Lecture 3: Differential Equations and Generating Functions

(Page numbers for the 2^{nd} and 3^{rd} editions of G. and S. are the same unless noted otherwise)

The two analytic tools we present today, can be applied in many different settings, not just to random walks.

Differential Equations

To illustrate the use of differential equations, we compute the probability that a biased random coin flipping walk returns to zero, i.e $p_{ret} = P(\exists n > 0 \ s.t. \ S_n = 0)$. By symmetry, we can assume that $p \leq \frac{1}{2}$. Moreover, we can assume that $p < \frac{1}{2}$ as we have already seen that the return probability is 1 when we toss a fair coin. By the solutions to Questions 4 and 1 on the assignment we know $p_{ret} < 1$.

We actually compute the probability R_i for each integer i that a random walk started at i passes through 0. We can then compute $p_{ret} = (1-p)R_{-1} + pR_1$.

For $i \geq 1$, by conditioning on decreasing to $i-1$ we have $R_i = R_1 R_{i-1}$, and hence $R_i = R_1^i$. By considering the first step from 1, we have $R_1 = (1-p) + pR_2$. The last two inequalities yield $pR_1^2 - R_1 + 1 - p = 0$. Solving we have $R_1 \in \{1, \frac{1-p}{p}\}$. A symmetrical argument yields $R_{-1} \in \{1, \frac{p}{1-p}\}$. Since $p < \frac{1}{2}$ and $R_1 \leq 1$ we see $R_1 = 1$. Now, if R_{-1} is also 1 then we obtain $p_{ret} = (1-p)R_{-1} + pR_1 = 1$, a contradiction. So, $R_{-1} = \frac{p}{1-p}$ and $p_{ret} = (1-p)\frac{p}{1-p} + p = 2p$.

As a byproduct, we see that the probability that the walk stays negative from time 1 on is $(1-p)(1-R_{-1}) = 1-2p$, the probability that it goes negative but returns to 0 is $(1-p)R_{-1} = p$ and the probability that it never goes positive is $\sum_{i=0}^{\infty}(1-2p)(p)^i = \frac{1-2p}{1-p}$.

We now consider the mean time M_+ until the walk goes negative. We can assume $p < \frac{1}{2}$; otherwise, as we have seen M_+ is infinite. We actually compute M_i the mean time to go negative starting at i for each nonnegative integer i ($M_+ = M_0$). Clearly, for $i \geq 1$, $M_i = M_0 + M_{i-1}$ and hence $M_i = (i+1)M_0$. Considering the first step, we have $M_0 = 1 + pM_1 = 1 + 2pM_0$ so $M_0 = \frac{1}{1-2p}$.

See Example 4 on page 17 of G. and S. and the start of example 6 on page 73 (pg 74 in 3^{rd} edition) for an analysis which uses difference equations to compute the probability R_k^N that the walk passes through 0 before it hits N for all k lying between 0 and N.

See the end of example 6 on page 73 (pg 74 in 3^{rd} edition) for an analysis which computes the mean time M_k^N until this walk passes through 0 or N for all k lying between 0 and N.

This is just the tip of the iceberg with respect to the use of differential equations in probabilistic analysis. But we leave the subject here.

Generating Functions

THIS PART OF THE LECTURE IS BASED ON SECTION 5.1 OF THE TEXT. WE COVERED ALL THE MATERIAL IN THE SECTION, EXCEPT MOMENT GENERATING FUNCTIONS + THEOREMS 23-25 WHICH WE DISCUSS NEXT WEEK AND JOINT GENERATING FUNCTIONS (PAGE 134 or pg 154 3^{rd} edition) WHICH WE WILL NOT COVER, SO YOU DO NOT NEED TO READ PAGE 134 (pg 154 3^{rd} edition).

A probability space on the non-negative integers is simply a sequence a_0,a_1, of nonnegative reals which sum to 1. As we show below, this sequence is uniquely determined by the function $G(S) = \sum_{i=0}^{\infty} a_i s^i$, which is called its *generating function*.

This definition extends to arbitrary seqeunces of reals provided the sequence converges for s near O(i.e. for some positive radius of convergence R, G converges for s strictly between -R and R.) Note that the generating function of a probability distribution converges for all s with $|s|$ strictly less than 1.

As remarked in the text, $G(s)$ can be differentiated and integrated term by term, any number of times for s between $-R$ and R.

One consequence of this fact is that we can recover the sequence, by setting $a_n = \frac{1}{n!}G^{(n)}(0)$. This shows that the generating function does indeed uniquely determine the sequence.

A second consequence, Theorem 18 in the text, is that the expected value of X is $G'(1)$ and more generally, $E[X(X-1)(X-2)...(X-k)] = G^{(k)}(1)$. This fact allows us to recover $E[X^k]$ for any k. For example, $E[X^2] = G^{(2)}(1)+G'(1)$. In doing so, we need to apply Abel's Theorem ((15) in the text), which tells us that $G^{(k)}(1)$ is computable for all k (since $\sum i^k s^i$ is finite for all s with $|s| < 1$.

Exercise

A random walk is symmetric if for all v, $Pr(X = v) = Pr(X = -v)$. Using the strong law of large numbers or otherwise, show that for any symmetric random walk in Z^1 whose step distribution X has finite mean, the probability that we return to the origin is 1.