ELC Workshop on Exponential Lower Bounds for Pivoting Algorithms

March 23, 2015

Oliver Friedmann wins a prize!

Starting point

(Courtesy: G. Ziegler)

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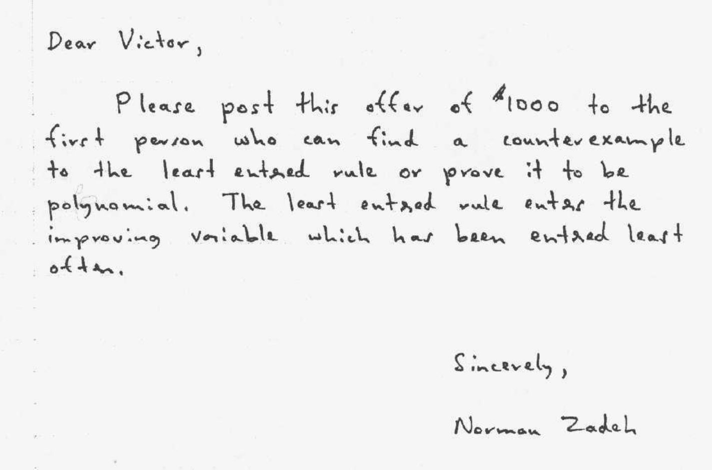

Dear Victor,

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Sincerely,

Norman Zadeh

Victor Klee (1925-2007)

Vic Klee at Oberwolfach in 1981
(photo: L. Danzer)

Mother of all pivoting algorithms: Simplex Method

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Mother of all pivoting algorithms: Simplex Method

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- "In terms of widespread application, Dantzig's algorithm is one of the most successful of all time: Linear programming dominates the world of industry..." (Top 10 Algorithms of the 20th century)
- It gave rise to the field of Operations Research (OR).

Operations Research faculty at Stanford (1969)

George Dantzig is on the far left, then Alan Manne, Frederick Hillier, Donald Iglehart, Arthur Veinott Jr., Rudolf E. Kalman, Gerald Lieberman, Kenneth Arrow and Richard Cottle.

Another OR graduate from Stanford

informs online
Institute for Operations Research and the Management Sciences

In The Media

 PM with OR degree steps down

"Japan's former prime minister, Yukio Hatoyama, could not apply math modeling to solving two pressing political problems......."

"Before entering politics, Hatoyama in the 1970s received a Ph.D in engineering in a field called operations research, which employs applied mathematics to solve complex problems, at Stanford University."

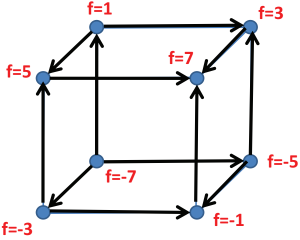
Linear programming in one slide

The Simplex Method and LP digraphs

Linear Programming (LP) **A**:

$$\min f := x + 2y + 4z$$

$$\begin{cases} -1 \leq x \leq 1 \\ -1 \leq y \leq 1 \\ -1 \leq z \leq 1 \end{cases}$$



LP digraph for A

The Simplex Method:

Algorithm of searching a sink of LP digraphs by some pivotting rules.

Strongly polynomial-time algorithms for LP?

Good characterizations for LP digraphs?

Klee-Minty paper (1970)

How Good Is the Simplex Algorithm?

VICTOR KLEE*

Department of Mathematics, University of Washington, Seattle, Washington

AND

GEORGE J. MINTY†

Department of Mathematics, Indiana University, Bloomington, Indiana

1. INTRODUCTION

By constructing long "increasing" paths on appropriate convex polytopes, we show that the simplex algorithm for linear programs (at least with its most commonly used pivot rule, Dantzig [*1*]) is not a "good algorithm" in the sense of Jack Edmonds. That is, the number of pivots or iterations that may be required is not majorized by any polynomial function of the two parameters that specify the size of the program. In particular, $2^d - 1$ iterations may be required in solving a linear program whose feasible region, defined by d linear inequality constraints in d nonnegative variables or by d linear equality constraints in $2d$ nonnegative variables, is projectively equivalent to a d-dimensional cube. Further, for each d there are positive constants α_d and

Klee-Minty Examples

- Squashed 3-cube (Chvátal, P.47)

$$\begin{array}{rcl} \textit{maximize} & 100x_1 + 10x_2 + x_3 & \\ s.t. & x_1 & \leq 1 \\ & 20x_1 + x_2 & \leq 100 \\ & 200x_1 + 20x_2 + x_3 & \leq 10000 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

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- Vertices:

0 0 0	0 100 8000
1 0 0	1 80 8200
1 80 0	1 0 9800
0 100 0	0 0 10000

Pivot Sequence (Dantzig's rule)

x_1	x_2	x_3	$z = 100x_1 + 10x_2 + x_3$
0	0	0	0
1	0	0	100
1	80	0	900
0	100	0	1000
0	100	8000	9000
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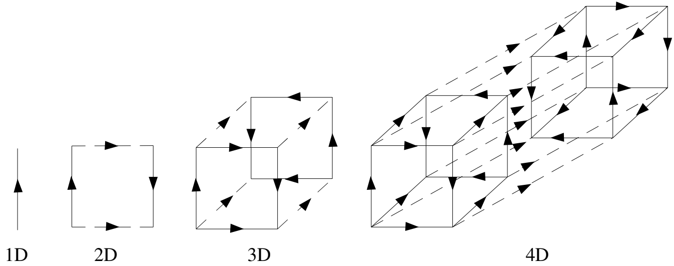
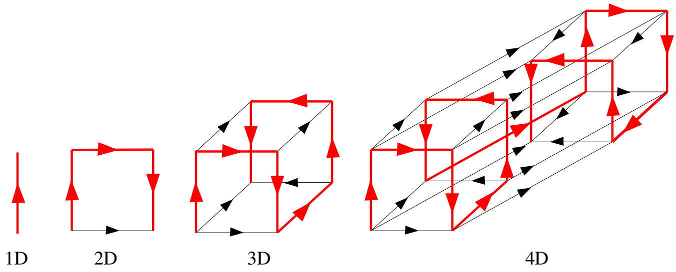
- Similar examples exist for each integer n: 2^n iterations!
- x_n pivots once
- x_1 pivots 2^{n-1} times.

Klee-Minty construction: combinatorial representation



Klee-Minty path



Similar results soon followed

- D.A. and V. Chvatal, "Notes on Bland's Pivoting Rule," Math Prog. Study, Vol 8, pp.24-34, 1978
- R.G. Jeroslow, "The simplex algorithm with the pivot rule of maximizing improvement criterion", Discrete Mathematics 4 (1973) 367-377.

Norm Zadeh

Norm Zadeh creator of Perfect Ten Magazine at his Beverly Hills Mansion November 2001 with his perfect 10 models (photo:Jonas Mohr)

For Sale!

Hot Property: Norm Zada - Hot Property: Norm Zada - Los Angeles Times http://www.latimes.com/classified/realestate/printedition/hm-hotpropzad...

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Los Angeles Times | Real Estate

SEARCH

You are here: LAT Home > Real Estate

1 of 6 Next Photo

(Erhard Pfeiffer)

Email

Hot property: Norm Zada, publisher of Perfect 10 magazine has listed his house in Beverly Park at $24.5 million

Zadeh's previously unpublished gem

- N. Zadeh, "What is the worst case behavior of the simplex algorithm," *Technical Report 27*, Dept. Operations Research, Stanford University, 1980.

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- Now published with postscript in: *Polyhedral Computation*, CRM-AMS Proceedings vol 48, eds. D.A., D. Bremner and A. Deza, 2009.
- Shows Klee-Minty examples can be defeated by history based rules

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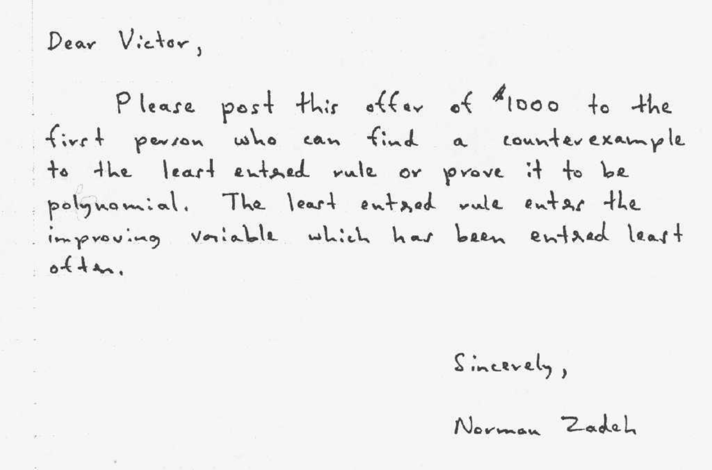

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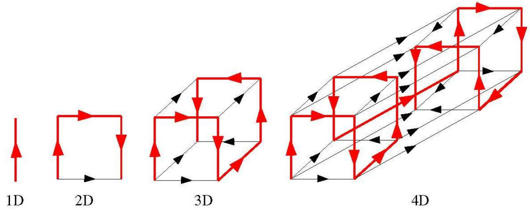

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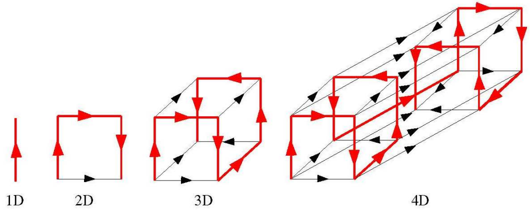

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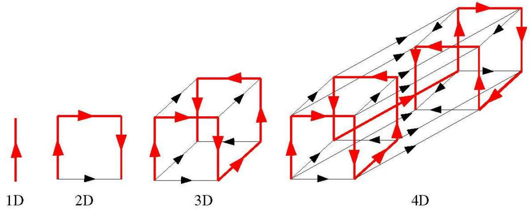

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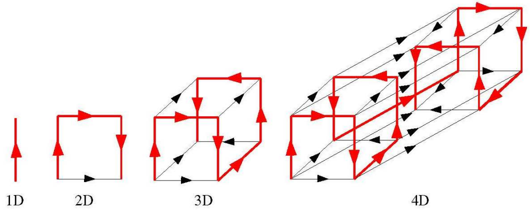

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- Eg 4D: after 3 steps we must enter the right cube and cannot return to the left
- After at most $O(n^2)$ steps the sink is found
- Oliver et al. showed Zadeh's rule requires exponential time

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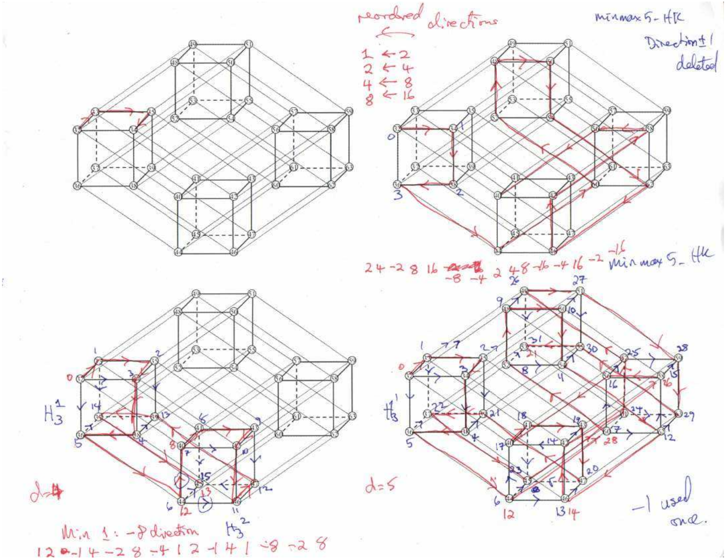
- ...they are not strongly polynomial time and cannot handle cutting planes
- Simplex method is very well adapted to handle cutting planes
- Lower bounds may apply to other problems: e.g. n-cube USOs
- Are there any polynomial time pivot selection methods?

Finding the sink of a USO

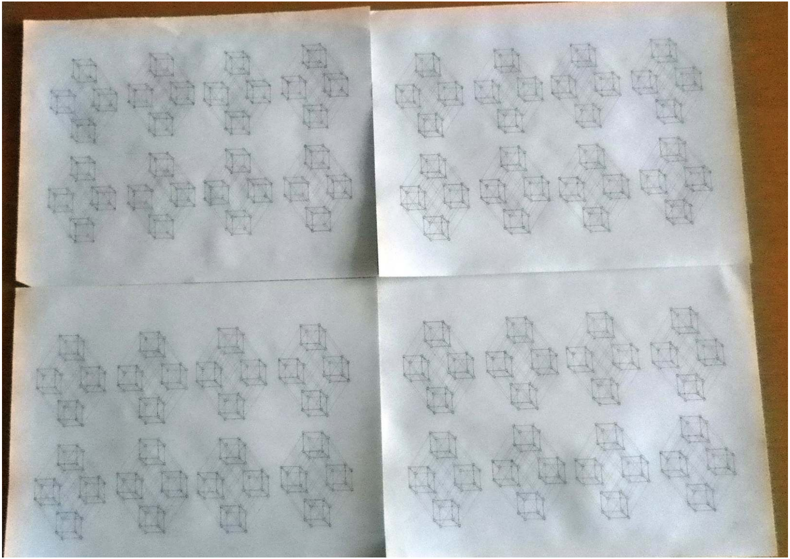
Warning: This problem may be addictive ...

Y. Aoshima, D. Avis, T. Deering, Y. Matsumoto, S. Moriyama, "On the Existence of Hamiltonian Paths for History Based Pivot Rules on Acyclic Unique Sink Orientations of Hypercubes", Discrete Applied Mathematics(2012) arXiv:1110.3014v2

Works for low dimensions ...



Let's try $d = 10$



Basic problem

Can we efficiently find the sink of an LP-digraph by following a directed path from any given vertex, using a given edge selection rule (pivoting)?

Necessary conditions for LP digraphs

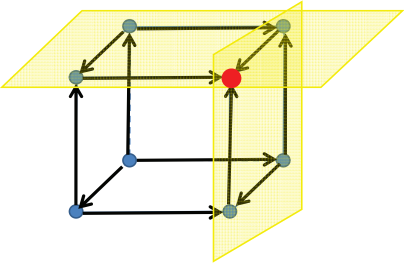
- Unique Sink Orientation (USO) [‘01 Szabo,Welzl]
- Acyclicity
- Holt Klee Property [‘99 Holt, Klee]
- Shelling Property [‘09 Avis, Moriyama]

Necessary conditions for LP digraphs

Unique Sink Orientation (USO)

['01 Szabo,Welzl]

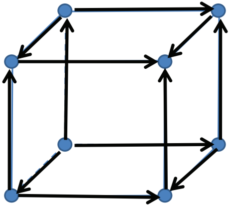
Each subgraph G(P,H) of G(P) induced by a face H of P has a unique sink (and then a unique source).



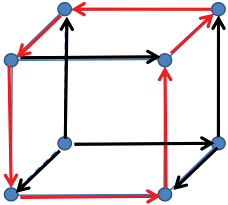
Necessary conditions for LP digraphs

Acyclicity

G(P) has no directed cycle.



Acyclic

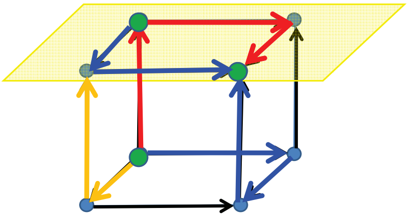


Not acyclic

Necessary conditions for LP digraphs

Holt Klee property ['99 Holt, Klee]

G(P) has a USO, and for every k-dimensional face H of P there are k disjoint paths from the unique source to the unique sink in G(P,H).



n-cube USOs

- Vertices $V = \{0, 1, ..., 2^n - 1\} = \{00..00, 00..01, ..., 11..11\}$

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- Facets $F_1, F_2, ..., F_{2n}$. For $i = 1, ..., n$,

$$F_i = \{(x_1, x_2, ..., x_n) | x_i = 0\}, \ \ F_{n+i} = \{(x_1, x_2, ..., x_n) | x_i = 1\}.$$

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- Cobasis $C(v) = \{i : v \in F_i, i = 1, ..., 2n\}, v \in V$

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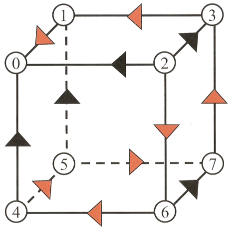
- Basis $B(v) = \{i : v \notin F_i, i = 1, ..., 2n\}, v \in V$
- Note $i \in B(v)$ iff $n + i \in C(v)$.
- A *pivot* interchanges a pair of indices i and $n + i$ between $B(v)$ and $C(v)$. (flips bit i of v)

3-cube acyclic USO



- Vertices $V = \{0, 1, ..., 7\} = \{000, 001, ..., 111\}$

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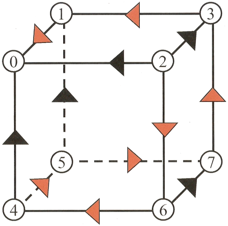

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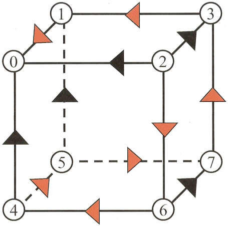

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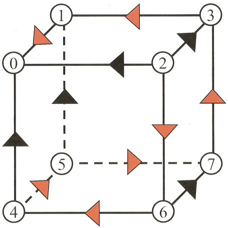
- $F_i = \{(x_1, x_2, x_3)|x_i = 0\}$, $F_{3+i} = \{(x_1, x_2, , x_3)|x_i = 1\}$
- $C(6) = C(110) = \{4, 5, 3\}$, $B(6) = B(110) = \{1, 2, 6\}$

3-cube acyclic USO



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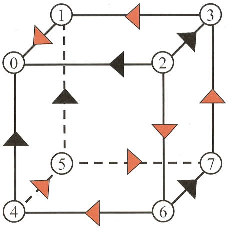

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- Pivots correspond to moves in the 4,2,1 *directions*

History based rules

Choose the improving variable that satisfies:

- Least recently basic (Johnson)

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- Least recently entered (Fathi-Tovey)
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)

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- Least number of times to enter basis (Zadeh)

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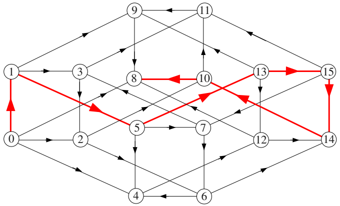
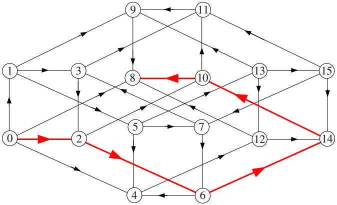
- Least number of iterations in basis (A-M-M)
- Least used direction (A-M-M)
- Least number of times to enter basis (Zadeh)
- All of the above break Klee-Minty type constructions
- We try to find an acyclic USO for which a given rule follows a Hamiltonian path

Least recently basic (Johnson)



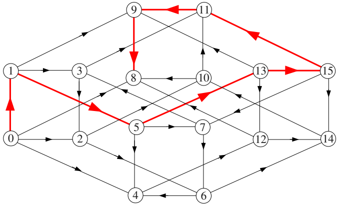
Vertex		(orientation, direction)-pair								Options
		+ 4	- 4	+ 3	- 3	+ 2	- 2	+ 1	- 1	
0	0 0 0 0		✓		✓		✓		✓	**+1**, +2, +3, +4
1	0 0 0 1		✓		✓		✓	✓		+2, **+3**, +4
5	0 1 0 1		✓	✓			✓	✓		+2, **+4**
13	1 1 0 1	✓		✓			✓	✓		**+2**
15	1 1 1 1	✓		✓		✓		✓		**-1**
14	1 1 1 0	✓		✓		✓			✓	**-3**
10	1 0 1 0	✓			✓	✓			✓	**-2**
8	1 0 0 0	✓			✓		✓		✓	

Least recently considered (Cunningham)



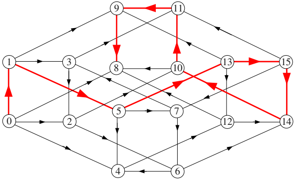
Vertex		Sequence	Options
0	0 0 0 0	+ 2, - 2, + 1, - 1, + 3, - 3, + 4, - 4	**+ 2**
2	0 0 1 0	- 2, + 1, - 1, + 3, - 3, + 4, - 4	**+ 3**
6	0 1 1 0	- 3, + 4, - 4, + 2, - 2, + 1, - 1	**+ 4**
14	1 1 1 0	- 4, + 2, - 2, + 1, - 1, + 3, - 3	**- 3**
10	1 0 1 0	+ 4, - 4, + 2, - 2, + 1, - 1, + 3	**- 2**
8	1 0 0 0	+ 1, - 1, + 3, - 3, + 4, - 4, + 2	

Least recently entered (Fathi-Tovey)



Vertex		(orientation, direction)-pair								Options
		+ 4	- 4	+ 3	- 3	+ 2	- 2	+ 1	- 1	
0	0 0 0 0		✓		✓		✓		✓	**+1**, +2, +3, +4
1	0 0 0 1		✓		✓		✓	✓		+2, **+3**, +4
5	0 1 0 1		✓	✓			✓	✓		+2, **+4**
13	1 1 0 1	✓		✓			✓	✓		**+2**
15	1 1 1 1	✓		✓		✓		✓		-1, **-3**, -4
11	1 0 1 1	✓			✓	✓		✓		**-2**
9	1 0 0 1	✓			✓		✓	✓		**-1**
8	1 0 0 0	✓			✓		✓		✓	

Least number of iterations in basis (A-M-M)



Vertex		(orientation, direction)-pair								Options
		+ 4	- 4	+ 3	- 3	+ 2	- 2	+ 1	- 1	
0	0 0 0 0	0	1	0	1	0	1	0	1	**+1**, +2, +3, +4
1	0 0 0 1	0	2	0	2	0	2	1	1	+2, **+3**, +4
5	0 1 0 1	0	3	1	2	0	3	2	1	+2, **+4**
13	1 1 0 1	1	3	2	2	0	4	3	1	**+2**
15	1 1 1 1	2	3	3	2	1	4	4	1	**-1**
14	1 1 1 0	3	3	4	2	2	4	4	2	**-3**
10	1 0 1 0	4	3	4	3	3	4	4	3	**+1**, -2
11	1 0 1 1	5	3	4	4	4	4	5	3	**-2**
9	1 0 0 1	6	3	4	5	4	5	6	3	**-1**
8	1 0 0 0	7	3	4	6	4	6	6	4	

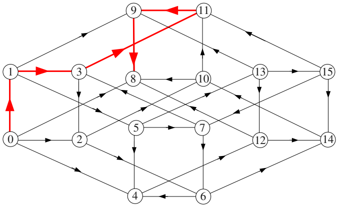
Least number of iterations in basis

- We searched a catalogue of acyclic USOs for n=3,4

Least number of iterations in basis

- We searched a catalogue of acyclic USOs for n=3,4
- We found no examples of Hamilton paths using this rule in dimensions 3 or 4.

Least used direction (A-M-M)



Vertex		Direction 4	3	2	1	Options
0	0 0 0 0	0	0	0	0	**1**, 2, 3, 4
1	0 0 0 1	0	0	0	1	**2**, 3, 4
3	0 0 1 1	0	0	1	1	**4**
11	1 0 1 1	1	0	1	1	**2**
9	1 0 0 1	1	0	2	1	**1**
8	1 0 0 0	1	0	2	2	

Least used direction

For n-cube H_n, directions $i = 1, ..., n$

- nv(i)= the number of times that direction i has been taken.

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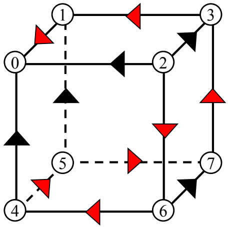
- Set nv(j)=nv(j)+1.
- Special case of Zadeh's rule.

Unique H_3

Hamilton path using least used direction rule
It satisfies the Holt-Klee condition



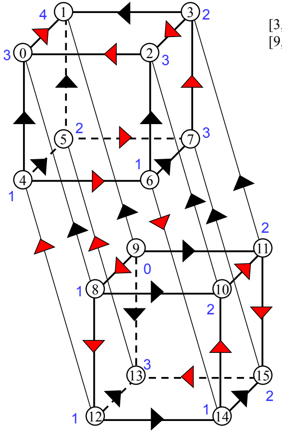
nv(1) = nv(2) = nv(4) = 0
4: nv(1) = nv(2) = 0, nv(4) = 1
2: nv(1) = 0, nv(2) = 1, nv(4) = 1
1: nv(1) = 1, nv(2) = 1, nv(4) = 1
2: nv(1) = 1, nv(2) = 2, nv(4) = 1
4: nv(1) = 1, nv(2) = 2, nv(4) = 2
2: nv(1) = 1, nv(2) = 3, nv(4) = 2
1

Least visited rule

Unique H_4

Hamilton path using least used direction rule

It satisfies the Holt-Klee condition

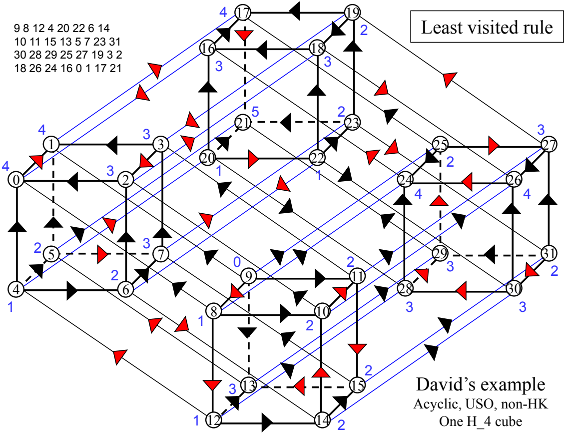


[3, 4, 3, 2, 1, 2, 1, 3, 1, 0, 2, 2, 1, 3, 2, 2]
[9, 8, 12, 4, 6, 14,10, 11, 15, 13, 5, 7, 3, 2, 0, 1]

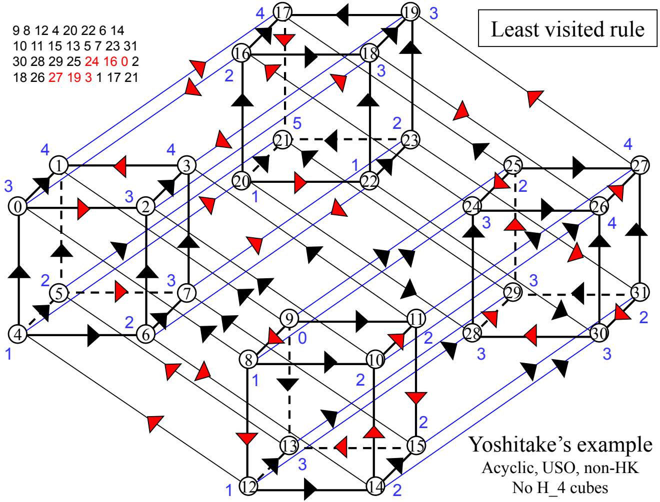
nv(1) = nv(2) = nv(4) = nv(8) = 0
1: nv(1) = 1, nv(2) = nv(4) = nv(8) = 0
4: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 0
8: nv(1) = 1, nv(2) = 0, nv(4) = 1, nv(8) = 1
2: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 1
8: nv(1) = 1, nv(2) = 1, nv(4) = 1, nv(8) = 2
4: nv(1) = 1, nv(2) = 1, nv(4) = 2, nv(8) = 2
1: nv(1) = 2, nv(2) = 1, nv(4) = 2, nv(8) = 2
4: nv(1) = 2, nv(2) = 1, nv(4) = 3, nv(8) = 2
2: nv(1) = 2, nv(2) = 2, nv(4) = 3, nv(8) = 2
8: nv(1) = 2, nv(2) = 2, nv(4) = 3, nv(8) = 3
2: nv(1) = 2, nv(2) = 3, nv(4) = 3, nv(8) = 3
4: nv(1) = 2, nv(2) = 2, nv(4) = 4, nv(8) = 2
1: nv(1) = 3, nv(2) = 2, nv(4) = 4, nv(8) = 2
2: nv(1) = 3, nv(2) = 3, nv(4) = 4, nv(8) = 2
1

Least visited rule

H_5


9 8 12 4 20 22 6 14
10 11 15 13 5 7 23 31
30 28 29 25 27 19 3 2
18 26 24 16 0 1 17 21
Least visited rule
David's example
Acyclic, USO, non-HK
One H_4 cube

Another candidate for H_5



Computational results: least used direction

dimension	2	3	4	5
number of Hamilton paths	1	1	1	2
Holt-Klee	1	1	1	0

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Computational results: least used direction

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number of Hamilton paths	1	1	1	2
Holt-Klee	1	1	1	0

- For $n \leq 4$, each example extends to the next dimension
- Since HK fails for $n = 5$, these examples are not LP-digraphs

Computational results: least used direction

But things do not go well for $n \geq 6$ ...

dimension	2	3	4	5	6	7	8
number of Hamilton paths	1	1	1	2	0	0	0
Holt-Klee	1	1	1	0	0	0	0

We did a computer search of all acyclic USOs that contain Hamiltonial paths.

Least times to enter basis (Zadeh's rule)

Facets $F_i, i = 1, ..., 2n$

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- Set nv(j)=nv(j)+1.

Computational results: least times to enter basis

The deluge!

dimension	2	3	4	5	6	7
Ham. paths	1	2	17	1,072	3,262,342	$\geq$ 42,500,000,000
Holt-Klee	1	2	12	79	360	none yet

Williamson Hoke's theorem (1988)

- Given an oriented n-cube H, let d_k = number of vertices with in-degree k

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- Given an oriented n-cube H, let d_k = number of vertices with in-degree k
- Theorem: H is an AUSO if and only if

$$d_k = \binom{n}{k}, \quad k = 0, 1, ..., n$$

How do we get the results?

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- Williamson-Hoke's theorem characterizes which orientations are AUSOs
- We generate only HPs consistent with Zadeh's rule using *nv* sequence
- Reject partial HP if it violates W-H theorem

How few times can a variable enter the basis?

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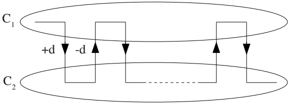
- In Klee-Minty examples, one variable never enters the basis.
- Theorem:
  Let H be a AUSO n-cube with a H.P. followed by Zadeh's rule.
  Each variable enters the basis at least $\frac{2^{n-2}}{n} - 1$ times.

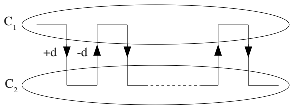
Proof of the lower bound

- Variable $-d$ enters the basis min number k times



Proof of the lower bound

- Variable $-d$ enters the basis min number k times
- At the sink $\sum_{i=1}^{2n} nv(i) = 2^n - 1$



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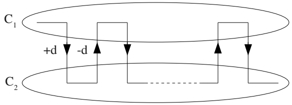
- Pivot $\pm d$ is blocked at v if v's twin already visited



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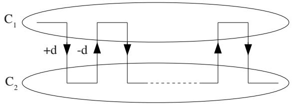

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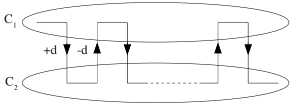
- $\sum_{i=1}^{2n} nv(i) \leq 2n(k+1) - 1 + 2^{n-1}$



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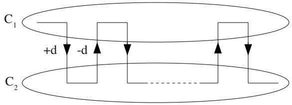
- $\sum_{i=1}^{2n} nv(i) \leq 2n(k+1) - 1 + 2^{n-1}$
- Combining, $k \geq \frac{2^{n-2}}{n} - 1$



Hamiltonian paths are special

- The theorem does not generalize to arbitrary exponential length Zadeh paths

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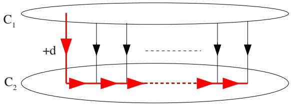
- Let C_1 and C_2 be copies of an AUSO with an exponential length Zadeh path.



And the open problems are ...

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- ... obvious