Lecture 4. More on Generating Functions

We recalled our results on generating functions and proved the theorem left over from the last lectures notes.

We then applied these results to branching processes, following the treatment in section 5.4, After that, we applied them to prove Theorem 1 and Corollary 4 of Section 5.3 in the text. Finally, we considered an application of our result to renewal theory. This application is from pages 160-161 of the text. We provide the details which were left out in the text next week, diverging slightly from the treatment given on pages 198-200 of the text.

The proof of Theorem 1 required us to prove that the sequence a with $a_{2k+1} = 0$ and $a_{2k} = \binom{2k}{k}(pq)^k$ had generating function $G(s) = (1-4pqs^2)^{\frac{-1}{2}}$. We do so here.

We note that $G(s) = H(s^2)$ where $H(s) = (1-4pqs)^{\frac{-1}{2}}$. Furthermore, $H(s) = F(s)^{\frac{-1}{2}}$ where $F(s) = (1-4pqs)$. Note that $F'(s) = -4pq$ and all further derivatives of F are 0.

Now $H'(s) = \frac{-1}{2}F(s)^{\frac{-3}{2}}F'(s)$. Because the second and higher derivatives of F are 0, we can prove by induction that

$$H^{(k)}(s) = (\prod_{j=1}^{k} -\frac{2k-1}{2})F(s)^{\frac{-2k-1}{2}}(F'(s))^k.$$

Substituting in, $F(0) = 1$ and $F'(0) = -4pq$, we get:

$$H^{(k)}(s) = \binom{2k}{k}k!(pq)^k$$

Thus, $H = H_b$ for the sequence $b_k = \binom{2k}{k}(pq)^k$.

So since $G = H(s^2)$, we do indeed obtain:

$$G(s) = \sum_{k=0}^{\infty}\binom{2k}{k}(pq)^k s^{2k}$$

as required.