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ON POP-STACKS IN SERIES*

David Avis and Monroe Newborn

ABSTRACT. A pop-stack is a stack with a restricted pop operation: a push operation is performed in the usual way but a pop operation causes all elements in the stack to be output at once. We consider m pop stacks in series and analyze the permutations that can be sorted.

1. *Introduction.*

The "power" of a *sorting primitive* can be measured by its ability to sort permutations. Increased power results when several primitives are connected together to form a network. For a given network, it is natural to ask how many permutations of n elements can be sorted. This problem has been studied for stacks, queues, and deques and partial results for the case of series and parallel networks have appeared in Knuth [1], [2], Nozaki [6], and Tarjan [5]. It is also of interest to study and characterize properties of permutations that can be sorted by a given network. For the case of a single stack, results have been obtained by Rotem [3], [4]. When several stacks are connected in series, the problem of characterizing the sortable permutations seems to be very difficult. In this paper, we consider a type of primitive called a "pop-stack" that has less power than a stack. A pop-stack is a stack with a restricted pop operation: a push operation is performed in the usual way, but a pop operation causes *all* elements in the stack to be output at once. We will consider m pop-stacks in series and analyze the permutations that result from an input stream consisting of the integers $1,2,\ldots,n$ after arbitrary sequences of pop-stack operations. This alternate formulation of the sorting problem above is easily seen to be equivalent.

We call a permutation m-*feasible* if it can be realized by a network of m pop-stacks in series, but not by m-1 (or fewer) pop-stacks in series. A permutation is called *feasible* if it is m-feasible for some m.

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The results of this paper are:

(i) a characterization of feasible permutations;

(ii) a formula for the number of feasible permutations of the integers $\{1,2,\ldots,n\}$, and exponential asymptotic bounds;

(iii) a characterization of the m-feasible permutations; and

(iv) a recursion for the number of m-feasible permutations.

2. *Feasible Permutations.*

We will denote by $\pi = \pi_1 \ldots \pi_n$ a permutation of the integers $\{1,2,\ldots,n\}$. The transpose, $\pi_n \ldots \pi_1$, is denoted $\bar{\pi}$. We define the set of *good* permutations recursively as follows:

(i) the permutation of length 1 is good;

(ii) if π is good then $\bar{\pi}$ is good;

(iii) for $n \geq 2$, π is good if there exists an integer $k < n$ such that $\pi_1,\ldots,\pi_k = \{1,2,\ldots,k\}$ and both $\pi_1 \ldots \pi_k$ and $\pi_{k+1} \ldots \pi_n$ are good permutations.

In (iii) above, we regard the permutation $\pi_{k+1} \ldots \pi_n$ as the translation of a permutation of $\{1,2,\ldots,n-k\}$. We call permutations that satisfy (iii) *forward decomposable*. Their transposes are called *backward decomposable*. As an example, consider the permutation 2167534. Figure 1 shows that it is *good*. Figure 2 shows that it is *feasible*. This suggests the following theorem.

THEOREM 2.1. *A permutation* π *is good if and only if it is feasible.*

Proof. (Good $\rightarrow$ Feasible). By induction on n. The assertion is true by inspection for $n \leq 2$. Let π be a good permutation of length $n \geq 3$. We may assume that π is forward decomposable since it is clear that π is feasible if and only if $\bar{\pi}$ is feasible. Let k be the smallest integer such that $\pi_1 \ldots \pi_k = \{1,2,\ldots,k\}$. Since this is a good permutation and $k < n$, it is feasible by induction. Similarly $\pi_{k+1} \ldots \pi_n$ is feasible. We may thus obtain π by first performing the necessary operations on $\{1,2,\ldots,k\}$ to obtain $\pi_1 \ldots \pi_k$ and following with the operations required to obtain $\pi_{k+1} \ldots \pi_n$.

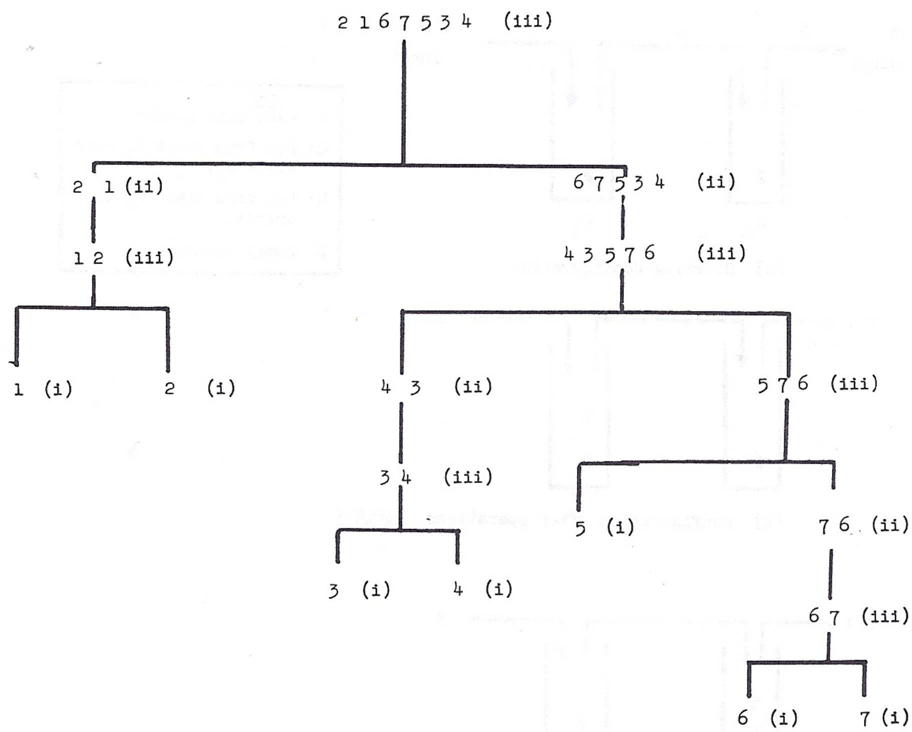

Figure 1. Decomposition of the permutation 2167534
(Numerals in parentheses refer to the step number given in the definition in Section 2.)

(Feasible → Good) Let π be the shortest counterexample, so that π is feasible but not good. Now in any series of pop-stacks, the first output from the last stack must be a set $\{1,2,\ldots,k\}$, of consecutive integers in a feasible permutation. Since π is the shortest counterexample, the first output string must have length n. Thus $\bar{\pi}$ was the contents of the last stack. But $\bar{\pi}$ is also not good, thus a similar argument shows that π must be the contents of the second last stack, and so on. Therefore π cannot be feasible.

From the proof of the first part of the theorem, it can be seen that the number of stacks required to output a feasible permutation of length n is at most one more than that required to output feasible permutations of length max $\{k, n-k\}$ for some k. This idea motivates the following recursive definition of a *stack number* $s(\pi)$ of a feasible permutation π. Let k be the smallest integer such that $\pi_1 \ldots \pi_k = \{1,2,\ldots,k\}$ or $\pi_n \ldots \pi_{n-k+1} = \{1,2,\ldots,k\}$. Define $s(1) = 0$

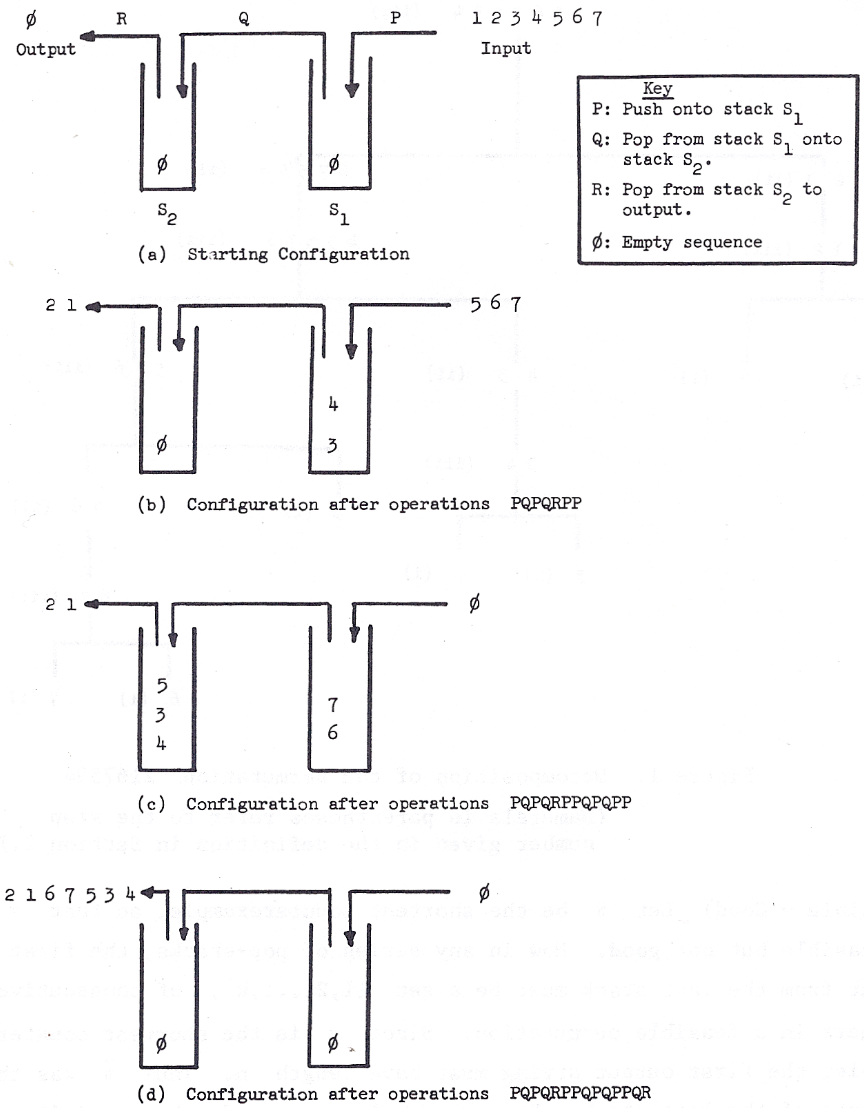

Figure 2. Illustration of the generation of the feasible permutation 2167534

and

$$(1)\quad s(\pi) = \begin{cases} \max\{s(\pi_1 \ldots \pi_k),\ s(\pi_{k+1} \ldots \pi_n)\}, & \pi \text{ forward decomposible,} \\ 1+\max\{s(\pi_{n-k} \ldots \pi_1),\ s(\pi_n \ldots \pi_{n-k+1})\}, & \pi \text{ backward decomposible.} \end{cases}$$

This definition provides us with an efficient means of deciding how many pop-stacks in series are required to output a feasible permutation, as the next theorem shows.

THEOREM 2.2. *A permutation* π *is* m-*feasible if and only if* $s(\pi) = m$.

Proof. By induction on n. For $n=2$, $s(12) = 0$ and $s(21) = 1$. It is easily verified that these are the minimum number of pop-stacks required. Consider a feasible permutation of length $n \geq 3$. Let k be defined as above.

Case (i). π is forward decomposible. $\pi_1 \ldots \pi_k = \{1,2,\ldots,k\}$. As in Theorem 2.1, we may output π by concatenating the operations for $\pi_1 \ldots \pi_k$ and $\pi_{k+1} \ldots \pi_n$. Thus the minimum number of stacks required is $\max\{s(\pi_1 \ldots \pi_k),\ s(\pi_{k+1} \ldots \pi_n)\}$, by the induction hypothesis.

Case (ii). π is backward decomposable. In this case, the entire string must be output in one operation, and so that last stack contains $\bar{\pi}$ which is forward decomposable. Thus we may use the results of Case (i) to verify that the number of stacks required is $1 + \max\{s(\pi_{n-k} \ldots \pi_1),\ s(\pi_n \ldots \pi_{n-k+1})\}$.

COROLLARY 2.3. $n-1$ *pop-stacks are required to output all feasible permutations of length* n.

Proof. That $n-1$ pop-stacks are sufficient follows immediately from the definition of $s(\pi)$ and Theorem 2.2. The permutations below show that $n-1$ pop stacks are necessary.

$n = 2k+1 \qquad \pi = 2k+1, 2k-1, \ldots, 3, 1, 2, 4, \ldots, 2k$

$n = 2k \qquad \pi = 2k, 2k-2, \ldots, 2, 1, 3, \ldots, 2k-1.$

In each case, an easy induction argument shows $s(\pi) = n-1$.

3. *Enumeration of Feasible Permutations.*

In this section we derive an exact formula for the number of feasible permutations of n integers, and derive an asymptotic bound. These results are based on the characterization of feasible permutations given by Theorem 2.1. We begin with a definition. Let f_n denote the number of feasible permutations of length n.

THEOREM 3.1.

(2) $$f_n = \sum_{k=1}^{n-1} f_k f_{n-k} + f_{n-1}, \; n \geq 2 .$$

Proof. We count the number of forward decomposable permutations. The total number of feasible permutations is precisely twice this number. For each k, $1 \leq k \leq n-1$, we wish to count the number of forward decomposable permutations π for which k is the *smallest* integer so that $\pi_1 \ldots \pi_k = \{1,2,\ldots,k\}$. Now for all such permutations π, the permutation $\pi_1 \ldots \pi_k$ *must* be backward decomposable. Otherwise π could be forward decomposed with a smaller value of k. For $k \geq 2$, the number of such permutations is thus $1/2\, f_k f_{n-k}$, since precisely one-half of the feasible permutations of length k are backwards decomposable, and the remaining $n-k$ integers may be arranged in any feasible permutation. The exceptional case is $k=1$. Here the number of forward decomposable permutations is f_{n-1}. Summing over k and multiplying by 2 gives the desired conclusion.

We now derive an exact formula for f_n. For any real number x and integers i and j, we denote by $\binom{x}{i,j}$ the multinomial coefficient

$$\binom{x}{i,j} = \frac{x(x-1)\ldots(x-i-j+1)}{i!j!} .$$

We will need the following lemma.

LEMMA 3.2.

(3) $$\binom{\frac{1}{2}}{i,j} = \binom{2i+2j-2}{i,j} \cdot \frac{2^{-2i-2j+1}}{i+j-1} \cdot (-1)^{i+j-1} .$$

Proof. $\binom{\frac{1}{2}}{i,j} = \dfrac{\frac{1}{2}(-\frac{1}{2})(-3/2)\dots\left(-\frac{2i+2j-3}{2}\right)}{i!j!}$

$$= \frac{1(-1)(-3)\dots(-2i-2j+3)\cdot 2\cdot 4\cdot \dots (2i+2j-2)}{i!j!\ (i+j-1)!\ 2^{2i+2j-1}(-1)^{i+j-1}} .$$

The formula (3) follows.

We may now derive the following formula for f_n.

THEOREM 3.3.

(4) $$f_n = \sum_{\substack{i\geq 0\\ j\geq 0\\ 2i+j=n}} (-1)^i \cdot \frac{(2i+2j-2)!}{i!j!(i+j-1)!} \cdot \frac{3^j}{2^n} , \quad n \geq 2 .$$

Proof. Let $f(x)$ be the generating function of f_n, so that $f(x) = \sum_{n=1}^{\infty} f_n x^n$. By multiplying each side of (2) by x^n and summing, we obtain

$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} f_k\, f_{n-k}\, x^n + x \sum_{n=2}^{\infty} f_{n-1}\, x^{n-1} .$$

We note that

$$f^2(x) = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} f_k\, f_{n-k}\, x^n ,$$

and hence we obtain the relation

$$f(x) - x = f^2(x) + xf(x) .$$

Rewriting, we obtain the formal quadratic expression for $f(x)$:

$$f^2(x) + (x-1)\ f(x) + x = 0 .$$

Hence

$$f(x) = \frac{(1-x) \pm \sqrt{(x-1)^2-4x}}{2} .$$

Since $f(0) = 0$, we obtain the expression

$$f(x) = \frac{(1-x) - \sqrt{x^2 - 6x + 1}}{2} .$$

For $n \geq 2$, we may apply the multinomial theorem to $\frac{1}{2}(x^2-6x+1)^{\frac{1}{2}}$ to obtain the coefficient f_n of x^n. Indeed,

$$\begin{aligned} f_n &= -\tfrac{1}{2} \sum_{2i+j=n} \binom{\frac{1}{2}}{i,j} (-6)^j \\ &= \sum_{2i+j=n} \frac{(2i+2j-2)!}{i!j!(i+j-1)!} (-1)^{i-1} \cdot \frac{6^j}{2^{2i+2j}} , \end{aligned}$$

where we have applied Lemma 3.2. The formula (4) follows.

The formula (4) does not give any simple estimate on the growth of f_n. This can be obtained by the observation that

$$f_n \leq \frac{3}{2} \sum_{k=1}^{n-1} f_k f_{n-k} . \tag{5}$$

We define the sequence g_n by

$$g_1 = 1, \quad g_n = \frac{3}{2} \sum_{k=1}^{n-1} g_k g_{n-k} .$$

It is easily seen that $f_n \leq g_n$.

THEOREM 3.4. $f_n \leq g_n = \frac{1}{n} \binom{2n-2}{n-1} \left(\frac{3}{2}\right)^{n-1} \leq \frac{6^{n-1}}{n}$.

Proof. Let $g(x)$ be the generating function of g_n. A similar argument to that of Theorem 3.3 shows that

$$g(x) - x = \frac{3}{2} g^2(x) .$$

Hence

$$g(x) = \frac{1 - \sqrt{1 - 6x}}{3} .$$

The binomial theorem then yields

$$\begin{aligned} g_n &= -\frac{1}{3} \binom{\frac{1}{2}}{n} (-6)^n = \frac{1}{3n} \binom{2n-2}{n-1} 2^{-2n+1} 6^n \\ &= \frac{1}{n} \binom{2n-2}{n-1} \left(\frac{3}{2}\right)^{n-1} \leq \frac{6^{n-1}}{n}. \end{aligned}$$

On the other hand, a simple computation shows that for $n \geq 11$, $f_n > 5 f_{n-1}$, and so we have the following

COROLLARY 3.5. $5^n < f_n < \frac{6^{n-1}}{n}$.

4. *Enumeration of* m-*feasible Permutations.*

In this section we develop a recursion for the number of m-feasible permutations of n elements. Let f_n^m denote this number. The main result of this section is:

THEOREM 4.1.

$$f_n^m = \sum_{k=2}^{n} f_{n-k}^m \left(\sum_{i=1}^{m} (-1)^{m-i} f_k^{i-1} \right) + f_{n-1}^m, \quad m \geq 1,\ n \geq 2,$$

where we establish the initial conditions

$$f_0^\alpha = f_\alpha^0 = f_1^\alpha = 1, \quad \alpha = 1,2,\ldots .$$

Before proving the theorem we look at the cases $m = 1$ and $m = 2$ as examples.

Case $m = 1$. For $n \geq 2$, applying Theorem 4.1 gives:

$$f_n^1 = \sum_{k=2}^{n} f_{n-k}^1 \ (-1)^0 \ f_k^0 + f_n^1 = \sum_{k=0}^{n-1} f_k^1 . \tag{6}$$

It is easily verified that $f_n^1 = 2^{n-1}$ solves (6).

Case $m = 2$. For $n \geq 2$, Theorem 4.1 yields

$$f_n^2 = \sum_{k=2}^{n} f_{n-k}^2 \ (f_k^1 - f_k^0) + f_{n-1}^2 = \sum_{k=2}^{n} f_{n-k}^2 \ (2^{k-1} - 1) + f_{n-1}^2 ,$$

from which f_n^2 may be readily computed. The expression can be simplified by some manipulation. Rearranging and temporarily dropping superscripts gives

$$\sum_{k=0}^{n} f_k = \sum_{k=1}^{n} 2^{k-1} f_{n-k} + f_{n-1} .$$

Taking first difference gives

$$f_n = \sum_{k=1}^{n} 2^{k-1} f_{n-k} - \sum_{k=1}^{n-1} 2^{k-1} f_{n-k-1} + f_{n-1} - f_{n-2}$$

$$= \sum_{k=1}^{n-1} 2^{k-1} f_{n-k-1} + f_{n-2} + 2f_{n-1} - 2f_{n-2}$$

$$= \sum_{k=0}^{n-1} f_k + 2f_{n-1} - 2f_{n-2} .$$

Therefore,

$$\sum_{k=0}^{n} f_k = 2f_n - 2f_{n-1} + 2f_{n-2} .$$

Another first difference yields

$$f_n = 2f_n - 4f_{n-1} + 4f_{n-2} - 2f_{n-3}$$

or, finally, inserting the superscripts

$$f_n^2 = 4f_{n-1} - 4f_{n-2}^2 + 2f_{n-3}^2 .$$

Proof of Theorem. For the purposes of the proof, it is useful to introduce two new functions

g_n^m = number of forward decomposable m-feasible permutations of n elements,

h_n^m = number of backward decomposable m-feasible permutations of n elements.

Thus $f_n^m = g_n^m + h_n^m$, $n \geq 2$. Consider any m-feasible forward decomposable permutation π, and let k be the smallest subscript so that $\pi_1 \ldots \pi_k = \{1,2,\ldots,k\}$. If $k \geq 2$, $\pi_1 \ldots \pi_k$ must be a backward decomposable m-feasible permutation of length k, and there are exactly h_k^m of these. The remaining permutation $\pi_{k+1} \ldots \pi_n$ is any m-feasible permutation, and there are exactly f_{n-k}^m of these. If $k = 1$, there are precisely f_{n-1}^m such permutations. Hence we obtain the expression:

(7) $$g_n^m = \sum_{k=2}^{n-1} h_k^m f_{n-k}^m + f_{n-1}^m, \quad m \geq 1, n \geq 1 ,$$

with the initial conditions:

$$g_\alpha^0 = f_0^\alpha = h_\alpha^1 = 1, \quad \alpha = 1,2,3,\ldots .$$

From the results of Section 2, we see that every backward decomposable m-feasible permutation is the transpose of a forward decomposable (m-1)-feasible permutation, and vice versa; hence:

$$h_n^m = g_n^{m-1}, \qquad m \geq 1,\ n \geq 2 .$$

Therefore,
$$f_n^m = g_n^m + g_n^{m-1}, \qquad m \geq 1,\ n \geq 2 ,$$

which may be inverted to give

$$g_n^m = \sum_{i=0}^{m} (-1)^{m-i} f_n^i, \quad n \geq 2 .$$

Making the indicated substitutions into (7) yields

$$\sum_{i=0}^{m} (-1)^{m-i} f_n^i = \sum_{k=2}^{n-1} \left(\sum_{i=0}^{m-1} (-1)^{m-i-1} f_k^i \right) f_{n-k}^m + f_{n-1}^m ,$$

and thus

$$f_n^m = \sum_{k=2}^{n-1} \left(\sum_{i=0}^{m-1} (-1)^{m-i-1} f_k^i \right) f_{n-k}^m + \sum_{i=0}^{n-1} (-1)^{m-i-1} f_n^i + f_{n-1}^m$$

$$= \sum_{k=2}^{n} f_{n-k}^m \left(\sum_{i=0}^{m-1} (-1)^{m-i-1} f_k^i \right) + f_{n-1}^m .$$

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School of Computer Science
McGill University
Montréal H3A 2K6

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