Visualizing and Constructing Cycles in The Simplex Method

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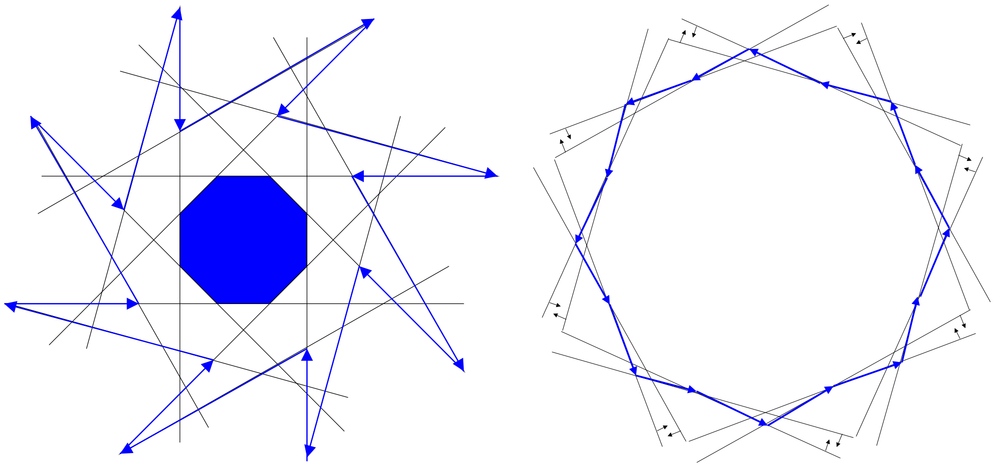


"Cycling is certainly not completely understood." - Hoffman, 1951.

Abstract

Most examples of cycling in the simplex method are given without explanation of how they were constructed. An exception is Beale's 1955 example built around the geometry of the dual simplex method in the plane. Using this approach we give a simple geometric explanation for a number of examples of cycling in the simplex method, including Alan Hoffman's original example. This gives rise to a simple method for generating examples with cycles.

La plupart des exemples du cycle dans la méthode simplexe sont donnés sans explication de la façon dont ils ont été construits. Une exception est l'exemple de Beale (1955) établi utilisant de la géométrie de la méthode simplexe duelle dans deux dimensions. En utilisant cette approche nous donnons une explication géométrique simple pour plusiers exemples du cycle dans la méthode simplexe, y compris l'exemple original de Alan Hoffman. Ceci provoque une méthode simple pour produire des exemples avec des cycles.

1 Introduction

What is the magic behind cycling in the simplex method? Every textbook on linear programming dedicates a section to cycling in the simplex method; however, no single textbook provides the reader with a geometric intuition behind cycling. This list includes the classic textbooks of Chvátal [3] and Dantzig [4], and recent books such as Sierksma [11]. The examples are usually presented as a sequence of *tableaux* or *dictionaries* - if we follow the terminology and notation used by Chvátal. Such presentations are sure to convince the reader that simplex methods can cycle, but do not provide any insight behind the construction. Consider the first cycling example, concocted by Hoffman in 1951 [7], [4, p.229], with starting dictionary

$$
\begin{aligned}
x_0 &= 1\\
x_1 &= -\cos\varphi x_3 + \omega\cos\varphi x_4 - \cos 2\varphi x_5 + 2\omega\cos^2\varphi x_6\\
&\quad - \cos 2\varphi x_7 - 2\omega\cos^2\varphi x_8 - \cos\varphi x_9 - \omega\cos\varphi x_{10}\\
x_2 &= -[(\tan\varphi\sin\varphi)/\omega]x_3 - \cos\varphi x_4 - [(\tan\varphi\sin 2\varphi)/\omega]x_5 - \cos 2\varphi x_6\\
&\quad + 2(\sin^2\varphi/\omega)x_7 - \cos 2\varphi x_8 + [(\tan\varphi\sin\varphi)/\omega]x_9 - \cos\varphi x_{10}\\
z &= [(1-\cos\varphi)/\cos\varphi]x_3 - \omega x_4 - 2\omega x_6 - 4\sin^2\varphi x_7\\
&\quad + 2\omega\cos 2\varphi x_8 - 4\sin^2\varphi x_9 - \omega(1-2\cos\varphi)x_{10}.
\end{aligned}
$$

If we set $\varphi = 5\pi/2, \omega > (1-\cos 2\varphi)/(1-2\cos\varphi) \approx 4.74$ the simplex method using Dantzig's rule cycles through a sequence of 10 dictionaries[1]. The left hand side variables cycle repeatedly through the sequence $\{x_0, x_1, x_2\}$, $\{x_0, x_2, x_3\}$, ... , $\{x_0, x_9, x_{10}\}$, $\{x_0, x_{10}, x_1\}$.

How was this example constructed? Strangely, even Hoffman forgot!

> "Finally, we regret that we are unable at this date to recall any details of the considerations that led to the construction of the example, beyond the fact that the geometric meaning of [a pivot rule for the simplex method] was very much in the foreground." - Hoffman, 1951.
>
> "Two months after I made up the example, I lost the mental picture which produced it..." - Hoffman, 1994.

This is a shame, as geometry can certainly help students understand the simplex method through a visual interpretation. In 1997 Lee [8] successfully untangled the algebraic derivation of Hoffman's example, and using *column geometry* (originally known as the *simplex interpretation* [4, p.160]) produced a three-dimensional picture. In Figure 1 we exhibit Hoffman's example in two-dimensions using the geometry of the dual simplex method, which was first explained and used to construct cycles by Beale [2], but not used to analyze Hoffman's example:

> "Unfortunately, the geometric motivation behind Hoffman's example is not easy to grasp" - Beale, 1955.

In the next section we see how cycling works in this setting, by giving a simplified version of this geometrical interpretation. We show that cycling examples obtained from various linear programmming

[1]Note that in [4], [6], [7], and [8] the bound on ω reads $\omega > (1-\cos\varphi)/(1-2\cos\varphi) \approx 1.81$. In fact, the tighter bound presented here is required for Hoffman's *LP* to cycle using Dantzig's pivot rule. See Appendix A for details.

texts have a similar structure, and many are combinatorially equivalent. In Section 3 we give a simple method of constructing examples that contain cycles of arbitrary even length.

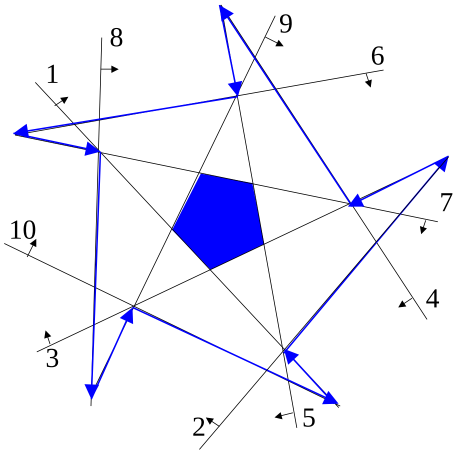


Figure 1: A picture of Hoffman's cycle

2 Visualizing Cycles

Given a linear program,

$$\max z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1 ... m$$

$$x_j \geq 0 \qquad j = 1 ... n,$$

we add nonnegative *slack* variables $x_{n+1}, ..., x_{n+m}$ to obtain a system of linear equations,

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \qquad i = 1, ..., m. \tag{1}$$

Solving these m equations for $m - n$ variables we obtain a dictionary. For example, the linear program

$$\begin{aligned} \max \;\; z = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0 \\ 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \leq 0 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \tag{2}$$

is encoded by the dictionary

$$\begin{aligned} x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4, \end{aligned} \tag{3}$$

and each variable x_j corresponds an inequality of (2). If the *basic solution*, obtained by setting the *cobasic* (right-hand side) variables to zero and evaluating the *basic* (left-hand side) variables, is nonnegative then the dictionary represents a *basic feasible solution* for the linear program, the value of z being the objective value. Simplex methods attempt to increase the objective value by exchanging a cobasic variable x_s with positive coefficient in the z-row with a basic variable x_r while maintaining a basic feasible solution. This is accomplished via a *ratio test* which chooses x_r such that 1) the coefficient of x_s in equation x_r is negative and 2) solving this equation for x_s and substituting for x_s in the other equations then yields another feasible basic solution. This basic operation is called a *simplex pivot*. *Pivot rules* (or refinements) dictate which variables to select; for example, following *Dantzig's pivot rule* [4] we choose the variable in the $z-$row with largest positive coefficient. We break ties in the ratio test arbitrarily (usually by selecting the variable with the smallest index). In our example we would pivot on variables x_1 and x_5, the former chosen as it has the largest positive coefficient in the $z-$row and the latter arbitrarily chosen from x_5 and x_6 due to a ratio test tie - caused by *degeneracy* - yielding the dictionary

$$\begin{aligned} x_1 &= 11x_2 + 5x_3 - 18x_4 - 2x_5 \\ x_6 &= -4x_2 - 2x_3 + 8x_4 + x_5 \\ z &= 53x_2 + 41x_3 - 204x_4 - 20x_5. \end{aligned} \tag{4}$$

Degeneracy may cause the simplex method to *stall*, or pivot from dictionary to dictionary without increasing the objective value. The simplex method cycles if it stalls indefinitely, revisiting previously computed dictionaries. Continuing with our example, the simplex method with Dantzig's rule visits the dictionaries

$$\begin{aligned} x_1 &= -0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\ x_2 &= -0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\ z &= 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6, \end{aligned} \tag{5}$$

$$\begin{aligned} x_2 &= x_1 - 2x_4 - 0.5x_5 + 2.5x_6 \\ x_3 &= -2x_1 + 8x_4 + 1.5x_5 - 5.5x_6 \\ z &= -29x_1 + 18x_4 + 15x_5 - 93x_6, \end{aligned} \tag{6}$$

$$\begin{aligned} x_3 &= 2x_1 - 4x_2 - 0.5x_5 + 4.5x_6 \\ x_4 &= 0.5x_1 - 0.5x_2 - 0.25x_5 + 1.25x_6 \\ z &= -20x_1 - 9x_2 + 10.5x_5 - 70.5x_6, \end{aligned} \tag{7}$$

$$\begin{aligned} x_4 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_6 \\ x_5 &= 4x_1 - 8x_2 - 2x_3 + 9x_6 \\ z &= 22x_1 - 93x_2 - 21x_3 + 24x_6, \end{aligned} \tag{8}$$

and returns to dictionary (3), cycling through the sequence of bases $\{5,6\}$, $\{1,6\}$, $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$, $\{5,6\}$... The six inequalities of the linear program (2) define a four-dimensional polyhedron, and the simplex method cycles among the bases of a single vertex. This geometry cannot help us visualize the cycle. In 1955, Beale [2] constructed a cycle for the simplex method by considering the geometry of the dual problem. The dual of (2) is

$$\begin{aligned} \min\ w &= 0y_1 + 0y_2 \\ &(1): 0.5y_1 + 0.5y_2 \geq 10 \\ &(2): -5.5y_1 - 1.5y_2 \geq -57 \\ &(3): -2.5y_1 - 0.5y_2 \geq -9 \\ &(4): 9y_1 + y_2 \geq -24 \\ &(5): y_1 \geq 0, \\ &(6): y_2 \geq 0 \end{aligned} \tag{9}$$

which has six inequalities in two dimensions. We can also write the dual problem in dictionary form. In fact, the dual dictionary is simply a disguised version of the (primal) dictionary for the primal: the transposed system with signs reversed.

$$\begin{array}{lcl} & & y_3 = -10 + 0.5y_1 + 0.5y_2 \\ x_5 = 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 & & y_4 = 57 - 5.5y_1 - 1.5y_2 \\ x_6 = 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 & \longleftrightarrow & y_5 = 9 - 2.5y_1 - 0.5y_2 \\ z = 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 & & y_6 = 24 + 9y_1 + y_2 \\ & & w = 0 + 0y_1 + 0y_2. \end{array}$$

Table 2. Primal and dual dictionaries

In the dual simplex method the $w-$row is assumed to have all coefficients non-negative. The leaving basic variable is chosen from the basic variables with negative constant on the right hand side of the dictionary. A ratio test is used to choose the entering variable so that the $w-$row remains non-negative. It is easy to verify that the dual simplex method applied to the dual problem will, with a suitably chosen pivot rule, follow the same sequence of pivots as the primal simplex method does on the primal problem. The sequence of bases for the primal problem becomes the sequence of cobases for the dual problem, and vice versa. Graphing (9), the geometry behind a simplex method cycle begins to unveil. We obtain an arrangement of lines, each bounding an inequality of (9). Figure 2 depicts this arrangement.

The two-dimensional geometric interpretation of the dual simplex method is simple when the objective function is zero. For the time being, let's ignore specific pivot rules and consider simplex pivots in general. Every line in the arrangement corresponds to a variable in the dual dictionary. The slack variable y_i for $i = 3...n$ corresponds to line $i-2$, while y_1 and y_2 correspond to lines $n-1$ and n respectively.

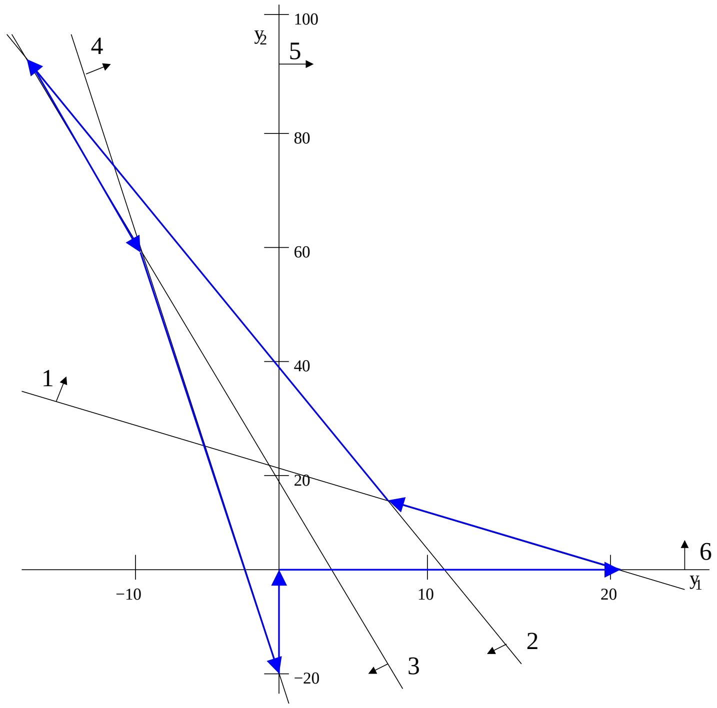


Figure 2: Chvátal [3]

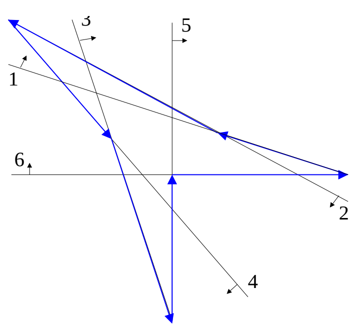
Each intersection of lines corresponds to a basic dual solution obtained by setting the two corresponding variables to zero. These are the two dual cobasic variables for a corresponding dual dictionary. For example, the dual dictionary in Table 2 with cobasis {1,2} corresponds to the intersection of lines labelled 5 and 6. The basic variable chosen in a pivot corresponds to a violated inequality: in our case the intersection point lies on the wrong side of line 1, so y_3 is the leaving basic variable. The entering variable is chosen from those with a positive coefficient in the equation for the leaving variable, so that that the new basic solution will satisfy the inequality corresponding to the entering variable. In the example we could choose either y_1 or y_2 as the entering variable. However from the cobasis {1,6} we could pivot to the cobasis {1,2} since this is on the correct side of line 4. We could not pivot to cobasis {3,6} as this is on the wrong side of line 5. A negative basic variable indicates a degree of infeasibility, for example $y_3 = -10 + 0.5y_1 + 0.5y_2$ indicates that at the intersection of line 5 and line 6, line 1 is infeasible by $\frac{10}{0.5}$ units along line 5 and by $\frac{10}{0.5}$ units along line 6 with respect to the vector space basis $\{y_1, y_2\}$. The dual simplex method terminates when it reaches a point where either no inequalities are violated (optimality) or when it cannot pivot to a violated inequality without violating one of the two intersecting lines (dual infeasible).

It is now routine to check that the cycles shown in Figure 1 for Hoffman's example, and Figure 2 for our example correspond to valid dual simplex pivots. It follows that the primal problems pivot through the same cycles, with bases/cobases interchanged.

Our example is derived from the following example found in Chvátal's book. The first dictionary of the cycle is presented as

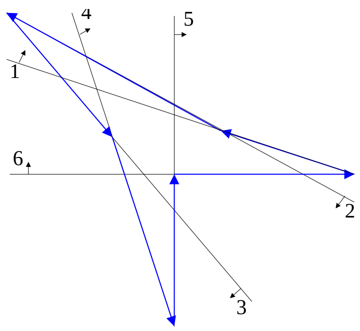
$$
\begin{aligned}
x_5 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1 \\
z &= 10x_1 - 57x_2 - 9x_3 - 24x_4,
\end{aligned} \tag{10}
$$

and consists of seven variables and three equations. However, variable x_7 never leaves the basis as the cycle only pivots on the variables x_1 through x_6. The equation $x_7 = 1 - x_1$ only serves to bound the linear program but can be removed from dictionary (10) without affecting the cycle. It turns out that many other examples found in literature, including Hoffman's, can be reduced in this way and then graphed in two-dimensions. In fact, many are then combinatorially equivalent.



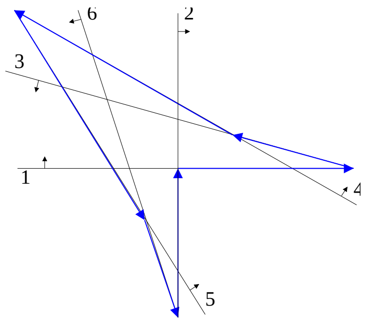
Beale [2]

$$
\begin{aligned}
x_5 &= 0 - \tfrac{1}{4}x_1 + 60x_2 + \tfrac{1}{25}x_3 - 9x_4 \\
x_6 &= 0 - \tfrac{1}{2}x_1 + 90x_2 + \tfrac{1}{50}x_3 - 3x_4 \\
z &= 0 + \tfrac{3}{4}x_1 + 150x_2 + \tfrac{1}{50}x_3 - 6x_4
\end{aligned}
$$



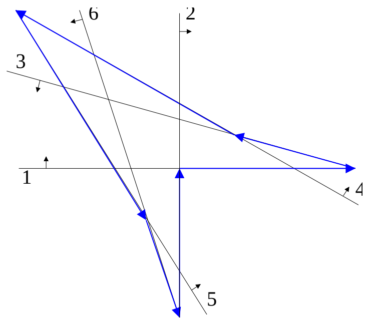
Sierksma [11]

$$
\begin{aligned}
x_5 &= 0 - x_1 + 32x_2 + 4x_3 - 36x_4 \\
x_6 &= 0 - x_1 + 24x_2 + x_3 - 6x_4 \\
z &= 0 + 3x_1 - 80x_2 + 2x_3 - 24x_4
\end{aligned}
$$



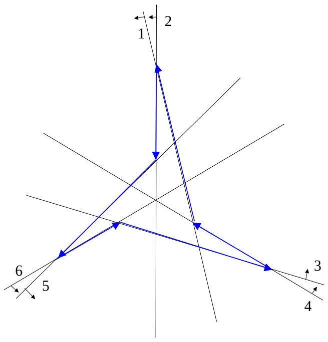
Yudin and Gol'shtein [13]

$$
\begin{aligned}
x_1 &= 0 - 2x_3 + 3x_4 + 5x_5 - 6x_6 \\
x_2 &= 0 - 6x_3 + 5x_4 + 3x_5 - 2x_6 \\
z &= 0 + x_3 - x_4 + x_5 - x_6
\end{aligned}
$$

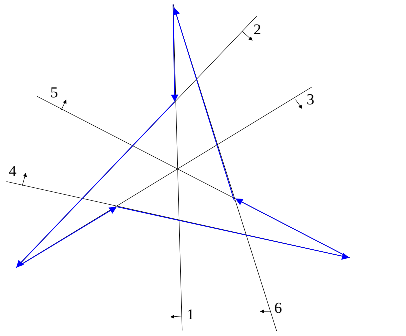


Yudin and Gol'shtein [13]

$$
\begin{aligned}
x_1 &= 0 - x_3 + 2x_4 + 3x_5 - 4x_6 \\
x_2 &= 0 - 4x_3 + 3x_4 + 2x_5 - x_6 \\
z &= 0 + x_3 - x_4 + x_5 - x_6
\end{aligned}
$$



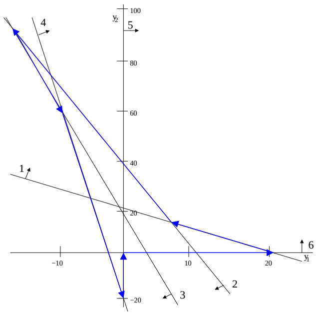
Solow [12]



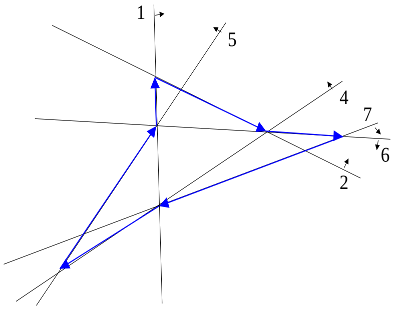
Marshall and Suurballe [9]

$$x_1 = 0 + 7x_3 + 3x_4 - 7x_5 - 2x_6$$
$$x_2 = 0 - 2x_3 - x_4 + 3x_5 + x_6$$
$$z = 0 + 2x_3 + 2x_4 - 8x_5 - 2x_6$$

$$x_1 = 0 - \tfrac{1}{2}x_3 + \tfrac{11}{2}x_4 + \tfrac{5}{2}x_5 - 9x_6$$
$$x_2 = 0 - \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4 + \tfrac{1}{2}x_5 - x_6$$
$$z = 0 - x_3 + 7x_4 + 1x_5 - 2x_6$$



Chvátal [3]



Kuhn [1]

$$x_5 = 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4$$
$$x_6 = 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4$$
$$z = 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4$$

$$x_1 = 0 + 2x_4 + 9x_5 - x_6 - 9x_7$$
$$x_2 = 0 - \tfrac{1}{3}x_4 - x_5 + \tfrac{1}{3}x_6 + 2x_7$$
$$z = 0 + 2x_4 + 3x_5 - x_6 - 12x_7$$

Table 2. Some examples of cycling found in literature

In all eight examples in Table 2, the primal linear program (after reduction to 2 equations) is unbounded. Thus, by duality theory, all six dual linear programs above are infeasible. Nonetheless, the simplex method cycles without detecting this fact. (Note that the shaded region in Figure 1 corresponds to the feasible set of Hoffman's dual linear program).

In no way is the geometry described in this section constrained to cycles with $m = 2$, as when $m = d$ oriented lines become d-dimensional oriented hyperplanes and a dual simplex pivot has an analogous description to its $m = 2$ counterpart. The geometry can be used to explain cycling exampes with $m \geq 3$ such as [5],[9] and [10]. We choose to restrict ourselves to the two-dimensional interpretation for simplicity.

3 Constructing Cycles

Every two-dimensional arrangement of k inequalities can be encoded as a primal dictionary with k equations and $k+2$ variables. For the time being, let's ignore specific pivot rules and consider simplex pivots in general. Only two basic rules need to be followed to build a simplex method cycle:

1. Each pivot must be a simplex pivot: the pivot step from the intersection of lines r and s to the intersection of lines r and t is a simplex pivot if and only if the inequality bounded by t is violated at $\{r, s\}$ and the inequality bounded by s is satisfied at $\{r, t\}$.

2. The cycle must be represented by closed loop in the diagram.

The path consists of an alternating series of left and right turns in the line arrangement, as the next point must always satisfy the two currently intersecting lines.

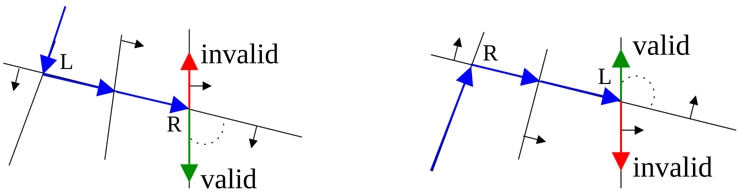
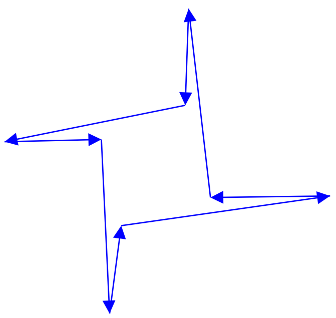
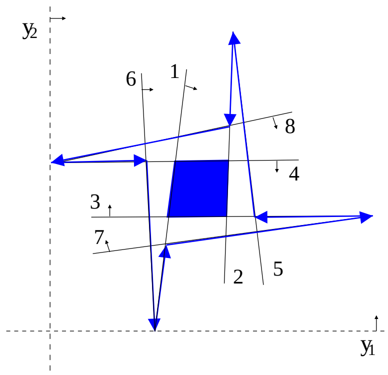


Table 3. Left and right turns

We use this fact to construct a cycle from scratch and the reader can follow the construction in Figure 3. First, using line segments we draw a path consisting of an alternating series of left and right turns that forms a closed loop. Note that the number of points must be even to form a closed loop, and the loop can indeed be self-crossing (as in Figure 4). Next, we choose coordinate axes (y_1, y_2), and lines bounding $y_1 \geq 0, y_2 \geq 0$ so that the loop lies in the positive quadrant.



$\rightarrow$



$\rightarrow$

$\min 0y_1 + 0y_2$
$(1) : -y_1 \leq -10$
$(2) : y_1 \leq 15$
$(3) : -y_2 \leq -10$
$(4) : y_2 \leq 15$
$(5) : 10y_1 + y_2 \leq 175$
$(6) : -10y_1 - y_2 \leq -100$
$(7) : y_1 - 10y_2 \leq -70$
$(8) : -y_1 + 10y_2 \leq 150$
$(9), (10) : y_1, y_2 \geq 0$

Figure 3

We then fit a line through each line segment in the loop. For each line assign a direction of feasibility which coincides with the path. Compute these inequalities with respect to (y_1, y_2), and add a zero objective function to obtain a dual LP. By duality, we convert to a primal LP, add slack variables, and compute an initial dictionary:

$$
\begin{aligned}
x_2 &= 0 + x_1 - 10x_3 + 10x_4 - 101x_7 + 101x_8 + x_9 - 10x_{10} \\
x_5 &= 0 + x_3 - x_4 + x_6 + 10x_7 - 10x_8 + x_{10} \\
z &= -5x_1 - 15x_3 + 10x_4 - 75x_6 - 165x_7 + 85x_8 - 15x_9 - 25x_{10}.
\end{aligned} \tag{11}
$$

Now simplex pivots in our diagram correspond to simplex pivots in the primal linear program, cycling repeatedly through the sequence $\{x_2, x_5\}$, $\{x_2, x_8\}$, $\{x_4, x_8\}$, $\{x_4, x_6\}$, $\{x_1, x_6\}$, $\{x_1, x_7\}$, $\{x_3, x_7\}$, $\{x_3, x_5\}$[2].

In order for the cycle to follow a specific pivot rule, an additional step is required. Note that for each inequality, $a_{i1}y_1 + a_{i2}y_2 \leq b_i$ for $i = 1...8$, we can scale both sides by a positive constant C_i at will as $a_{i1}y_1 + a_{i2}y_2 \leq b_i \Leftrightarrow C_i(a_{i1}y_1 + a_{i2}y_2) \leq C_i b_i$. These constants in turn scale the primal variables, yielding the dictionary,

$$
\begin{aligned}
x_2 &= 0 + \frac{1}{C_2}x_1 - \frac{10}{C_2}x_3 + \frac{10}{C_2}x_4 - \frac{101}{C_2}x_7 + \frac{101}{C_2}x_8 \\
x_5 &= 0 + \frac{1}{C_5}x_3 - \frac{1}{C_5}x_4 + \frac{1}{C_5}x_6 + \frac{10}{C_5}x_7 - \frac{10}{C_5}x_8 \\
z &= -5C_1x_1 - 15C_3x_3 + 10C_4x_4 - 75C_6x_6 - 165C_7x_7 + 85C_8x_8.
\end{aligned} \tag{12}
$$

By pivoting along the desired cycle, the corresponding dictionaries constrain the values of C_i we can choose so that the pivot rule is followed. For example, for the first pivot to follow Dantzig's rule we need $85C_8 > 10C_4$ in the primal dictionary. We accumulate all the constraints on C_i in this manner, and only if we find satisfying values for the C_i's then following the pivot rule yields the cycle. (The reader can easily check that for any dictionary the coefficient for variable x_j in the $z-$row is strictly a multiple of C_j). In our example we need $85C_8 > 10C_4$, $85C_6 > 10C_1$, $80C_7 > 10C_3$, $135C_5 > 15C_2$, thus setting $C_i = 1$ for $i = 1...8$ results in a simplex method cycle following Dantzig's pivot rule.

This process has a geometric analogue that can guide us to construct a diagram to ensure that Dantzig's rule (or any other coefficient based rule) will follow the loop. The sufficient condition is as follows: given a diagram, for each point along the loop we list the pivot rule requirements for the cycle to be followed. For example, at point A in Figure 4 we want to pivot to violated line 1 instead of line 2 or 6 which are also violated at A. This translates into constraints $C_1 > C_2$ and $C_1 > C_6$. Similar constraints are obtained from all points B, C, D, E and F along the cycle. If the set of constraints are satisfiable, as is the case here, then there exist C_i's large enough such that Dantzig's rule will cycle on the resulting LP. Of course the challenge is to draw a diagram with this in mind!

[2] We can ignore variables x_9 and x_{10} as they are not involved in the cycle. They correspond, respectively, to inequalities $y_1 \geq 0$ and $y_2 \geq 0$ of the dual LP which are only used to specify the coordinate axes

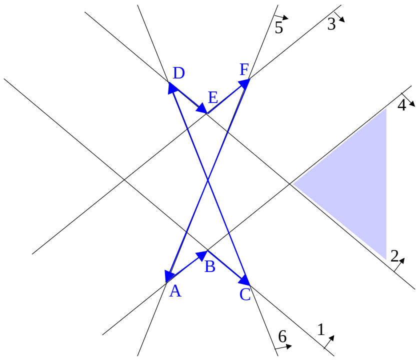


Figure 4: **Crossing loop.**

point A : C_1 bigger than C_2 and C_6
point B : C_6 bigger than C_2
point C : only line 2 violated
point D : C_3 bigger than C_4 and C_5
point E : C_5 bigger than C_4
point F : only line 4 violated

References

[1] M. Balinski, A. Tucker, Duality theory of linear programs - a constructive approach with applications, *SIAM Review* **11** (3) (1969) 347–77.

[2] E. Beale, Cycling in the dual simplex method, *Naval Research Logistics Quarterly* **2** (4) (1955) 269–75.

[3] V. Chvátal, *Linear Programming*, Freeman, New York, 1983.

[4] G. Dantzig, *Linear Programming and Extensions*, Princeton University Press, NJ, 1963

[5] K. Fukuda, H-J Lüthi, *Optimization Techniques* (course notes), 1999.

[6] S. Gass, *Linear Programming: Methods and Applications*, 5th edition, McGraw-Hill Book Company, New York, 1985.

[7] A. Hoffman, Cycling in the simplex algorithm, *National Bureau of Standards*, Washington, 1953.

[8] J. Lee, Hoffman's circle untangled, *SIAM Review* **39** (1997) 98–105.

[9] K. Marshall, J. Suurballe, A note on cycling in the simplex method, *Naval Research Logistics Quarterly* **16** (1) (1969) 121–37.

[10] E. Nering, A. Tucker, *Linear Programs and Related Problems*, Academic Press, Boston, 1993.

[11] G. Sierksma, *Linear and Integer Programming*, 2nd ed. Marcel Dekker Inc., New York, 1996.

[12] D. Solow, *Linear Programming: An Introduction to Finite Improvement Algorithms*, North-Holland Press, Amsterdam, 1984.

[13] D. Yudin, E. Gol'shtein, *Linear Programming*, Israel Program of Scientific Translations, Jerusalem, 1965.

A Hoffman's ω: The Correct Bound

All references to Hoffman's example, for example [4], [6], [7], and [8], state an incorrect lower bound $\omega > \frac{1-cos(\phi)}{1-2\cdot cos(\phi)} \approx 1.81$. If we set the value of ω close to this bound the simplex method with Dantzig's rule terminates. We now compute the correct lower bound. Consider the objective function of the initial dictionary,

$$z = \frac{1-cos(\phi)}{cos(\phi)} \cdot x_3 - \omega \cdot x_4 - 2 \cdot w \cdot x_6 - 4 \cdot sin^2(\phi) \cdot x_7$$
$$+ 2 \cdot w \cdot cos(2\phi) \cdot x_8 - 4 \cdot sin^2(\phi) \cdot x_9 + \omega \cdot [\,2 \cdot cos(\phi) - 1\,] \cdot x_{10}.$$

For positive values of ω, x_3 is the only choice to enter the basis since $\frac{1-cos(\phi)}{cos(\phi)}$ is the only positive coefficient in the z-row ($cos(2\phi) < 0$ and $[2 \cdot cos(\phi) - 1] < 0$ because $\frac{\pi}{3} < \phi < \frac{\pi}{2}$).
We pivot x_3 with x_1 and obtain the second dictionary with objective function

$$z = \frac{cos(\phi)-1}{cos(\phi)} \cdot x_1 + \omega \cdot \frac{1-2\cdot cos(\phi)}{cos(\phi)} x_4 - 2 \cdot sin(\phi) \cdot tan(\phi) \cdot x_5 - 2 \cdot \omega \cdot cos(\phi) \cdot x_6$$
$$+ \frac{1-cos(\phi)}{cos(\phi)} \cdot x_7 - 3 \cdot \omega \cdot x_8 - 2 \cdot sin(\phi) \cdot tan(\phi) \cdot x_9 - \omega \cdot [\,4 \cdot cos^2(\phi) - 3\,] \cdot x_{10}.$$

For x_4 to enter the basis, using Dantzig's rule to pivot, we require both

$$\omega \cdot \frac{1-2\cdot cos(\phi)}{cos(\phi)} \geq 2 \cdot sin(\phi) \cdot tan(\phi)$$

and

$$\omega \cdot \frac{1-2\cdot cos(\phi)}{cos(\phi)} \geq \frac{1-cos(\phi)}{cos(\phi)},$$

together yielding a tighter bound

$$\omega \geq 2 \cdot sin(\phi) \cdot tan(\phi) \cdot \frac{cos(\phi)}{1-2\cdot cos(\phi)} = \frac{1-cos(2\phi)}{1-2\cdot cos(\phi)} \approx 4.74.$$

Proceeding in this manner, analyzing all ten dictionaries of Hoffman's cycle, the correct lower bound $\omega > \frac{1-cos(\phi)}{1-2\cdot cos(\phi)} \approx 4.74$ is obtained.

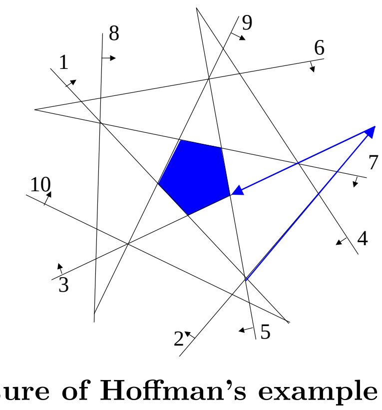


Figure 7 A picture of Hoffman's example with $0 < \omega < 4.74$