Bounding the Mixing Time

We considered finite irreducible aperiodic chains. Every such chain $\mathcal{M} = (\Omega, P)$ has a unique stationary distribution Π. This distribution is also a limit distribution (ie. $TVD(x_t, \Pi) -> 0$ for every intial distribution x_o). We want to bound how quickly this happens.

To this end, we defined $T_{mix}(\epsilon)$ to be the maximum over all x_0 of the minimum t such that $TVD(x_{t'}, \Pi) \leq \epsilon$ for all $t' \geq t$. By taking convex combinations, we saw that we need only take the maximum over all x_o such that $x_o(i) = 1$ for some i and hence since Π is a limit distribution and Ω is finite, for every ϵ, T_{mix} is well defined and finite. Again, by taking convex combinations, we saw that we obtained an equivalent definition if we replaced $t' \geq t$ by $t' = t$.

We then bound T_{mix} for some example Markov chains defined by the standard random walks on various graphs.

Next, we proved that $T_{mix}(\epsilon) \leq T_{mix}(\frac{1}{e}) log_{2/e} \epsilon^{-1}$. For this reason, we will focus to a large extent just on $T_{mix}(\frac{1}{e})$. We use T_{Mix} to denote this value.

You should be able to reproduce all these (short) proofs.

We next considered an arbitrary subset S of Ω, such that $\Pi(S) = \sum_{i \in S} \Pi(i) \leq \frac{1}{2}$.

We noted that for any state i in S, if the probability we have not left S by time t, given that we start at i is less than $\frac{1}{2} - \frac{1}{e}$ then $T_{Mix} > t$.

So, for every j in S, the probability that we leave S in T_{Mix} steps given that we start at j is at least $1/10$. So, letting E_i be the expected time until we leave S given that we start at i and $MLT(S)$ to be the maximum of E_i for the i in S, we obtain the recurrence:

$$MLT(S) \leq T_{Mix} + \frac{9}{10} MLT(S).$$

That is, $T_{Mix} \geq \frac{MLT(S)}{10}$.

Thus, $MLT(S)$ is finite for all such S, and hence so is E_i for every i in S.

For every S, we define $H(S) = \sum_{i \in S, j \notin S} \Pi(i) p_{ij}$. We note that $\Pi(S) = \Pi P(S) = \Pi(S) - H(S) + H(\Omega - S)$ so $H(S) = H(\Omega - S)$.

If we set $x_o = \Pi$ and run the Markov Chain, then $H(S)$ is the probability that we leave S at any given step, and $\Pi(S)$ is the probability we are in S, so the expected time of a sojourn in S is $\frac{H(S)}{\Pi(S)}$. Hence for S with $\Pi(S) \leq \frac{1}{2}$, we would expect that $T_{Mix} \geq \frac{H(S)}{10\Pi(S)}$. We now prove this formally.

We consider starting our Markov chain in Π and running it forever.

Then for any sequence $i_0, i_1, i_2, ..., i_N$, the probability these are the first N states visited is $\Pi(i_o) \prod_0^{N-1} p_{i,i+1}$. Hence the expected length of the sojurns in S which start with an initial state in S is clearly $\sum_{i \in S} \pi(i) E_i$. The expected length of the sojourns in S which start with the second state visited in S but the first state not in S is

$$\sum_{i \in S, j \notin S} \Pi(j) p_{ji} E_i.$$

Each such sojourn contains some states which are initial or second states and some which occur later. The contribution to the length of these sojourns from the first two states is $2\Pi(S)$. Since the probability the second state is i is $\Pi(i)$, the contribution from the latter parts of the sojourns is clearly $\sum_{i \in S} \Pi(i)(E_i - 1)$. Thus we have:

$$2\Pi(S) + \sum_{i \in S} \Pi(i)(E_i - 1) = \sum_{i \in S, j \notin S} \Pi(j) p_{ji} E_i + \sum_{i \in S} \Pi(i) E_i.$$

Which yields:

$$\Pi(S) \leq \sum_{i \in S, j \notin S} \Pi(j) p_{ji} MLT.$$

This gives

$$MLT \geq \frac{\Pi(S)}{H(\Omega - S)} = \frac{\Pi(S)}{H(S)}.$$

We defined conductance as in the Jerrum and Sinclair Chapter and noted this gave a lower bound on the mixing time in terms of conductance.

We mentioned that Jerrum and Sinclair had proved a similar upper bound.