Computational Intractability	2010/4/22
Lecture 3	
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1 Linear Programming & Simplex Method

solve Maximize $5x_1 + 6x_2 + 9x_3$ where,

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 5 \\ x_1 + x_2 + 2x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \tag{1}$$

We know how to solve equations, but not inequalities.

1.1 Simplex Method

1.1.1 Step 1

1. Convert to equations.
2. Rewrite by putting everything in the form $\cdots \geq 0$.
3. The new variables are slack variables.
4. Original variables are decision variables.
5. Variable on LHS of dictionary are called basic, on RHS of dictionary are called co-basic or non-basic.

$$\begin{aligned} x_4 &= 5 - x_1 - 2x_2 - 3x_3 \geq 0 \\ x_5 &= 3 - x_1 - x_2 - 2x_3 \geq 0 \\ z &= 5x_1 + 6x_2 + 9x_3 \end{aligned} \tag{2}$$

where, $x_1, x_2, x_3, x_4, x_5 \geq 0$.

- **remarks**

1. x_1, x_2, x_3 solves (1) iff x_1, x_2, x_3, x_4, x_5 solves (2) where x_4, x_5 defined by (2)
2. I.e : Forget about (1) and work with (2)

$$x_1 = 1, x_2 = 0, x_3 = 1 \text{ solves } (1)$$
$$\Rightarrow x_4 = 1, x_5 = 0$$

3. A dictionary represent all solutions, but we will choose exactly one called a basic solution. We get it by setting all RHS variable to zero.

   If all variables are non-negative (ie. the b-vector is non-negative) the dictionary is called feasible and we have a basic feasible solution. Roughly speaking, in the simplex method we start with a basic feasible solution and improve it until no improvements are possible. This is described in Step 2.

current basic feasible solution : $x_1 = x_2 = x_3 = 0, x_4 = 5, x_5 = 3, z = 0$

$$x_4, x_5 : \text{basic, } x_1, x_2, x_3 : \text{non-basic}$$

1.1.2 Step 2

1. Improve the current solution.
2. choose any non-basic variable x_j with positive coefficient in z-row (if none of terminate optimal).

   Choose x_1. How much can we increase x_1, when $x_2, x_3 = 0$

   $$\begin{aligned} x_4 &= 5 - x_1 \\ x_5 &= 3 - x_1 \end{aligned}$$

   The max value of $x_1 = 3$ or $x_5 < 0$

   $x_j = x_1 = 0 \Rightarrow$ increase to 3
   $x_i = x_1 = 3 \Rightarrow$ decrease to zero

3. Increase x_j until some basic variable x_i become zero.
4. Let x_i enter basis and x_j enter non-basis by solving the equations.
5. Eliminate x_i from RHS.

$$\begin{aligned} x_4 &= 2 - x_2 - x_3 + x_5 \\ x_1 &= 3 - x_2 - 2x_3 - x_5 \\ z &= 15 + x_2 - x_3 - 5x_5 \end{aligned} \tag{3}$$

current basic solution : $x_2 = x_3 = x_5 = 0, x_1 = 3, x_4 = 2, z = 15$

$$x_1, x_4 : \text{basic, } x_2, x_3, x_5 : \text{non-basic}$$

Choose $x_j = x_2$, Fix $x_3 = x_5 = 0$

$$\begin{aligned} x_4 &= 2 - x_2 \\ x_1 &= 3 - x_2 \end{aligned}$$

Increasing x_2 to 2 will lead to x_4 decrease to zero.
x_4 leaves basis and x_2 enters.

$$\begin{aligned} x_4 &= 2 - x_2 - x_3 + x_5 \\ x_1 &= 3 - x_2 - 2x_3 + x_5 \\ z &= 15 + x_2 - x_3 - 5x_5 \end{aligned}$$

$$\begin{aligned} x_2 &= 2 - x_3 - x_4 + x_5 \\ x_1 &= 3 - x_2 - 2x_3 - x_5 \\ &= 1 - x_3 - x_4 - 2x_5 \\ z &= 17 - 2x_3 - x_4 - 4x_5 \end{aligned} \tag{4}$$

current solution : $x_3 = x_4 = x_5 = 0, x_1 = 1, x_2 = 2, z = 17$

Claim On termination we have an optimum solution.

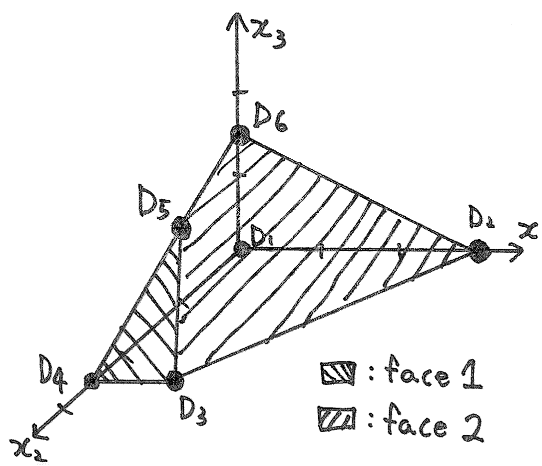
why feasible? It is feasible for the last dictionary, which is equivalent to initial dictionary (2), Which is equivalent to the original LP (1). However, we can also directly check feasibility by substituting into (1) above.

why optimum? Consider any other feasible solution to the last dictionary (4).
I.e. at least one of non-basic variable must be larger than zero.
In this case, $z < 17$ because all co-efficients are negative in z-row.

$$x_3 = 1 \Rightarrow z = 15 \text{ (not optimum solution)}$$
$$x_4 = 2 \Rightarrow z = 15 \text{ (not optimum solution)}$$

1.2 Geometry

The solution set to (1) is the polyhedron sketched below. It has 5 faces of dimension 2:



$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 5 \\
x_1 + x_2 + 2x_3 &= 3 \\
x_1 &= 0 \\
x_2 &= 0 \\
x_3 &= 0
\end{aligned}
$$

note : the maximum dimension faces are often called facets

$D_1 : (0,0,0)$ Origin
$D_2 : (3,0,0)$ Hit the face 2
$D_3 : (1,2,0)$ Hit the face 1 and 2
$D_4 : (0,\frac{5}{2},0)$ Hit the face 1
$D_5 : (0,1,1)$ Hit the face 1 and 2
$D_6 : (0,0,\frac{3}{2})$ Hit the face 2

Each dictionary represents a vertex, which is given by a basic feasible solution for some dictionary. The non-basic variables tell us which faces we are on. So each vertex is the intersection of 3 faces.

Our path was $D_1 \Rightarrow D_2 \Rightarrow D_3$. And there are another paths that the simplex method could follow: $D_1 \Rightarrow D_4 \Rightarrow D_3$ and $D_1 \Rightarrow D_6 \Rightarrow D_5 \Rightarrow D_3$ for example.

Geometric interpretation of Simplex method

Start at any vertex v of the polyhedron P. Choose any outgoing edge from v that increases z. Stop when no such edge exists.

- **remarks**

1. Basic feasible solutions to the LP give vertices of P. Each vertex of P gives a basic feasible solution for some dictionary. When you reach a face a slack variable or decision variable reaches zero. In this example we could have increased x_2 In this case we stop when $x_1 + 2x_2 + 3x_3 = 5$, i.e. $x_4 = 0$
2. Dantzig invented simplex method during WW2, but it was not published until 1947-48.
3. Work by L. V. Kantorovich (1939) and T. C. Koopmans (1942-1943) lead up to Dantzig's invention. They received the nobel prize in economics in 1975.
4. Many ideas were anticipated even earlier in work by Fourier in short papers published in 1826-7.

References

[1] V.Chvatal *Linear Programming* (W.H.FREEMAN AND COMPANY, New York, 1983)