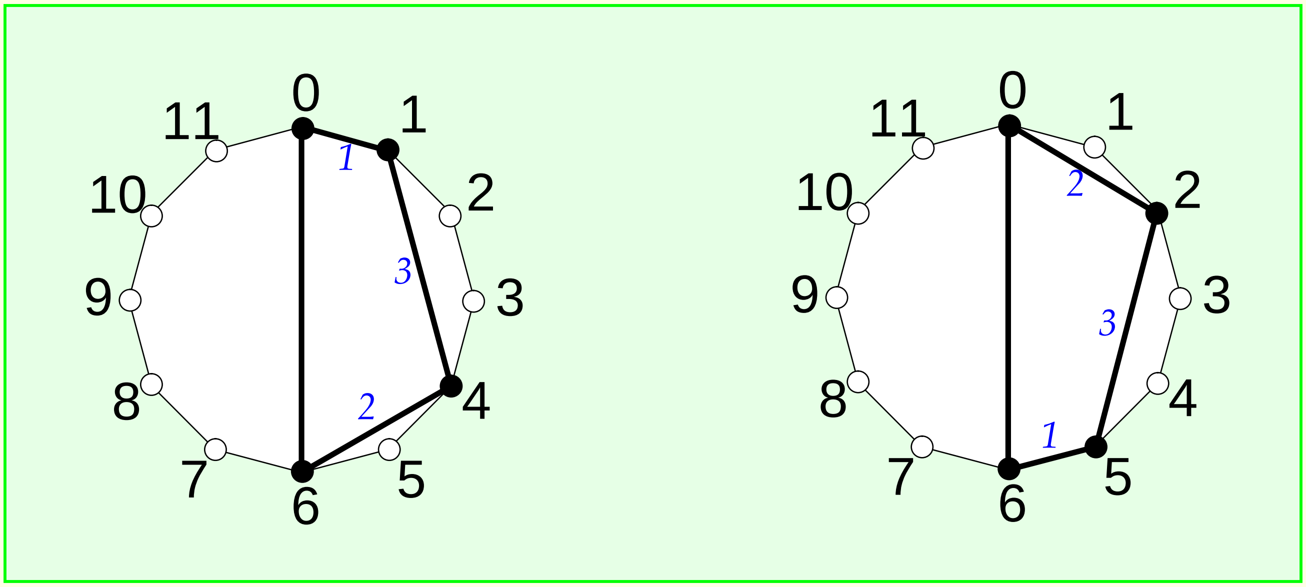


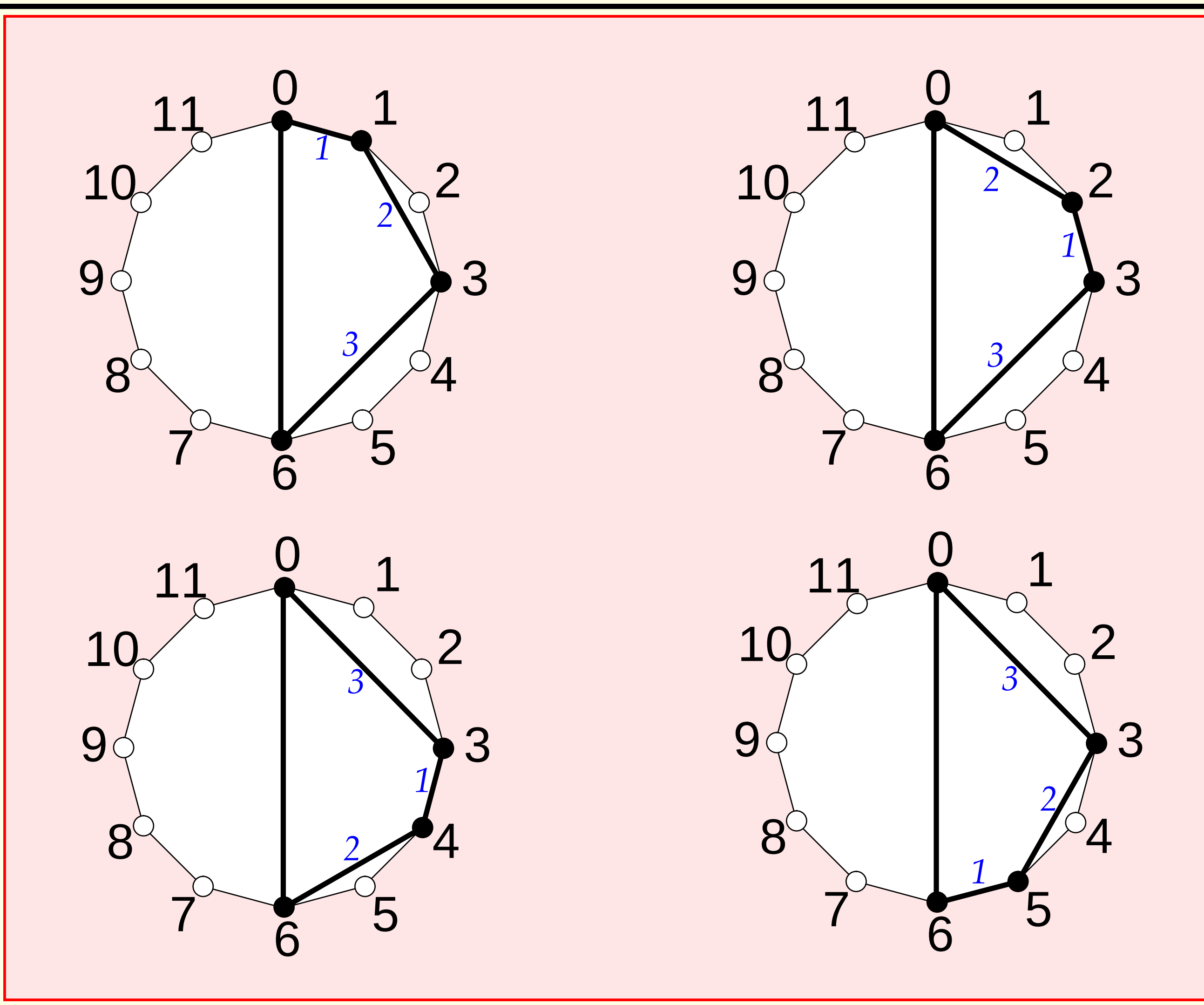
case 2: *diameter determined by edge*





case 2: *diameter determined by edge*



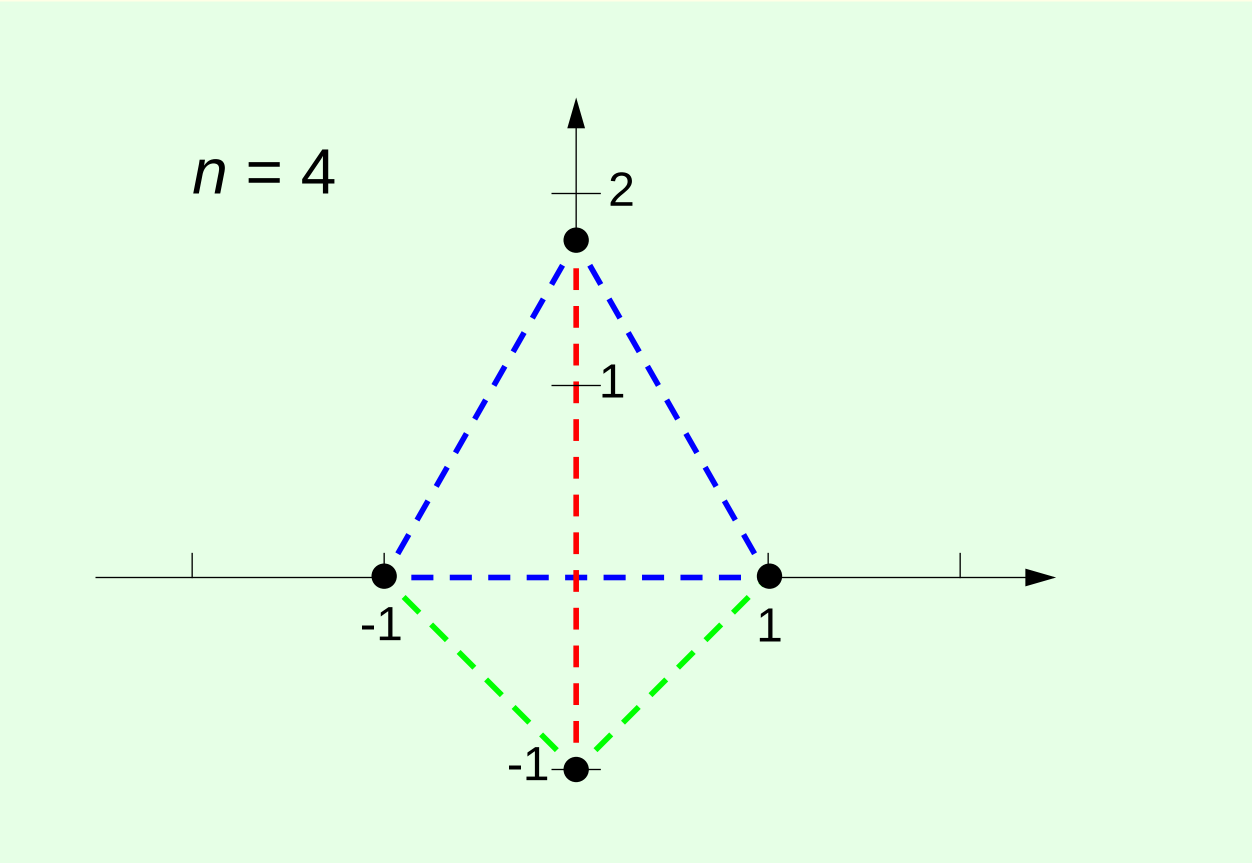


Points with specified distance multiplicities

Paul Erdös - 1986

Can one find n points in the plane (no 3 on a line and no 4 on a circle) so that for every i, $i = 1, 2, ..., n-1$ there is a distance determined by these points that occurs exactly i times?

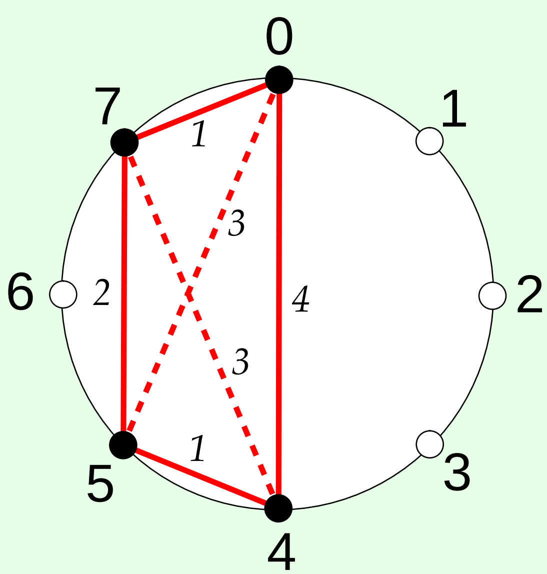
Solutions have been found for $n = 2, 3, ..., 8$.
Ilona Palásti for $n = 7$ and 8.

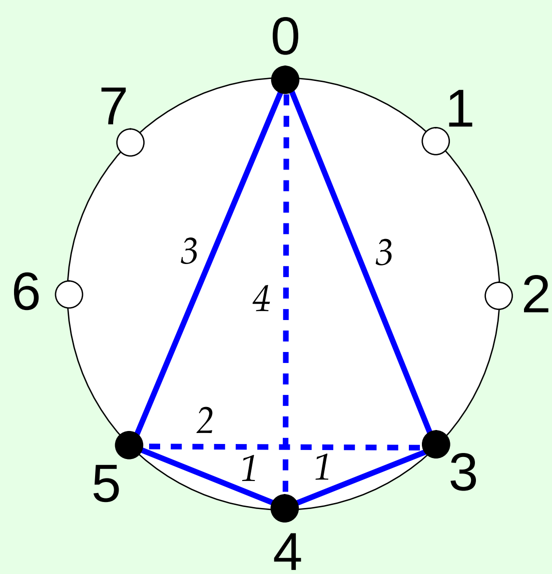


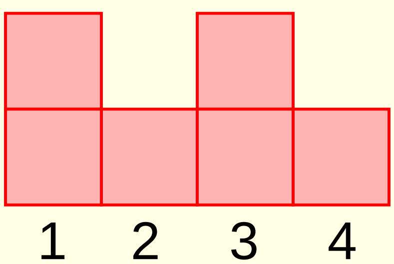
Patterson's example of a homometric pair

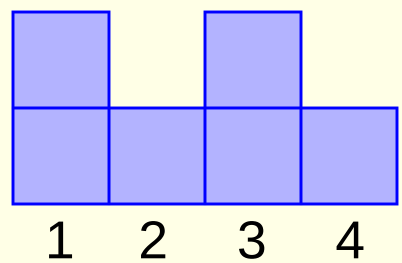
A. Lindo Patterson,
"Ambiguities in the X-ray analysis of crystal structures," *Physical Review*, March, 1944.

A simple homometric pair.





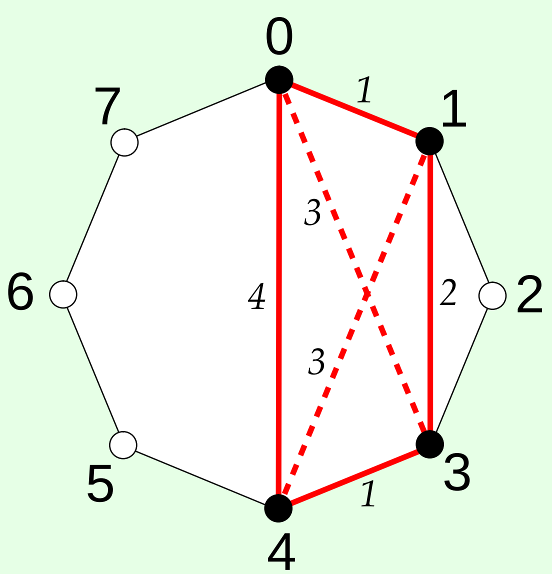




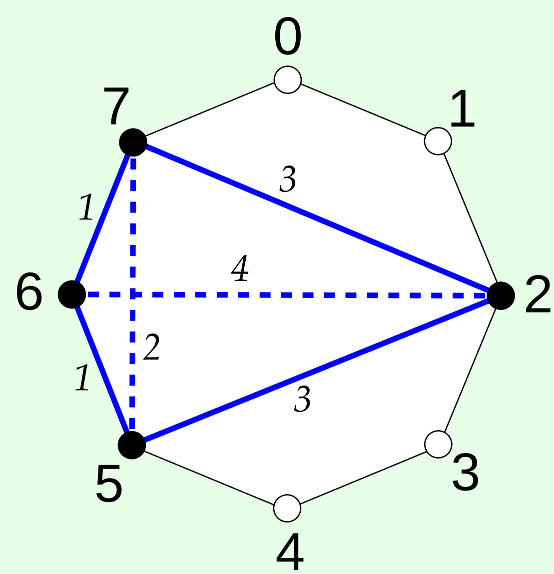
Complementary homometric rhythms

V. G. Rau, L. G. Parkhomov, V. V. Ilyukhin and N. V. Belov, 1980

Every *n*-point subset of a regular 2*n*-gon is homometric to its complement.



Low Conga

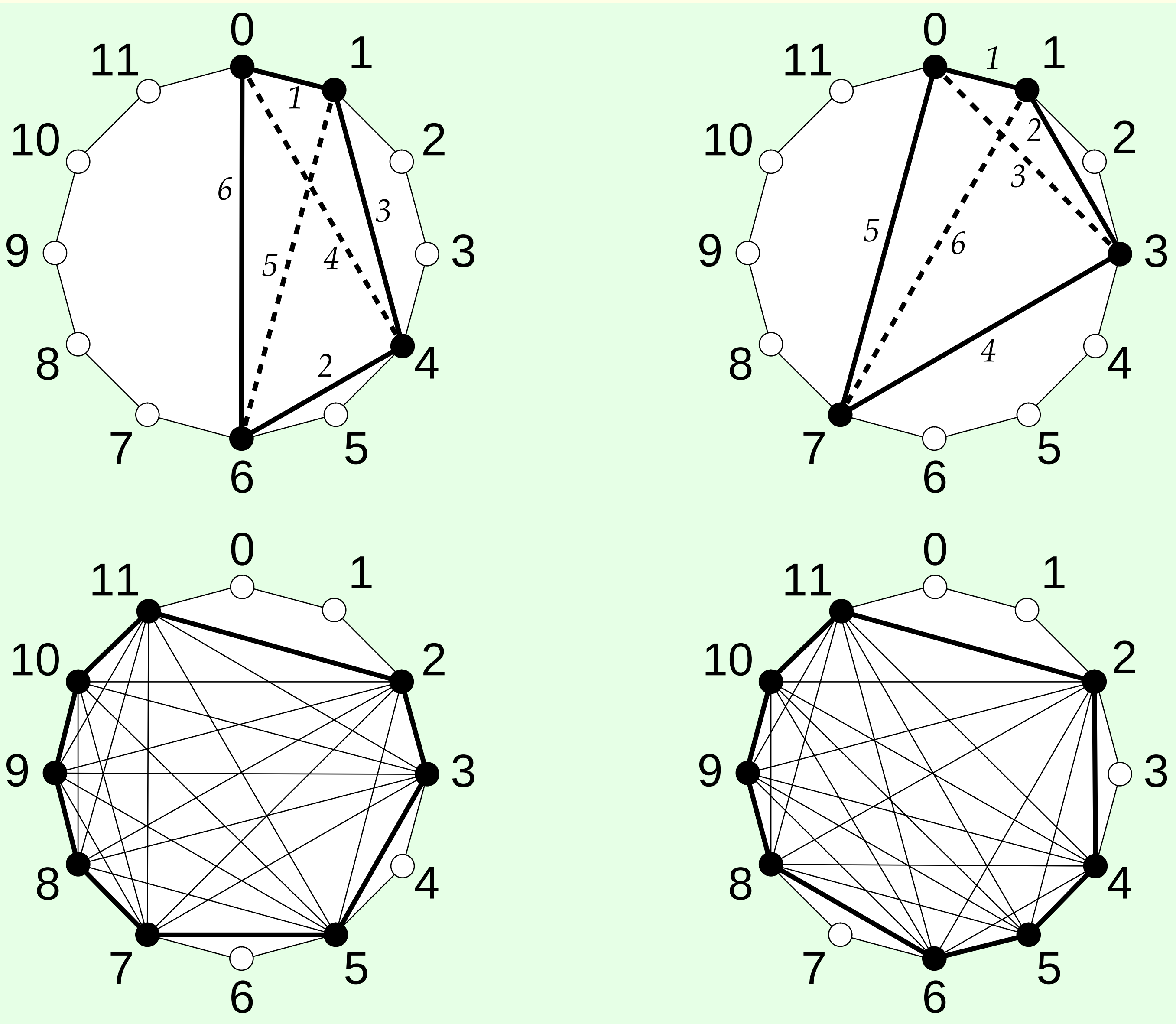


High Conga

Patterson's first theorem

A. Lindo Patterson,
"Ambiguities in the X-ray analysis of crystal structures," *Physical Review*, March, 1944.

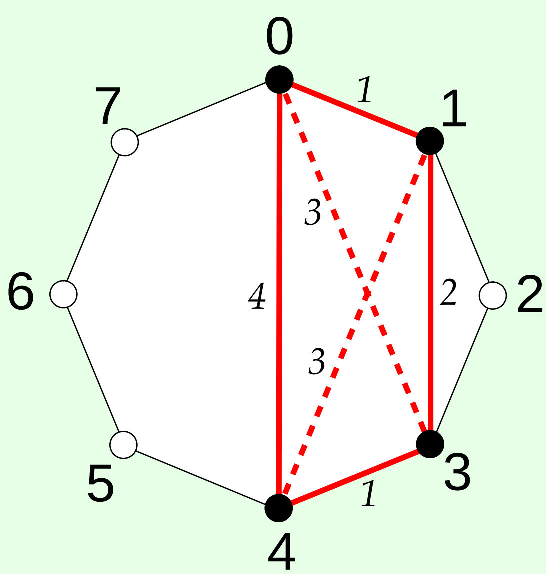
If two subsets of a regular *n*-gon are homometric, then their complements are.

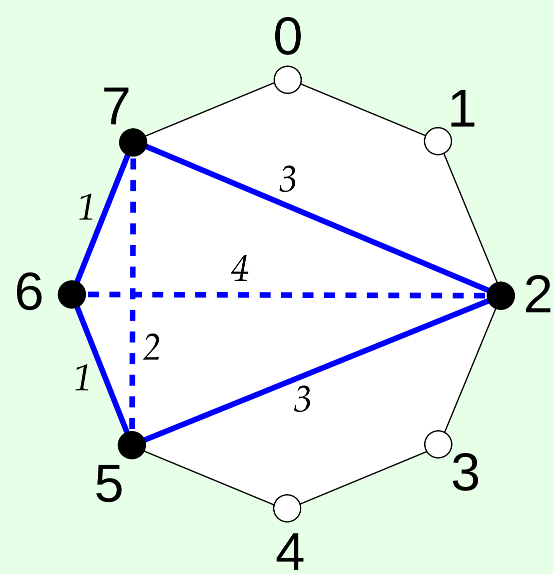


Patterson's second theorem

A. Lindo Patterson,
"Ambiguities in the X-ray analysis of crystal structures," *Physical Review*, March, 1944.

Every *n*-point subset of a regular 2*n*-gon is homometric to its complement.

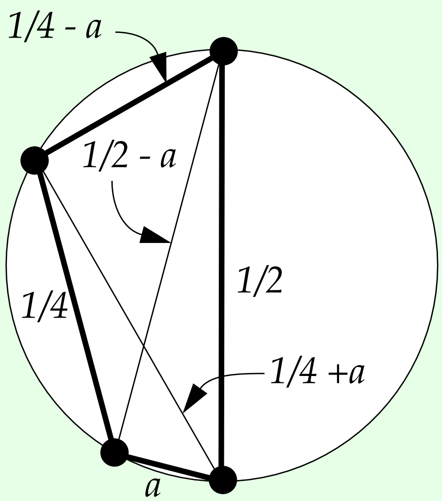


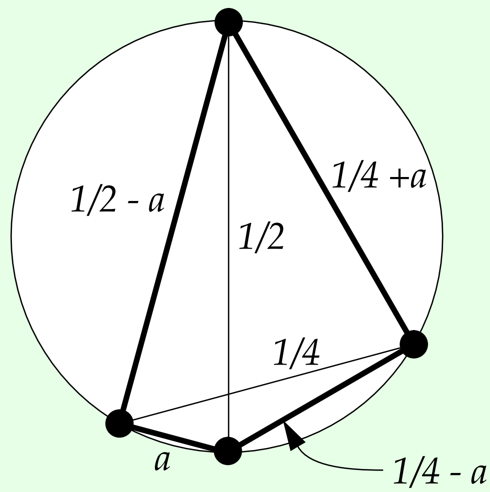


Erdös infinite family of homometric pairs

Paul Erdös,
in personal communication to A. Lindo Patterson,
Physical Review, March, 1944.

$a < 1/4$

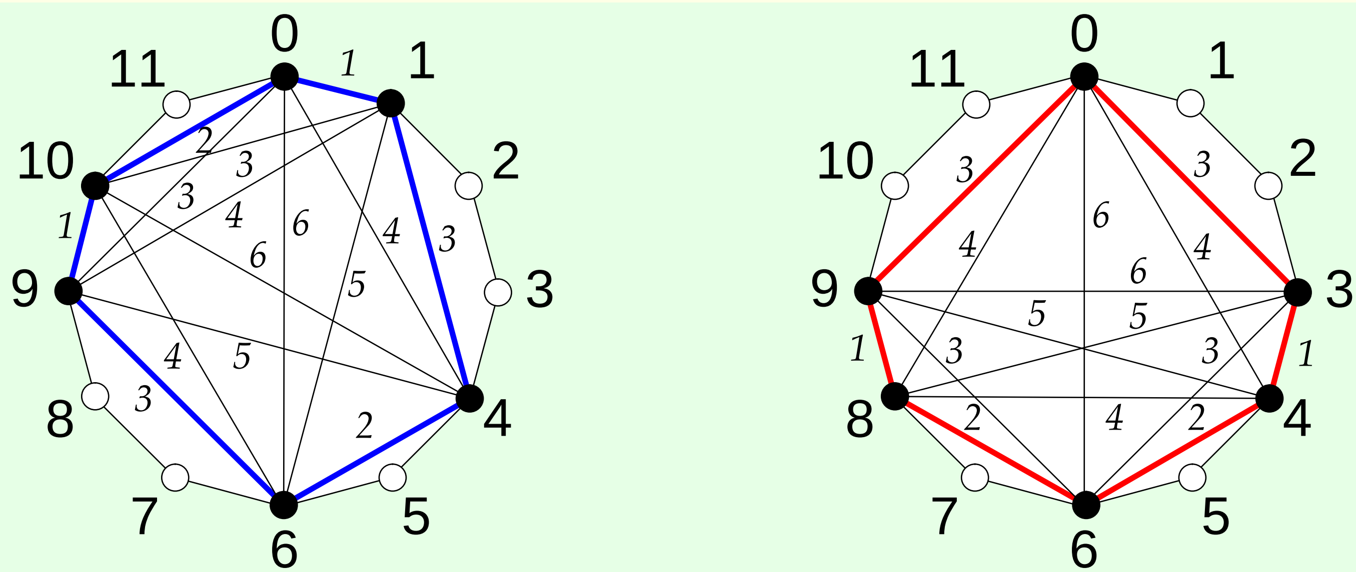




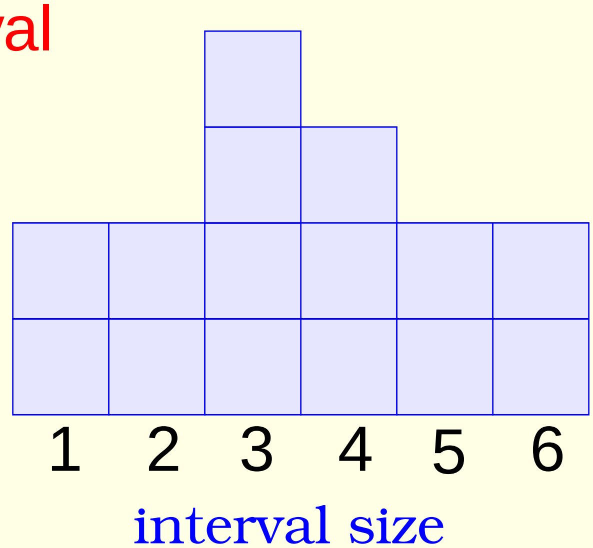
The Hexachordal Theorem

Theorem: Two *complementary* hexachords have the same *interval content*.

First observed empirically: Arnold Schoenberg, ~ 1908.



pitch interval histogram



The Hexachordal Theorem: Music-Theory Proofs

Theorem: Two *complementary* hexachords have the same *interval content*.
First observed empirically: Arnold Schoenberg, 1908.

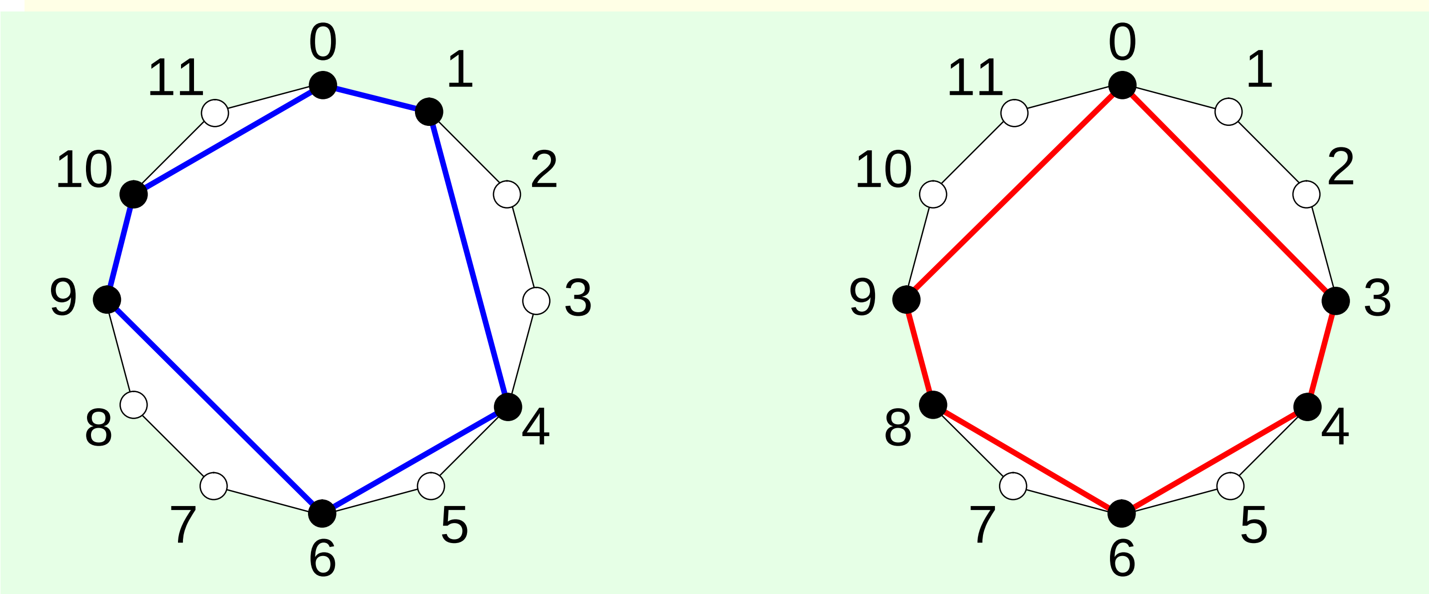
Proofs:

1. Milton Babbitt and David Lewin - 1959, *topology*

2. David Lewin - 1960, *group theory*

3. Eric Regener - 1974, *elementary algebra*

4. Emmanuel Amiot - 2006, *discrete fourier transform*



The Hexachordal Theorem: Crystallography Proofs

First observed experimentally: Linus Pauling and M. D. Shappell, 1930.

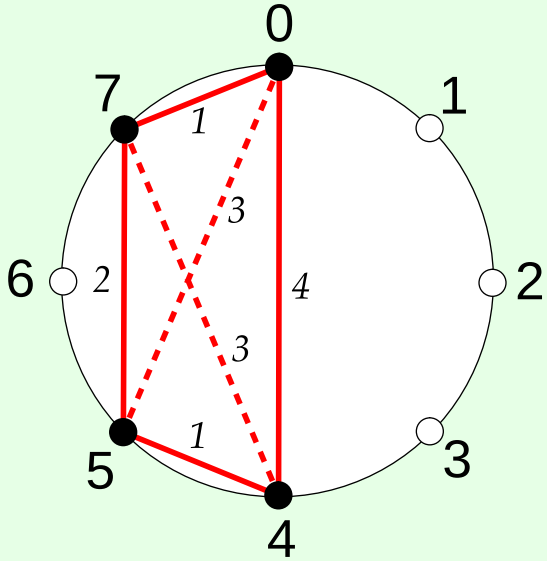
Proofs:

1. Lindo Patterson - 1944, *claimed proof not published*

2. Martin Buerger - 1976, *image algebra*

3. Juan Iglesias - 1981, *elementary induction*

4. Steven Blau - 1999, *elementary induction*



The interval-content theorem of Iglesias

Juan E. Iglesias,
"On Patterson's cyclotomic sets and how to count them," *Zeitschrift für Kristallographie*, 1981.

Theorem: Let p of the N vertices of a regular polygon inscribed on a circle be black dots, and the remaining $q = N - p$ vertices be white dots. Let n_{ww}, n_{bb}, and n_{bw} denote the multiplicity of the distances ***of a specified length*** between *white*-*white*, *black*-*black*, and *black*-*white*, vertices, respectively.

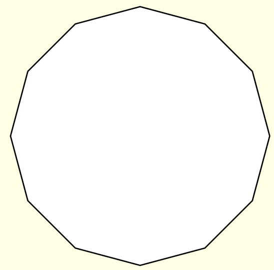
Then the following relations hold:

$$p = n_{bb} + (1/2)n_{bw}$$

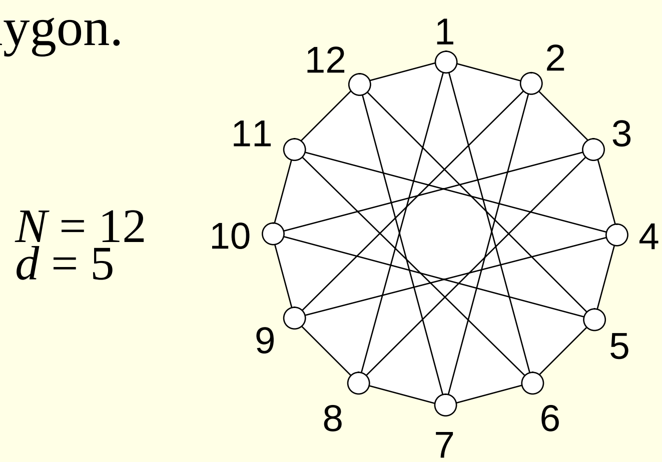
$$q = n_{ww} + (1/2)n_{bw}$$

Lemma: Any given duration value *d* occurs with multiplicity *N*.

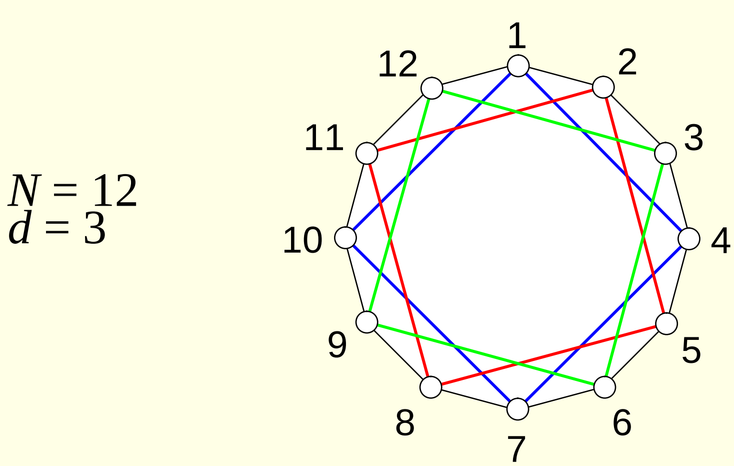
(1) If $d = 1$ or $d = N$-1 the multiplicity equals the number of sides of an *N-vertex regular* polygon.



(2) If $1 < d < N$-1, and *d* and *N* are *relatively prime*, the multiplicity equals the number of sides of an *n-vertex regular star*-polygon.



(3) If *d* and *N* are *not relatively prime* then the multiplicity equals the total number of sides of a group of convex polygons. There are *g.c.d.*(*d*,*N*) polygons with *N/g.c.d*(*d*,*N*) sides each.

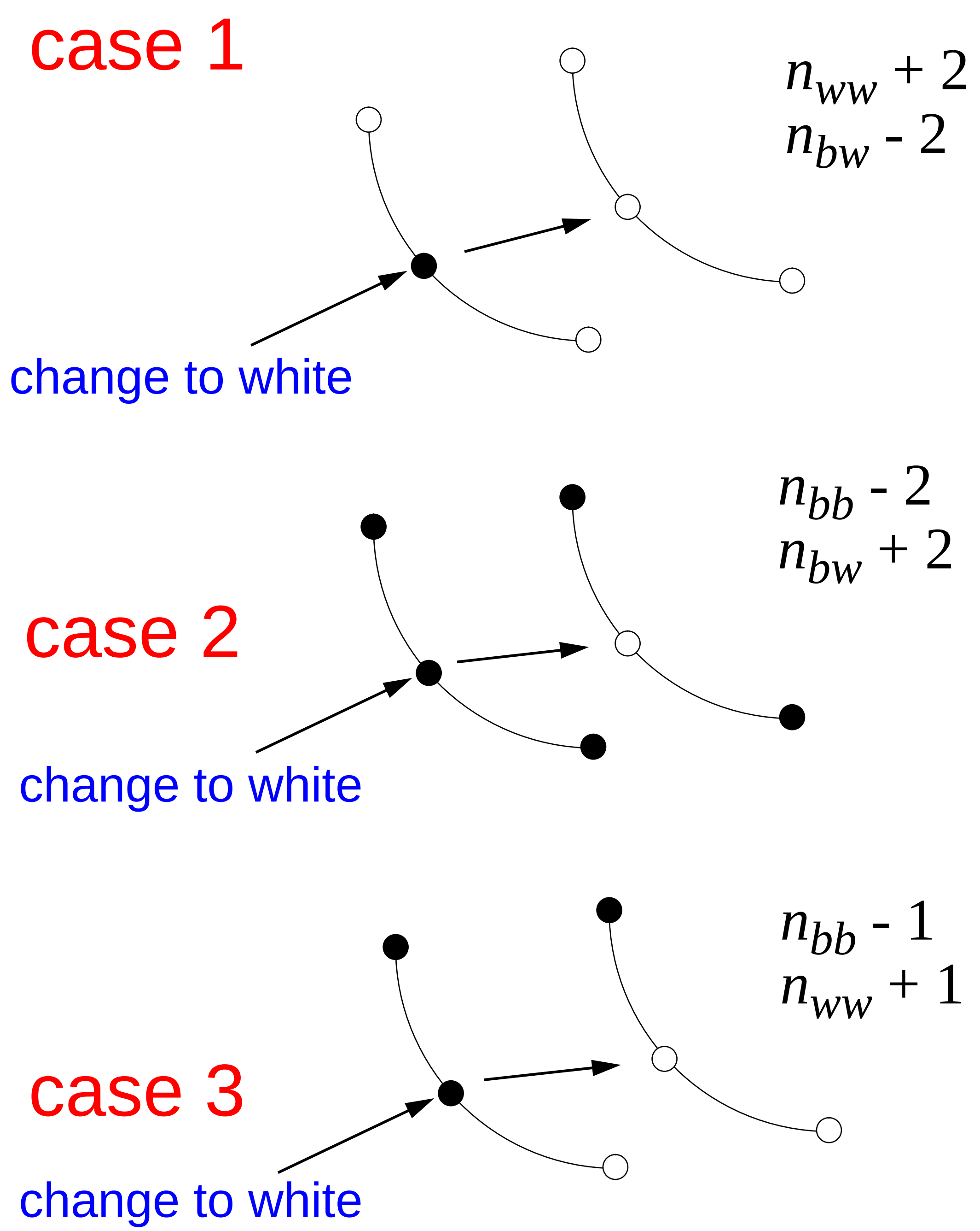


Proof of Iglesias' theorem:

For each duration value *d*

$$p = n_{bb} + (1/2)n_{bw}$$

$$q = n_{ww} + (1/2)n_{bw}$$



Iglesias' Proof of Patterson's Theorems

Theorem 1: If two different black sets form a homometric pair, then their corresponding complementary white sets also form a homometric pair.

Proof: If the black sets are homometric they must have the same number of points.
Then, for each duration value d

$$p = n_{bb} + (1/2)n_{bw} = n^*_{bb} + (1/2)n^*_{bw}$$

$$q = n_{ww} + (1/2)n_{bw} = n^*_{ww} + (1/2)n^*_{bw}$$

and thus

$$p - q = n_{bb} - n_{ww} = n^*_{bb} - n^*_{ww}$$

Since the black sets are homometric $n_{bb} = n^*_{bb}$
and thus $n_{ww} = n^*_{ww}$

Theorem 2: If $p = q$ the two sets are homometric.

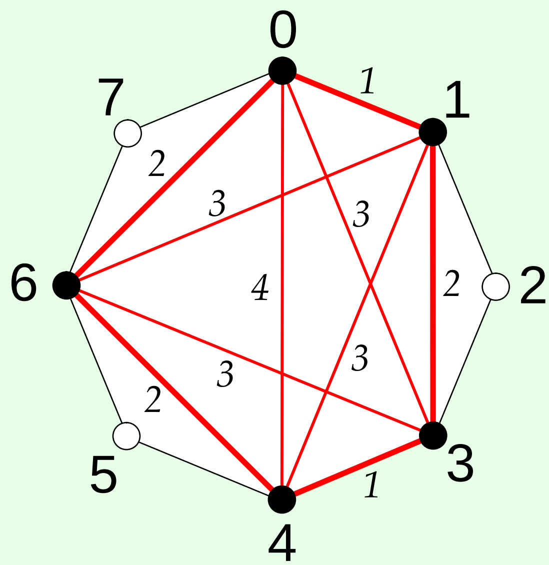
Proof: If $p = q$ then

$$n_{bb} + (1/2)n_{bw} = n_{ww} + (1/2)n_{bw}$$

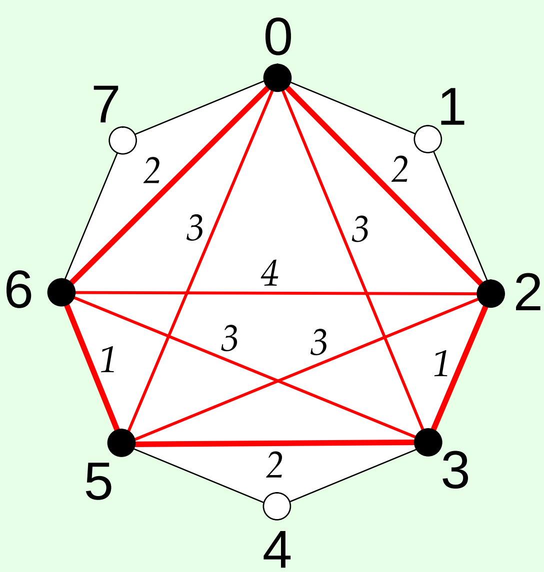
and thus

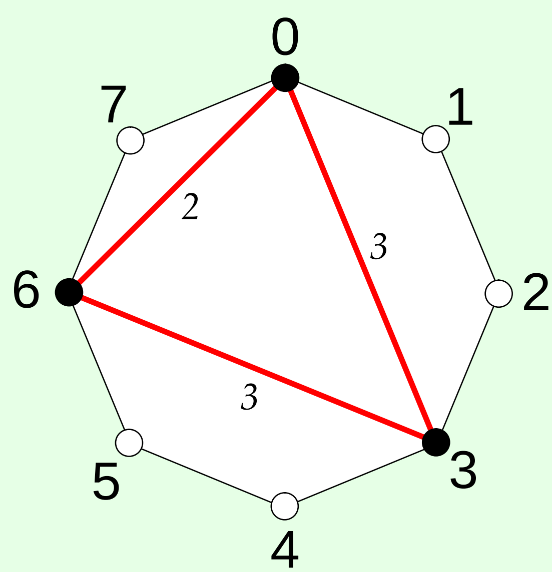
$$n_{bb} = n_{ww}$$

Popular (2/4)-time folk-dance rhythms of northern Transylvania



Ubiquitous rhythms in *African*, *rockabilly*, and *world music*. The *Habanera* rhythms.

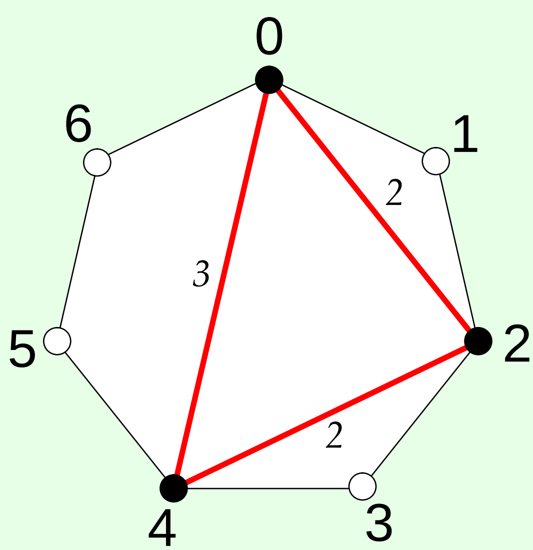




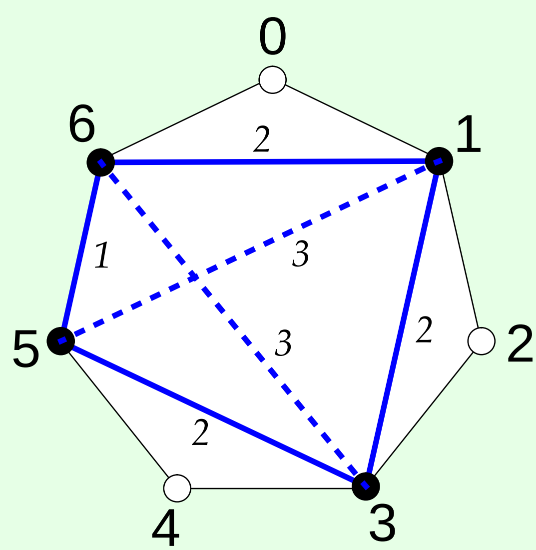
Which *necklaces* have the property that they are *deep* and have *deep complementary necklaces*?

Complementary deep rhythms

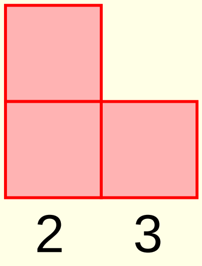
Dave Brubek, *Unsquare Dance*,
in Time Further Out, 1961.
Columbia Records, CS 8490 (stereo).

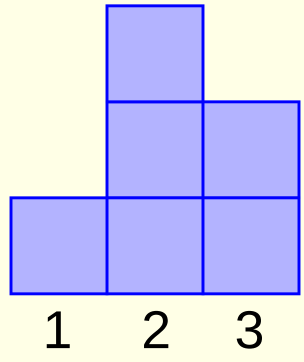


Bass



Clap





Deep Scales in Music Theory

Deep scales have been studied in music theory at least since 1966 by Terry Winograd.
Carlton Gamer, *Journal of Music Theory*, 1967.