The optimal linear programming solution is $x = (\frac{20}{7}, 3, 0, 0, \frac{23}{7}) \notin Z^5_+$, so we use the first row, in which the basic variable x_1 is fractional, to generate the cut:

$$\tfrac{1}{7}x_3 + \tfrac{2}{7}x_4 \geq \tfrac{6}{7}$$

or

$$s = -\tfrac{6}{7} + \tfrac{1}{7}x_3 + \tfrac{2}{7}x_4$$

with $s, x_3, x_4 \geq 0$ and integer.

Adding this cut, and reoptimizing leads to the new optimal tableau

$$\begin{array}{lllllllll}
z = \max \frac{15}{2} & & & & & -\frac{1}{2}x_5 & -3s & & \\
 & x_1 & & & & & +s & = & 2 \\
 & & x_2 & & & -\frac{1}{2}x_5 & +s & = & \frac{1}{2} \\
 & & & x_3 & & -x_5 & -5s & = & 1 \\
 & & & & x_4 & +\frac{1}{2}x_5 & +6s & = & \frac{5}{2} \\
 & x_1, & x_2, & x_3, & x_4, & x_5, s & \geq & & 0 \text{ and integer.}
\end{array}$$

Now the new optimal linear programming solution $x = (2, \frac{1}{2}, 1, \frac{5}{2}, 0)$ is still not integer, as the original variable x_2, and the slack variable x_4 are fractional. The Gomory fractional cut on row 2, in which x_2 is basic, is $\frac{1}{2}x_5 \geq \frac{1}{2}$ or $-\frac{1}{2}x_5 + t = -\frac{1}{2}$ with $t \geq 0$ and integer. Adding this constraint and reoptimizing, we obtain

$$\begin{array}{llllllllll}
z = \max 7 & & & & & & -3s & -t & & \\
 & x_1 & & & & & +s & & = & 2 \\
 & & x_2 & & & & +s & -t & = & 1 \\
 & & & x_3 & & & -5s & -2t & = & 2 \\
 & & & & x_4 & & +6s & +t & = & 2 \\
 & & & & & x_5 & & -t & = & 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, s, & & t & \geq & 0 \text{ and integer.}
\end{array}$$

Now the linear programing solution is integral, and optimal, and thus $(x_1, x_2) = (2, 1)$ solves the original integer program. ■

It is natural to also look at the cuts in the space of the original variables.

Example 8.10 (cont) Considering the first cut, and substituting for x_3 and x_4 gives:

$$\tfrac{1}{7}(14 - 7x_1 + 2x_2) + \tfrac{2}{7}(3 - x_2) \geq \tfrac{6}{7}$$

or $x_1 \leq 2$.

In Figure 8.2 we can verify that this inequality is valid and cuts off the fractional solution $(\frac{20}{7}, 3)$. Similarly, substituting for x_5 in the second cut $\frac{1}{2}x_5 \geq \frac{1}{2}$ gives the valid inequality $x_1 - x_2 \leq 1$ in the original variables. ■

To find a general formula that gives us the cut in terms of the original variables, one can show:

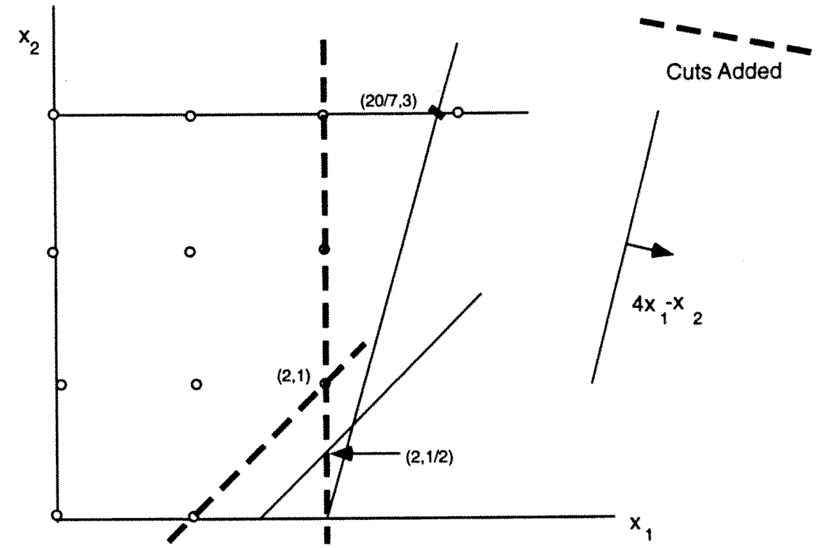

Fig. 8.2 Gomory cutting planes

Proposition 8.5 *Let β be row u of B^{-1}, and $q_i = \beta_i - \lfloor \beta_i \rfloor$ for $i = 1, \cdots, m$. The Gomory cut $\sum_{j \in NB} f_{uj}x_j \geq f_{u0}$, when written in terms of the original variables, is the Chvátal-Gomory inequality*

$$\sum_{j=1}^{n} \lfloor qa_j \rfloor x_j \leq \lfloor qb \rfloor.$$

Looking at the first Gomory cut generated in Example 8.10, β is given by the coefficients of the slack variables in row $u = 1$, so $\beta = (\frac{1}{7}, \frac{2}{7}, 0)$. Thus $q = (\frac{1}{7}, \frac{2}{7}, 0)$ and we obtain $1x_1 + 0x_2 \leq \lfloor \frac{20}{7} \rfloor = 2$.

8.7 MIXED INTEGER CUTS

8.7.1 The Basic Mixed Integer Inequality

We saw above that when $y \leq b, y \in Z^1$, the rounding inequality $y \leq \lfloor b \rfloor$ suffices to generate all the inequalities for a pure integer program. Here we examine if there is a similar *basic* inequality for mixed integer programs.

Proposition 8.6 *Let $X^{\geq} = \{(x, y) \in R^1_+ \times Z^1 : x + y \geq b\}$, and $f = b - \lfloor b \rfloor > 0$. The inequality*

$$x \geq f(\lceil b \rceil - y) \text{ or } \frac{x}{f} + y \geq \lceil b \rceil$$

is valid for $X^{\geq}$.

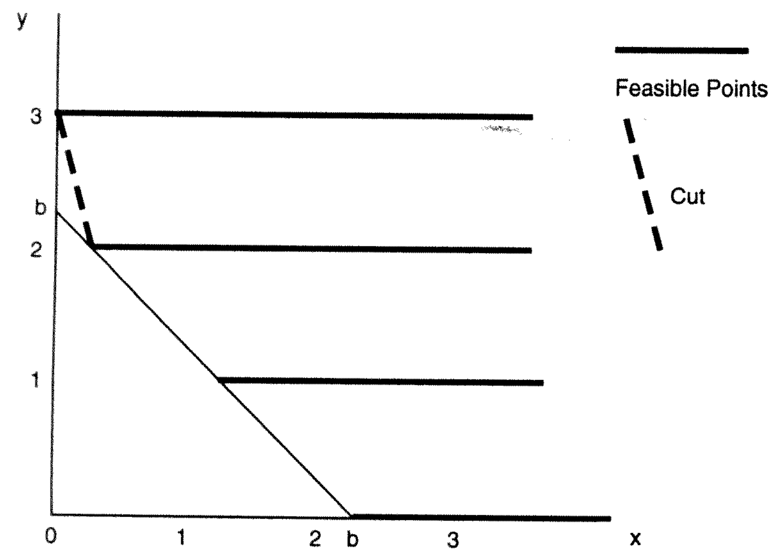

Fig. 8.3 Basic mixed inequality

Proof. If $y \geq \lceil b \rceil$, then $x \geq 0 \geq f(\lceil b \rceil - y)$. If $y < \lceil b \rceil$, then

$$\begin{aligned} x &\geq b - y = f + (\lfloor b \rfloor - y) \\ &\geq f + f(\lfloor b \rfloor - y), \text{ as } \lfloor b \rfloor - y \geq 0 \text{ and } f < 1, \\ &= f(\lceil b \rceil - y). \end{aligned}$$

■

The situation is shown in Figure 8.3. The following corollary allows us to compare more directly with the all-integer case.

Corollary If $X^{\leq} = \{(x, y) \in R^1_+ \times Z^1 : y \leq b + x\}$, and $f = b - \lfloor b \rfloor > 0$, the inequality

$$y \leq \lfloor b \rfloor + \frac{x}{1 - f}$$

is valid for $X^{\leq}$.

Proof. Rewriting $y \leq b + x$ as $x - y \geq -b$ and observing that $-b - \lfloor -b \rfloor = 1 - f$, we obtain from Proposition 8.6 that $\frac{x}{1-f} - y \geq \lceil -b \rceil = -\lfloor b \rfloor$. ■

Thus we see that when the continuous variable $x = 0$, we obtain the basic integer rounding inequality.

Example 8.6 (cont) The trucking example discussed earlier led to the set $2y_1 + 2y_2 + y_3 + y_4 + \frac{s}{11} \geq \frac{72}{11}$ with $y \in Z^4_+$ and $s \geq 0$. Using Proposition 8.6 with $\lceil b \rceil = 7$ and $f = \frac{6}{11}$, we obtain immediately that

$$\frac{s}{11} \geq \frac{6}{11}(7 - 2y_1 - 2y_2 - y_3 - y_4)$$

is a valid inequality. ■

8.7.2 The Mixed Integer Rounding (MIR) Inequality

To obtain a slight variant of the basic inequality, we consider a set

$$X^{MIR} = \{(x, y) \in R^1_+ \times Z^2_+ : a_1 y_1 + a_2 y_2 \leq b + x\},$$

where a_1, a_2 and b are scalars with $b \notin Z^1$.

Proposition 8.7 *Let $f = b - \lfloor b \rfloor$ and $f_i = a_i - \lfloor a_i \rfloor$ for $i = 1, 2$. Suppose $f_1 \leq f \leq f_2$, then*

$$\lfloor a_1 \rfloor y_1 + (\lfloor a_2 \rfloor + \frac{f_2 - f}{1 - f}) y_2 \leq \lfloor b \rfloor + \frac{x}{1 - f} \tag{8.12}$$

is valid for X^{MIR}.

Proof. $(x, y) \in X^{MIR}$ satisfies $\lfloor a_1 \rfloor y_1 + \lceil a_2 \rceil y_2 \leq b + x + (1 - f_2) y_2$ as $y_1 \geq 0$, and $a_2 = \lceil a_2 \rceil - (1 - f_2)$. Now the Corollary to Proposition 8.6 gives

$$\lfloor a_1 \rfloor y_1 + \lceil a_2 \rceil y_2 \leq \lfloor b \rfloor + [x + (1 - f_2) y_2]/(1 - f),$$

which is the required inequality. ■

Example 8.11 Consider the set $X = \{(y, x) \in Z^3_+ \times R^1_+ : \frac{10}{3} y_1 + 1 y_2 + \frac{11}{4} y_3 \leq \frac{21}{2} + x\}$. Using Proposition 8.7, we have $f = 1/2, f_1 = 1/3, f_2 = 0, f_3 = 3/4$, and thus

$$3y_1 + y_2 + \frac{5}{2} y_3 \leq 10 + 2x$$

is valid for X. ■

8.7.3 The Gomory Mixed Integer Cut*

Here we continue to consider mixed integer programs. As for all integer programs in Section 8.6, any row of the optimal linear programming tableau, in which an integer variable is basic but fractional, can be used to generate a cut removing the optimal linear programming solution. Specifically, such a row leads to a set of the form:

$$X^G = \{(y_{B_u}, y, x) \in Z^1 \times Z^{n_1}_+ \times R^{n_2}_+ : y_{B_u} + \sum_{j \in N_1} \overline{a}_{uj} y_j + \sum_{j \in N_2} \overline{a}_{uj} x_j = \overline{a}_{u0}\},$$

where $n_i = | N_i |$ for $i = 1, 2$.

Proposition 8.8 *If $\bar{a}_{u0} \notin Z^1$, $f_j = \bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor$ for $j \in N_1 \cup N_2$, and $f_0 = \bar{a}_{u0} - \lfloor \bar{a}_{u0} \rfloor$, the Gomory mixed integer cut*

$$\sum_{f_j \le f_0} f_j y_j + \sum_{f_j > f_0} \frac{f_0(1-f_j)}{1-f_0} y_j + \sum_{\bar{a}_{uj}>0} \bar{a}_{uj} x_j + \sum_{\bar{a}_{uj}<0} \frac{f_0}{1-f_0} \bar{a}_{uj} x_j \ge f_0$$

is valid for X^G.

Proof. The mixed integer rounding inequality (8.12) for X^G is

$$y_{B_u} + \sum_{f_j \le f_0} \lfloor \bar{a}_{uj} \rfloor y_j + \sum_{f_j > f_0} (\lfloor \bar{a}_{uj} \rfloor + \frac{f_j - f_0}{1-f_0}) y_j + \sum_{\bar{a}_{uj}<0} \frac{\bar{a}_{uj}}{1-f_0} x_j \le \lfloor \bar{a}_{u0} \rfloor.$$

Substituting for y_{B_u} proves the claim. ■

Example 8.12 Consider the mixed integer program:

$$\begin{array}{rrcrcl}
z = \max & 4x_1 & - & x_2 & & \\
 & 7x_1 & - & 2x_2 & \le & 14 \\
 & & & x_2 & \le & 3 \\
 & 2x_1 & - & 2x_2 & \le & 3 \\
 & x_1 & \in & Z^1_+, & x_2 & \ge 0.
\end{array}$$

Note that this is the same as Example 8.10 except that variable $x_2 \in R^1_+$ is a real variable. Solving as a linear program gives:

$$\begin{array}{lccccccc}
z = \max \frac{59}{7} & & & -\frac{4}{7}x_3 & -\frac{1}{7}x_4 & & & \\
 & x_1 & & +\frac{1}{7}x_3 & +\frac{2}{7}x_4 & & = & \frac{20}{7} \\
 & & x_2 & & +x_4 & & = & 3 \\
 & & & -\frac{2}{7}x_3 & +\frac{10}{7}x_4 & +x_5 & = & \frac{23}{7} \\
 & x_1 \in Z^1_+, & x_2, & x_3, & x_4, x_5 & & \ge & 0.
\end{array}$$

The basic variable x_1 is fractional and the first row gives the MIR cut $x_1 \le 2$, which after elimination of x_1 becomes the Gomory mixed integer cut:

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \ge \frac{6}{7}.$$

Adding this cut and reoptimizing leads to the solution $x = (2, \frac{1}{2})$, which is feasible and hence optimal for the mixed integer program. This can also be seen graphically in Figure 8.2 with just the addition of the cut $x_1 \le 2$. ■

8.8 DISJUNCTIVE INEQUALITIES*

The set $X = X^1 \cup X^2$ with $X^i \subseteq R^n_+$ for $i = 1, 2$ is a *disjunction* (union) of the two sets X^1 and X^2. The following simple result has already been used implicitly in Proposition 8.6 in deriving the basic mixed integer inequality.

Proposition 8.9 *If $\sum_{j=1}^n \pi^i_j x_j \le \pi^i_0$ is valid for X^i for $i = 1, 2$, then t inequality*

$$\sum_{j=1}^n \pi_j x_j \le \pi_0$$

is valid for X if $\pi_j \le \min[\pi^1_j, \pi^2_j]$ for $j = 1, \ldots, n$ and $\pi_0 \ge \max[\pi^1_0, \pi^2_0]$.

Proof. If $x \in X$, then $x \in X^1$ or $x \in X^2$. If $x \in X^i$, then as $x \ge$ $\sum_{j=1}^n \pi_j x_j \le \sum_{j=1}^n \pi^i_j x_j \le \pi^i_0 \le \pi_0$ for $i = 1, 2$. Thus the inequality is vali for all $x \in X$.

Disjunctions of polyhedra are particularly interesting. Modeling such se is easy; see Exercises 1.3 and 8.10. Using Proposition 8.2, it is also easy t characterize valid inequalities for such disjunctions.

Proposition 8.10 *If $P^i = \{x \in R^n_+ : A^i x \le b^i\}$ for $i = 1, 2$ are nonemp polyhedra, then (π, π_0) is a valid inequality for $conv(P^1 \cup P^2)$ if and only there exist $u^1, u^2 \ge 0$ such that $\pi \le u^i A^i$ and $\pi_0 \ge u^i b^i$ for $i = 1, 2$.*

Example 8.13 Let $P^1 = \{x \in R^2_+ : -x_1 + x_2 \le 1, x_1 + x_2 \le 5\}$ and $P^2 = \{x \in R^2_+ : x_2 \le 4, -2x_1 + x_2 \le -6, x_1 - 3x_2 \le -2\}$. Tak ing $u^1 = (2, 1)$ and $u^2 = (\frac{5}{2}, \frac{1}{2}, 0)$ and applying Proposition 8.10 gives tha $-x_1 + 3x_2 \le 7$ is valid for $P^1 \cup P^2$. See Figure 8.4.

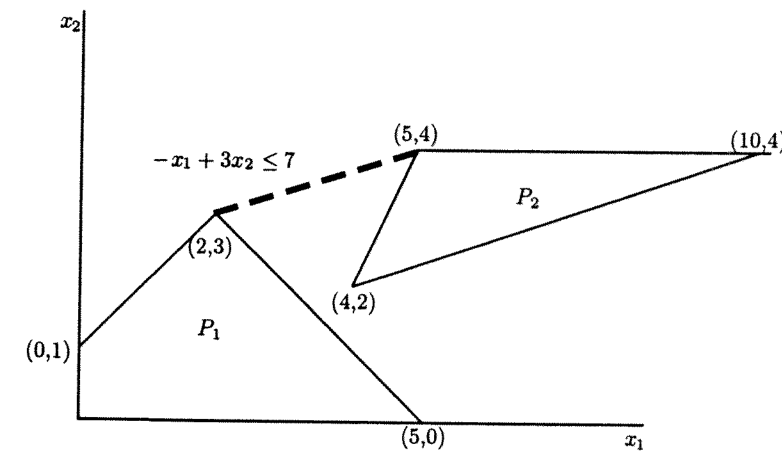

Fig. 8.4 Disjunctive inequality

Specializing further, we consider 0–1 problems, where $X = P \cap Z^n \subseteq \{0,1\}^n$ with $P = \{x \in R^n : Ax \le b, \mathbf{0} \le x \le \mathbf{1}\}$. Let $P^0 = P \cap \{x \in R^n : x_j = 0\}$ and $P^1 = P \cap \{x \in R^n : x_j = 1\}$ for some $j \in \{1, \ldots, n\}$.

Proposition 8.11 *The inequality* (π, π_0) *is a valid for* $conv(P^0 \cup P^1)$ *if there exists* $u^i \in R^m_+, v^i \in R^n_+,\ w^i \in R^1_+$ *for* $i = 0, 1$ *such that*

$$\pi \le u^0 A + v^0 + w^0 e_j, \pi \le u^1 A + v^1 - w^1 e_j,$$
$$\pi_0 \ge u^0 b + \mathbf{1}.v^0, \pi_0 \ge u^1 b + \mathbf{1}.v^1 - w^1.$$

Proof. Apply Proposition 8.10 with $P^0 = \{x \in R^n_+ : Ax \le b, x \le \mathbf{1}, x_j \le 0\}$ and $P^1 = \{x \in R^n_+ : Ax \le b, x \le \mathbf{1}, -x_j \le -1\}$. ∎

Example 8.14 Consider the 0–1 knapsack problem

$$\begin{aligned} \max 12x_1 + 14x_2 + 7x_3 + 12x_4 \\ 4x_1 + 5x_2 + 3x_3 + 6x_4 \le 8 \\ x \in B^4 \end{aligned}$$

with linear programming solution $x^* = (1, 0.8, 0, 0)$.

As $x^*_2 = 0.8$ is fractional, we choose $j = 2$ in defining P^0 and P^1, and then look for the most violated valid inequality (π, π_0) given by Proposition 8.11. To do this, we solve a linear program consisting of maximizing $\pi x^* - \pi_0$ over the polyhedron describing the coefficients of the valid inequalities given in the proposition, namely

$$\begin{aligned} \max 1.0\pi_1 + 0.8\pi_2 - \pi_0 \\ \pi_1 \le 4u^0 + v^0_1, \pi_1 \le 4u^1 + v^1_1 \\ \pi_2 \le 5u^0 + v^0_2 + w^0, \pi_2 \le 5u^1 + v^1_2 - w^1 \\ \pi_3 \le 3u^0 + v^0_3, \pi_3 \le 3u^1 + v^1_3 \\ \pi_4 \le 6u^0 + v^0_4, \pi_4 \le 6u^1 + v^1_4 \\ \pi_0 \ge 8u^0 + v^0_1 + v^0_2 + v^0_3 + v^0_4 \\ \pi_0 \ge 8u^1 + v^1_1 + v^1_2 + v^1_3 + v^1_4 - w^1 \\ u^0, u^1, v^0, v^1, w^0, w^1 \ge 0. \end{aligned}$$

Note that for the linear program to have a bounded optimal value, it is necessary to normalize the inequality. Two possibilities are $\sum_{j=1}^n \pi_j \le 1$ or $\pi_0 = 1$. The resulting inequality is

$$1x_1 + \tfrac{1}{4}x_2 \le 1,$$

with violation of $\frac{1}{5}$. For P^0, it is a combination of constraints $x_1 \le 1$ and $x_2 \le 0$ with $v^0_1 = 1$ and $w^0 = \frac{1}{4}$ respectively. For P^1, it is a combination of the knapsack inequality $4x_1 + 5x_2 + 3x_3 + 6x_4 \le 8$ and $-x_2 \le -1$ with $u^1 = \frac{1}{4}$ and $w^1 = 1$ respectively. Both normalizations lead to the same inequality. ∎

The idea of looking for the most violated inequality will be pursued in the next chapter, when we try to obtain "strong" inequalities.

8.9 NOTES

8.2 The inequality (8.4), called a *blossom inequality*, is from [Edm65a].

8.3 The rounding procedure to generate cuts is from [Gom58]. The general procedure described here and the proof of Theorem 8.4 for bounded integer programs is from [Chv73]. In [Sch80], the result is extended to unbounded polyhedra.

8.4 The strong formulation for UFL is used computationally in [Spi69].

8.5 The first cutting plane algorithm reported is the procedure used to solve a 54-city TSP in [DanFulJoh54].

8.6 The fractional cutting plane algorithm is presented in [Gom58],[Gom63]. The latter paper also contains a beautiful theoretical result, namely that the algorithm converges finitely if the rows off which the cuts are generated are properly chosen.

8.7 Gomory mixed integer cuts are proposed in [Gom60]. The presentation of mixed integer rounding inequalities is from [NemWol90]. The theory of superadditive valid inequalities and superadditive duality [Joh80] and Chapter II.1 in [NemWol88] provides a complete explanation of cuts for integer and mixed integer programs.

Gomory has shown that finite convergence can be attained with his mixed integer cuts if the objective function is integer valued. It is an open question whether this is true for 0–1 mixed integer programs with an arbitrary objective function. In [CooKanSch90], the question of how finite convergence might be obtained is reexamined.

Recently [GunPoc98] present a new way to combine basic mixed integer inequalities. Gomory mixed integer cuts have also been recently revived as a computational tool; see [Balasetal96].

8.8 Disjunctive and Gomory mixed integer cuts are closely related. Proposition 8.9 was already used implicitly by Gomory in developing the mixed integer cut. In the same way that the Chvátal-Gomory procedure can be used to generate all valid inequalities for an integer program, it can be shown that a simple disjunctive procedure repeated finitely (see Exercise 8.11) can be used to generate all valid inequalities for a 0–1 mixed integer program.

The approach here is based on the disjunction of polyhedra developed by Balas [Balas75a] in the 1970s; see also [Jer72]. In particular, Balas shows the beautiful result that to obtain the convex hull of a 0–1 MIP, it suffices to take the convex hulls of each 0–1 variable one at a time. Related to this result, a variety of extended formulations have recently been proposed to obtain tighter formulations for 0–1 MIPs [LovSch91], [BalasCerCor93], [SheAda90].

In practice, under the name *lift and project*, Proposition 8.11 is used as in Example 8.14 to develop a disjunctive cutting plane algorithm. Computational results are given in [BalasCerCor96].

8.10 EXERCISES

1. For each of the three sets below, find a missing valid inequality and verify graphically that its addition to the formulation gives conv(X).

(i) $X = \{x \in B^2 : 3x_1 - 4x_2 \leq 1\}$
(ii) $X = \{(x, y) \in R^1_+ \times B^1 : x \leq 20y, x \leq 7\}$
(iii) $X = \{(x, y) \in R^1_+ \times Z^1_+ : x \leq 6y, x \leq 16\}$

2. In each of the examples below a set X and a point x or (x, y) are given. Find a valid inequality for X cutting off the point.

(i)

$$X = \{(x, y) \in R^2_+ \times B^1 : x_1 + x_2 \leq 2y, x_j \leq 1 \text{ for } j = 1, 2\}$$

$$(x_1, x_2, y) = (1, 0, 0.5)$$

(ii)

$$X = \{(x, y) \in R^1_+ \times Z^1_+ : x \leq 9, x \leq 4y\}$$

$$(x, y) = (9, \frac{9}{4})$$

(iii)

$$X = \{(x_1, x_2, y) \in R^2_+ \times Z^1_+ : x_1 + x_2 \leq 25, x_1 + x_2 \leq 8y\}$$

$$(x_1, x_2, y) = (20, 5, \frac{25}{8})$$

(iv)

$$X = \{x \in Z^5_+ : 9x_1 + 12x_2 + 8x_3 + 17x_4 + 13x_5 \geq 50\}$$

$$x = (0, \frac{25}{6}, 0, 0, 0)$$

(v)

$$X = \{x \in Z^4_+ : 4x_1 + 8x_2 + 7x_3 + 5x_4 \leq 33\}$$

$$x = (0, 0, \frac{33}{7}, 0).$$

3. Prove that $y_2 + y_3 + 2y_4 \leq 6$ is valid for

$$X = \{y \in Z^4_+ : 4y_1 + 5y_2 + 9y_3 + 12y_4 \leq 34\}.$$

4. Consider the problem

$$\begin{aligned} \min x_1 + 2x_2 \\ x_1 + x_2 \geq 4 \\ \tfrac{1}{2}x_1 + \tfrac{5}{2}x_2 \geq \tfrac{5}{2} \\ x \in Z^2_+. \end{aligned}$$

Show that $x^* = (\frac{15}{4}, \frac{1}{4})$ is the optimal linear programming solution and find an inequality cutting off x^*.

5. Solve $\min\{5x_1 + 9x_2 + 23x_3 : 20x_1 + 35x_2 + 95x_3 \geq 319, x \in Z^3_+\}$ using Chvátal-Gomory inequalities or Gomory's cutting plane algorithm.

6. Solve $\max\{5x_1 + 9x_2 + 23x_3 - 4s : 2x_1 + 3x_2 + 9x_3 \leq 32 + s, x \in Z^3_+, s \in R^1_+\}$ using MIR inequalities.

7. (i) Show that the inequality $x_t \leq d_t y_t + s_t$ is valid for ULS.
(ii) Show that $x_t + x_{t+1} \leq (d_t + d_{t+1})y_t + d_{t+1}y_{t+1} + s_{t+1}$ is valid.
(iii) For $l \leq n$, $L = \{1, \ldots, l\}$ and $S \subseteq L$, show that the inequality

$$\sum_{j \in S} x_j \leq \sum_{j \in S} (\sum_{t=j}^{l} d_t) y_j + s_l$$

is valid for ULS.

8. Consider the stable set problem. An *odd hole* is a cycle with an odd number of nodes and no edges between nonadjacent nodes of the cycle. Show that if H is the node set of an odd hole,

$$\sum_{j \in H} x_j \leq (| H | -1)/2$$

is a valid inequality.

9. Use the mixed integer rounding procedure to show that

$$(y_1 + 6y_2)/4 + y_3 + 4y_4 \geq 16$$

is a valid inequality for

$$X = \{y \in Z^4_+ : y_1 + 6y_2 + 12y_3 + 48y_4 \geq 184\}.$$

10. Use the mixed integer rounding procedure to show that

$$x_2 + x_4 \leq 20 + 4(y - 2)$$

is a valid inequality for $X =$

$$\{(x, y) \in R^4_+ \times Z^1_+ : x_1 + x_2 + x_3 + x_4 \leq 10y, x_1 \leq 13, x_2 \leq 15, x_3 \leq 6, x_4 \leq 9\}.$$