Lecture 9: Markov Chains I

Lectures 9 was based on Sections 6.1-6.3 of the text and part of Section 6.4. We skipped some of the examples in 6.1. We did not prove (3) in Section 6.2, and proved (4) by extending our proof that the mean number of returns to 0 for a random walk is infinite precisely if the probability of return is 1. As in the book, we delayed the proof of (9). We proved (2) of 6.3 differently but along similar lines as those followed in the text. We also left (2c) for later. We did not prove (5) of section 6.3.

We introduced a directed graph $D(\mathcal{M})$ associated with a chain $\mathcal{M}$ in which there is an edge from i to j precisely if p_{ij} exceeds 0.

We did not follow the treatment in Section (6.4). Instead, we proved that for any positive persistent state i, the limit as n goes to infinity of $p_{ii}(n)$ is $\frac{1}{\mu_i}$ by simply applying Theorem (24) on page 160 of the text.

We next proved

(A) for any i and j such that $i --> j$ and both i and j are persistent, the limit as n^* goes to infinity of $\sum_{n=1}^{n^*} f_{ij}(n)$ goes to one. I.e., the probability that we visit j given that we start at i is 1.

To do so we noted that:

the probility p^* that we visit j before returning to i is greater than 0, so the probability p_k that we visit i k times before we visit j which is $(1-p^*)^k$ goes to 0 as k goes to infinity, and we can find a k_ϵ such that the probability we return to i k_ϵ times before visiting j is less than ϵ.

Combining this with the fact that for every k the probability that we visit i at least k times is 1, proves (A).

We combined this with the fact that for any positive persistent state j, we have $p_{jj}(n)$ goes to $\frac{1}{\mu_j}$ to prove:

(B) for any i and j such that $i --> j$ and both i and j are positive persistent, the limit as n^* goes to infinity of $p_{ij}(n)$ goes to $\frac{1}{\mu_j}$.

To do so, we defined $N_{i,j,\epsilon}$ so that $\sum_{n=1}^{N_{i,j,\epsilon}} f_{ij}(n) > (1-\frac{\epsilon}{2})$ and $N_{j,\epsilon}$ such that for $n > N_{j,\epsilon}$ we have: $|p_{jj}(n) - \frac{1}{\mu_j}| \leq \frac{\epsilon}{2}$. Then, for any $n > N_{i,j,\epsilon} + N_{j,\epsilon}$, we have:

$$p_{ij}(n) = \sum_{k<n} f_{ij}(k)p_{jj}(n-k) = \sum_{k=1}^{N_{i,j,\epsilon}} f_{ij}(k)p_{jj}(n-k) + \sum_{k=N_{i,j,\epsilon}+1}^{n} f_{ij}(k)p_{jj}(n-k).$$

Now, $\sum_{k=N_{i,j,\epsilon}}^{n} f_{ij}(k)p_{jj}(n-k)$ is at least 0 and at most $\sum_{k=N_{i,j,\epsilon}}^{n} f_{ij}(k)$ which, by definition, is bounded above by $\frac{\epsilon}{2}$.

On the other hand for $k < N_{i,j,\epsilon}$, the definition of $N_{j,\epsilon}$ ensures that $p_{jj}(n-k)$ lies between $\frac{1}{\mu_j} - \frac{\epsilon}{2}$ and $\frac{1}{\mu_j} + \frac{\epsilon}{2}$, so since $\sum_{k<N_{i,j,\epsilon}} f_ij(k)$ lies between $1-\frac{\epsilon}{2}$ and 1, we see that the first sum lies between $\frac{1}{\mu_j} - \epsilon$ and $\frac{1}{\mu_j} + \frac{\epsilon}{2}$. Thus $|p_{ij}(n) - \frac{1}{\mu_j}| < \epsilon$ and (B) holds.