# Verifying Nash Equilibria in PageRank Games on Undirected Web Graphs

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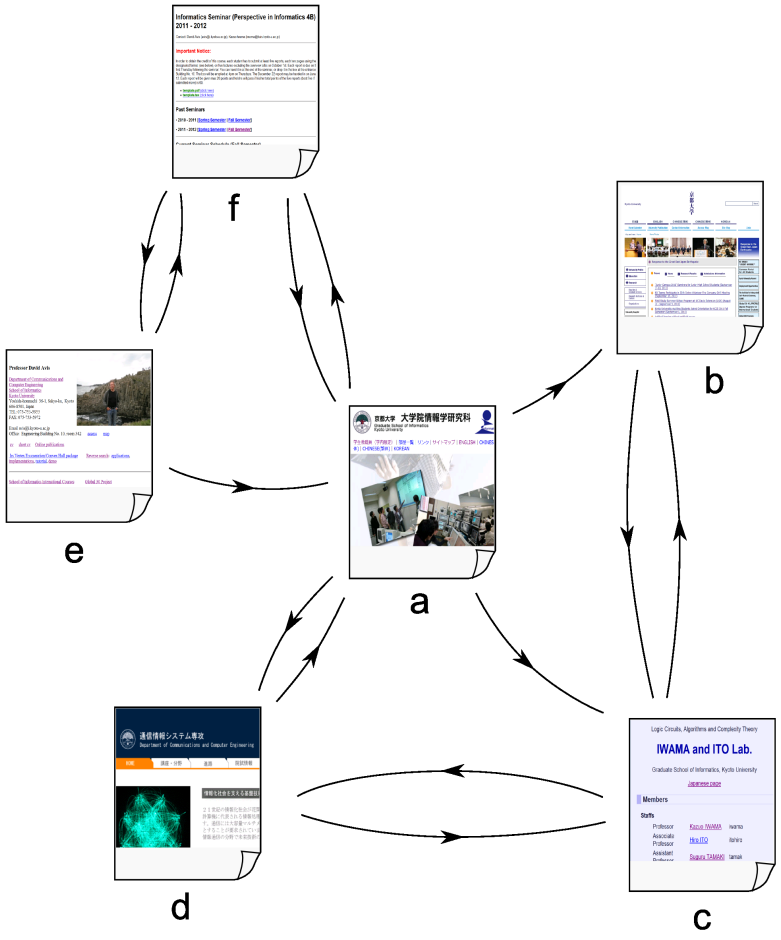
# Directed graphs: WWW



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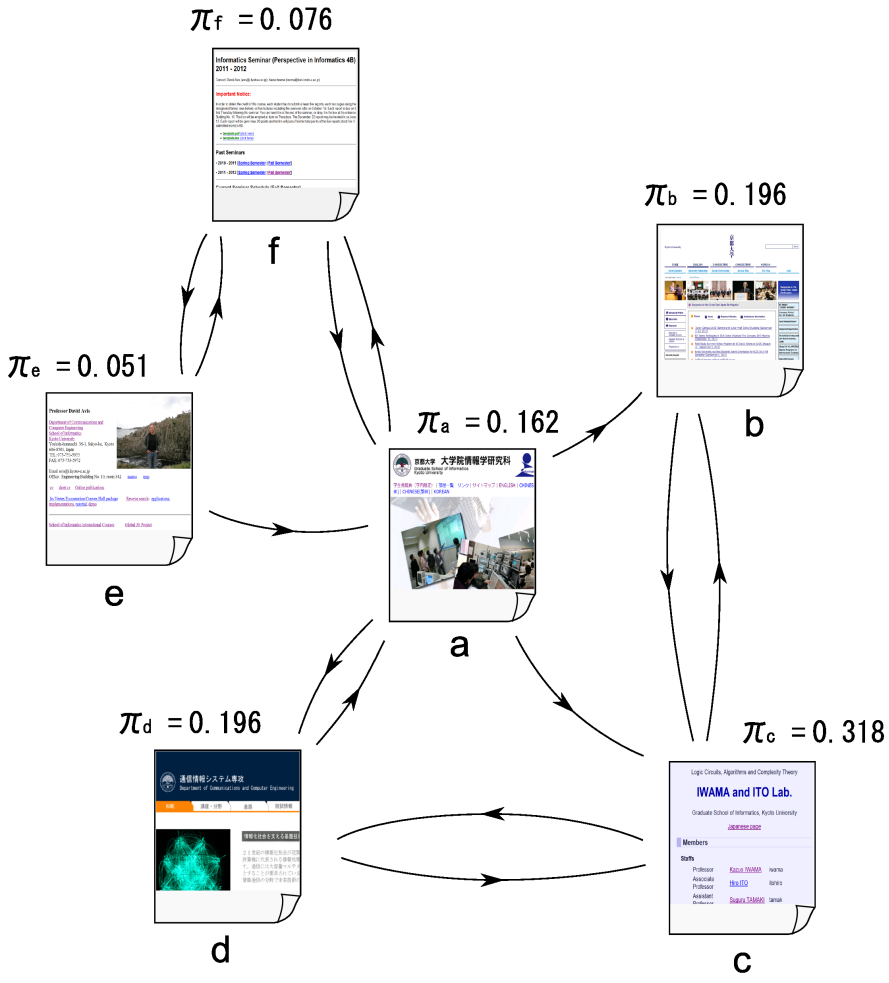
- In this talk we assume $q$ is uniform.

# PageRank



# Computing PageRank

- Potential:

$$\begin{aligned}\phi_{uv} &= P\{\alpha \textit{ random walk from } u \textit{ visits } v \textit{ before jumping}\} \\ &= \frac{1-\alpha}{|\Gamma(u)|}\sum_{i\in\Gamma(u)}\phi_{iv}.\end{aligned}$$

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- PageRank = stationary distribution $\pi$

$$\pi_v = \alpha\frac{\sum_{u\in V} q_u\phi_{uv}}{1-\frac{(1-\alpha)}{|\Gamma(v)|}\sum_{i\in\Gamma(v)}\phi_{iv}}.$$

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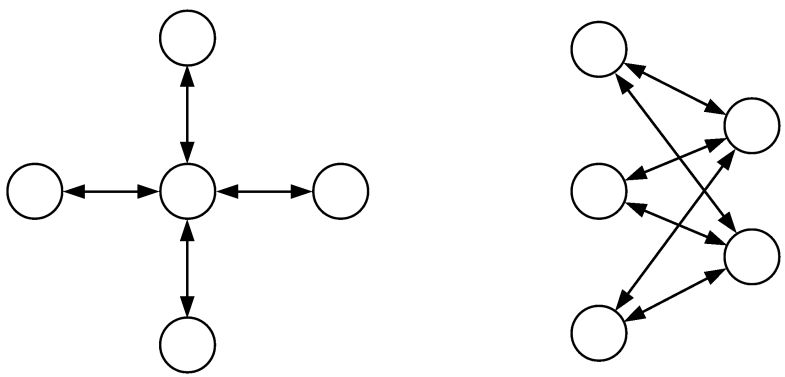
- $\alpha$-insensitive: NE for all $0 < \alpha < 1$.



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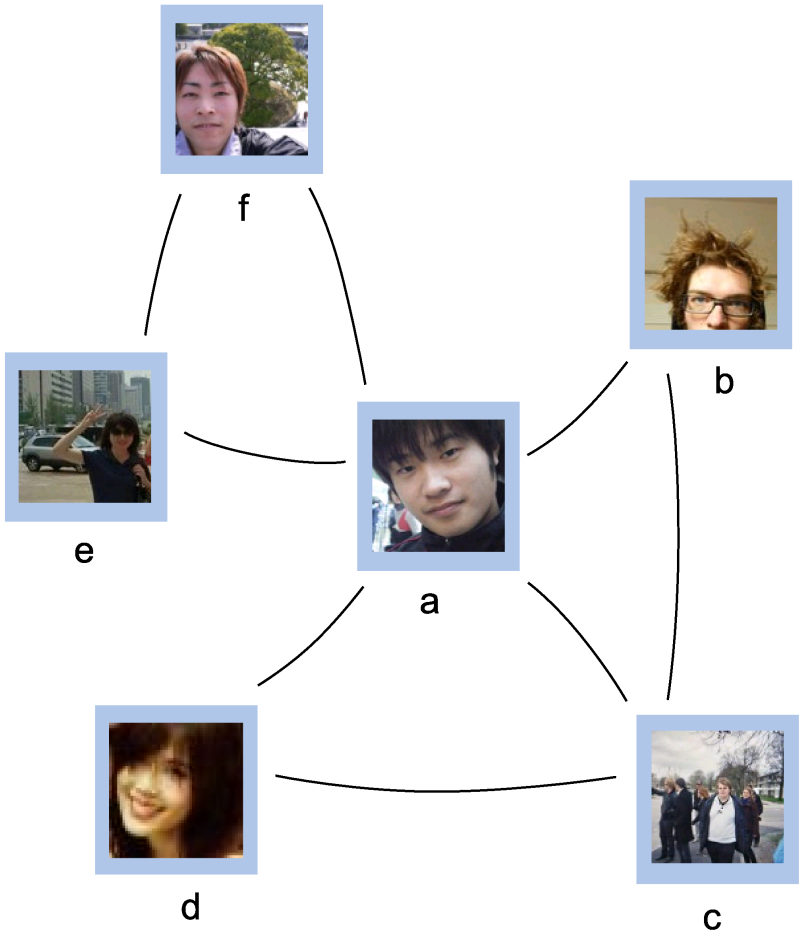
- If a strongly connected graph $G$ is a NE all edges are bidirected.
- $\alpha$-sensitive NE exist (Chen et al.'09)

# Undirected graphs

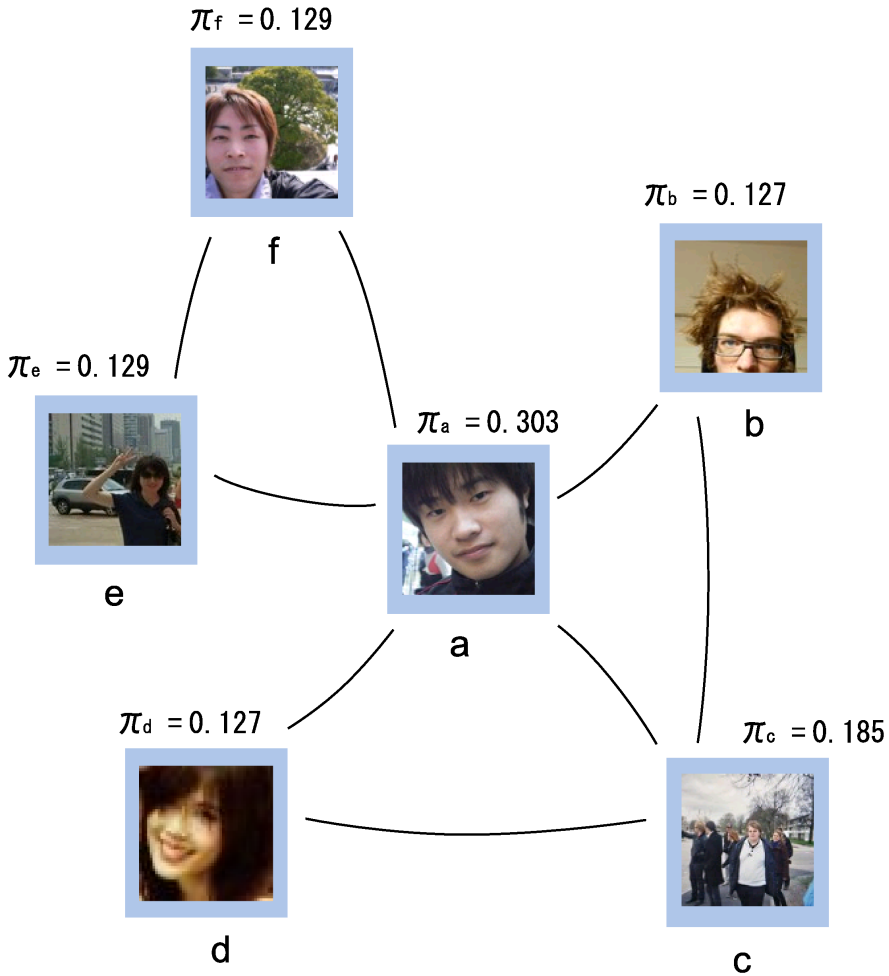
We study PageRank for undirected graphs, such as:

- Friends on Facebook
- Co-authorship (Erdös graph)
- Bilateral agreements

# Undirected graphs: Facebook



# Undirected graphs: PageRank



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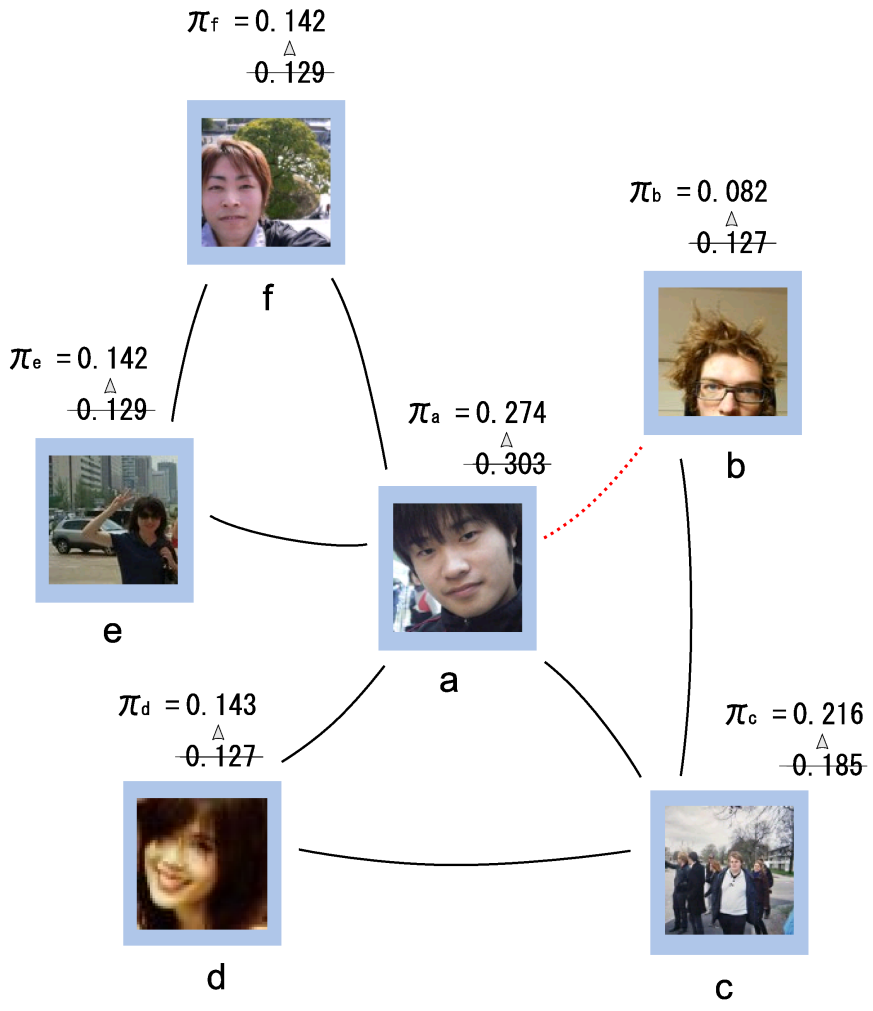

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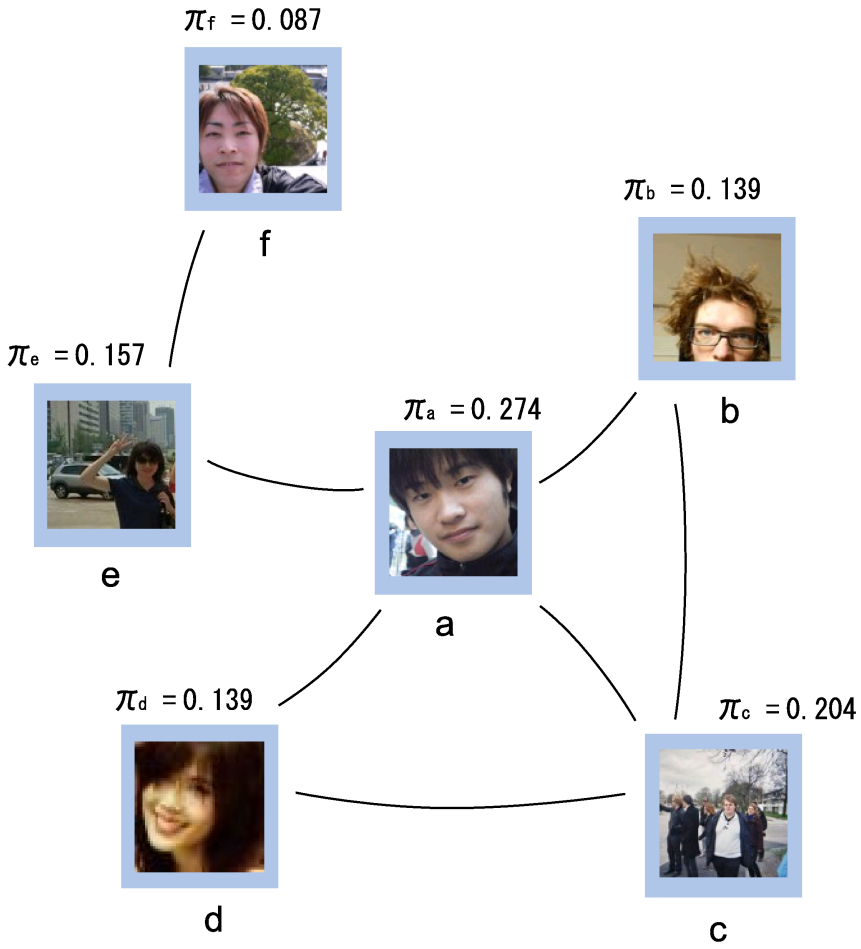

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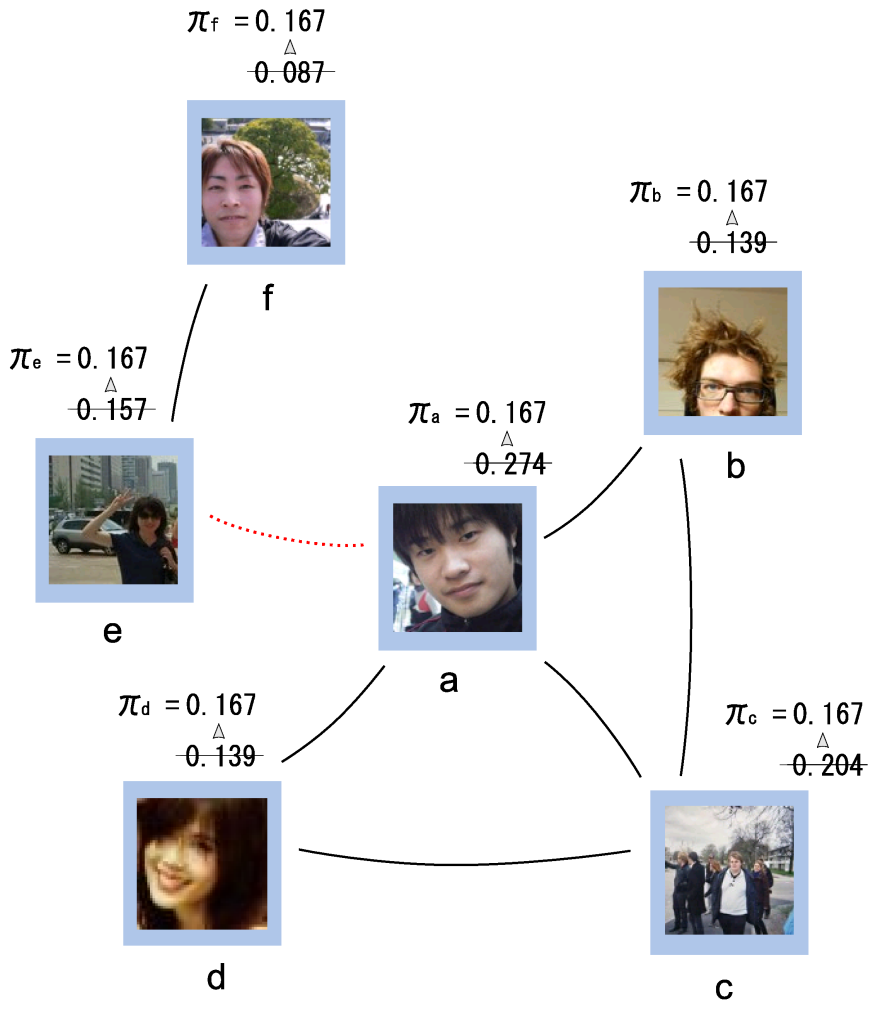

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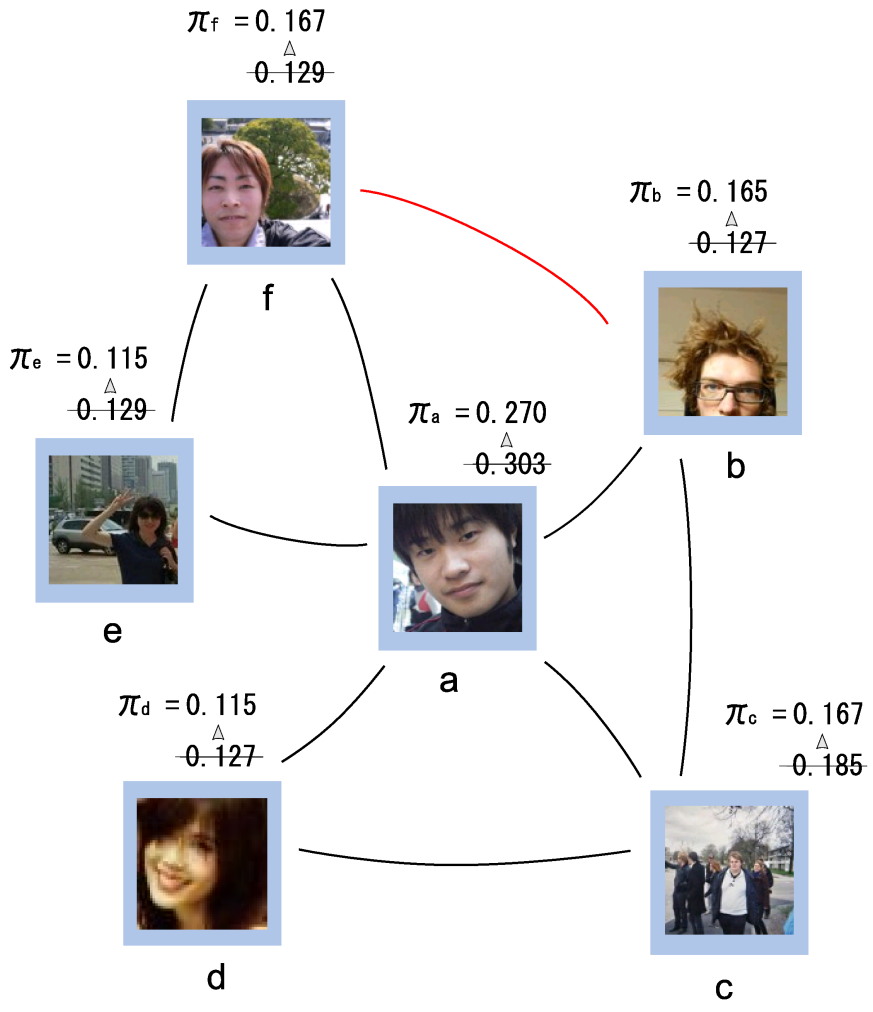

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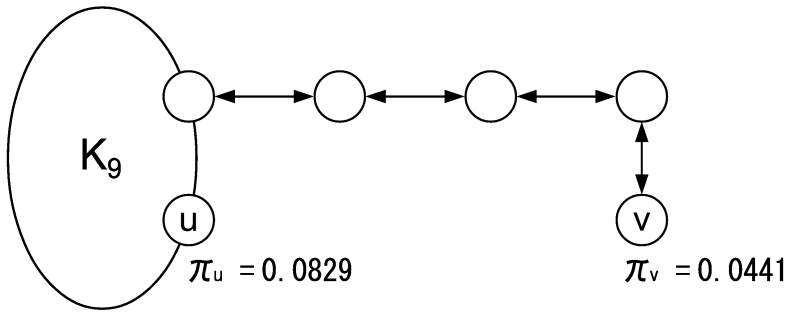

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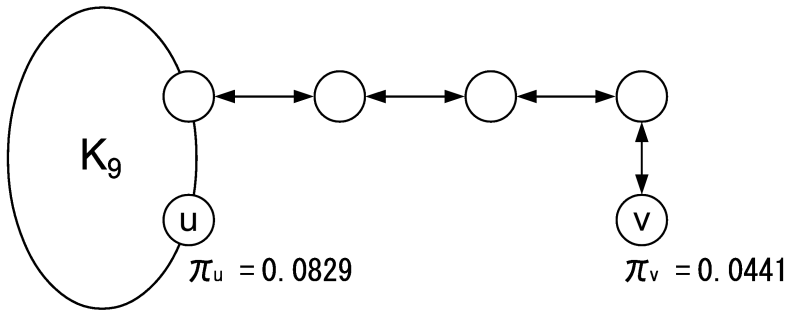
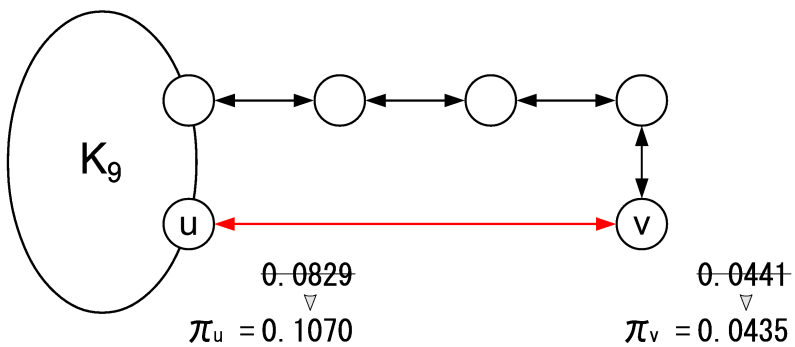



- ... but $v$ refuses!



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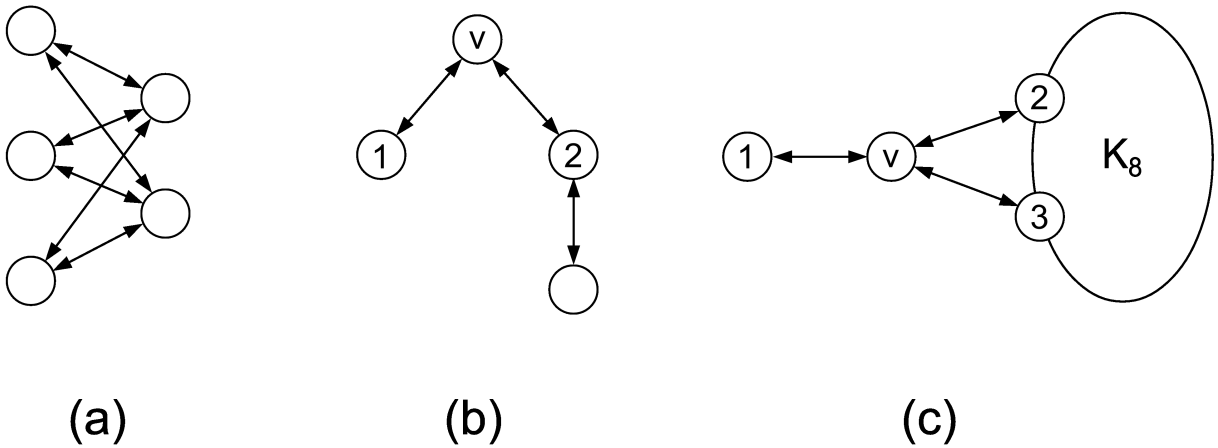
# Directed vs Undirected NE




- (a) $\alpha$-insensitive NE in both models

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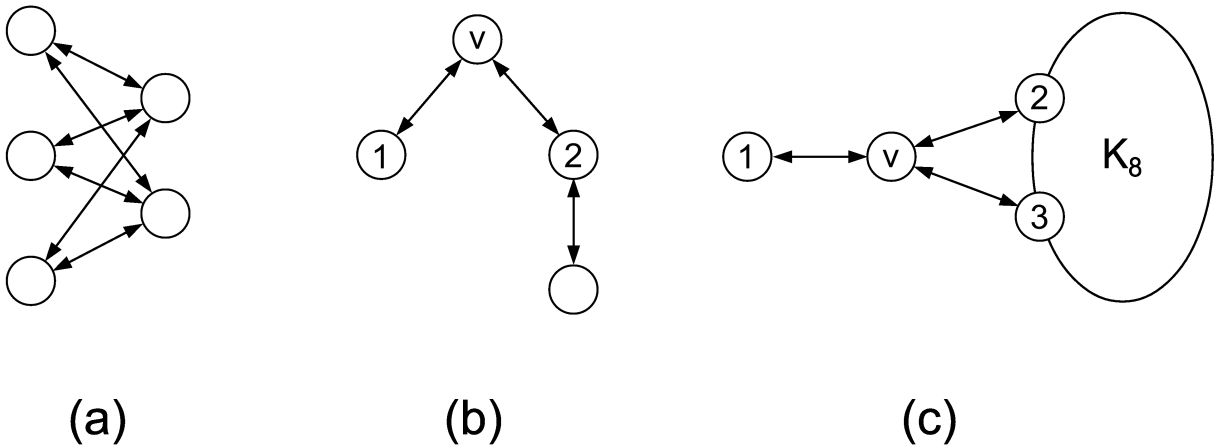


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- (b) Undirected: $\alpha$-insensitive NE
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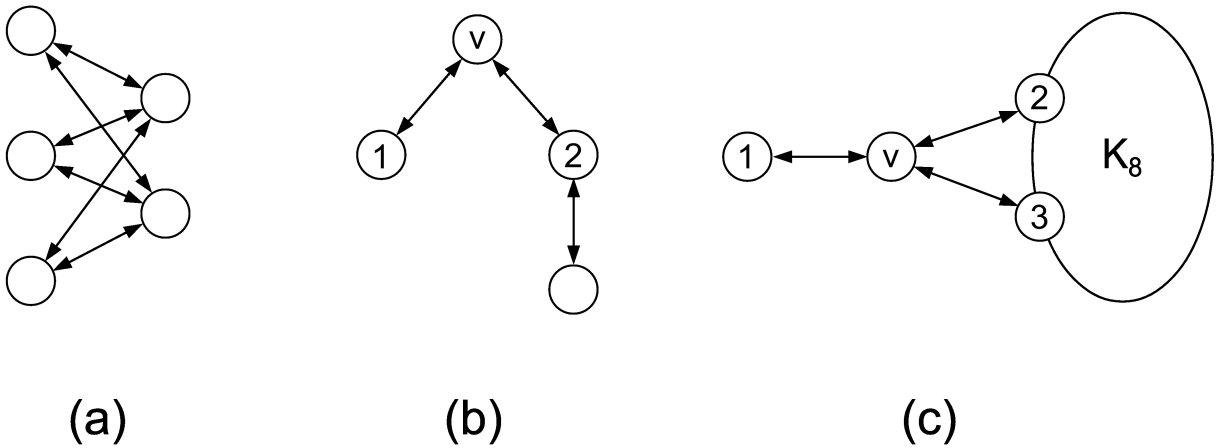


- (a) $\alpha$-insensitive NE in both models
- (b) Undirected: $\alpha$-insensitive NE
  Directed: delete v2
- (c) Undirected: NE only for $4/7 \leq \alpha < 1$
  else delete both v2 and v3
  Directed: Not NE

# Our results

Input: An undirected graph $G$, $\alpha$, $q$.

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- $G$ is a tree: $O(n^2)$ algorithm.
- Let $k = k(G)$ the max vertex degree in any biconnected component
- $G$ is a graph: $O(2^k n^4)$ algorithm.

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- $G$ is a NE iff for any strategy $x$

$$\pi_v(\mathbf{1}) \geq \pi_v(x)$$

# Fractional programming (Megiddo's method)

- To solve:

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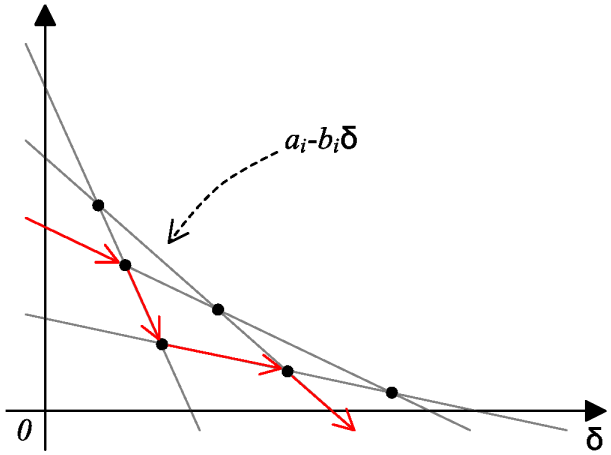
- $z^*$ is the optimum solution to $Q$ iff $h(z^*) = 0$.

# Root finding

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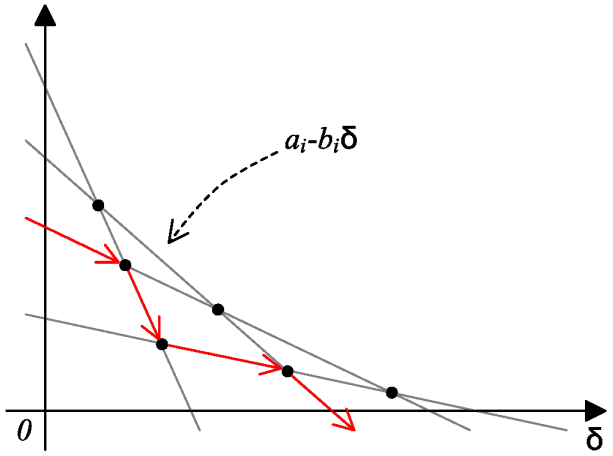


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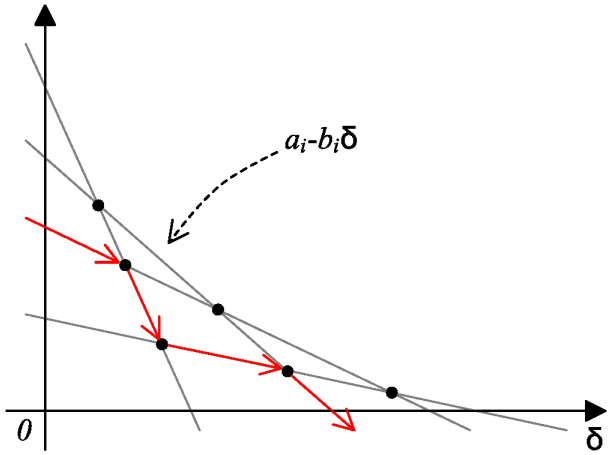


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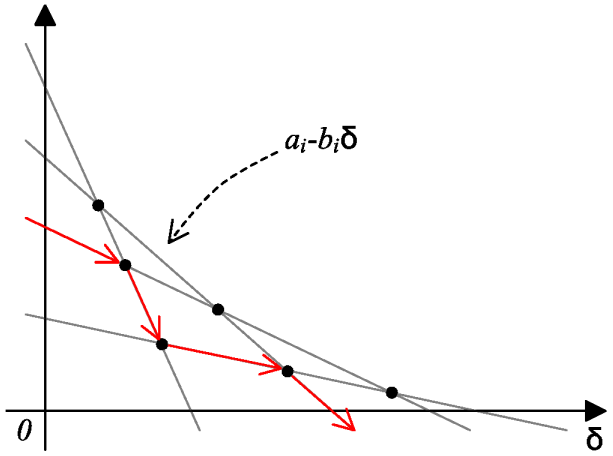


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- We find all roots in $O(|\Gamma(v)|^2)$-time for given vertex $v$
- Since $G$ is a tree we have

$$\sum_{v \in V} (|\Gamma(v)|^2 \leq 4n^2$$

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- How about edge additions?