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An $O(n \log n)$ procedure for identifying facets of the knapsack polytope☆

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Abstract

An $O(n \log n)$ procedure is presented for obtaining facets of the knapsack polytope by lifting the inequalities induced by the extensions of strong minimal covers. The procedure does not require any sequential lifting of the inequalities. In contrast, it utilizes the information from the maximal cliques implied by the knapsack constraint for determining the combination of the lifting coefficients to generate each facet.

Keywords: Knapsack polytope; Facet-defining inequalities; Strong minimal covers; Tightening procedures

1. Introduction and preliminaries

Consider the 0–1 knapsack inequality

$$\sum_{j \in N} a_j x_j \leqslant a_0, \tag{1}$$

where $a_j \leqslant a_0$, such that $a_0, a_j, \in Z^+$ and $x_j \in \{0,1\}$, $j \in N = \{1,\dots,n\}$. The knapsack polytope P given by (1) is the convex hull of the 0–1 points satisfying (1):

$$P = conv\left\{x_j \in \{0,1\}, \sum_{j \in N} a_j x_j \leqslant a_0\right\}.$$

An inequality is said to be a *facet* of P if it is satisfied by every $x \in P$, and it is satisfied with strict equality by exactly n affinely independent points $x \in P$ (see e.g. [11]).

A set $S \subseteq N$ is called a *cover* implied by the knapsack constraint (1), or equivalently by P, provided that

$$\sum_{j \in S} a_j > a_0.$$

The cover S is called a *minimal cover* provided that

$$\sum_{j \in S-\{i\}} a_j \leqslant a_0 \quad \forall i \in S.$$

It is well known that the inequality

$$\sum_{j \in S} x_j \leqslant k \tag{2}$$

is induced by the minimal cover S, where $k = |S| - 1$ and, then, it is a valid inequality for any solution that satisfies (1).

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The set $E(S) = S \cup S'$ from the minimal cover S, where $S' = \{j \in N - S : a_j \geqslant a_i,\ \forall i \in S\}$, is called the *extension* of S to N. Note that $\sum_{j\in E(S)} x_j \leqslant k$ is a valid inequality for the feasible set of (1) (see [11], among others).

It is also well known that a clique-induced inequality is a cover with $k = 1$. A trivial clique C is a clique with $|C| = 1$. A clique from a given set is called a *maximal clique* if it is not dominated by any other member of the set.

A minimal cover S is called a *strong minimal cover*, if either $E(S) = N$ or

$$\sum_{j\in S-\{j1\}} a_j + a_{i1} \leqslant a_0, \tag{3}$$

where $j1 = \mathrm{argmax}\{a_j : j \in S\}$ and $i1 = \mathrm{argmax}\{a_j : j \in N - E(S)\}$.

Consider the following proposition for a useful characterization of strong minimal covers.

Proposition 1.1. *A minimal cover S is a strong minimal cover if there exists no other minimal cover with the same size as S, whose extension strictly contains the extension $E(S)$.*

Proof. Trivial. □

In this note we are interested in the family of facets obtained from strong minimal covers. The existence, properties and characterizations of such valid inequalities have been studied in great detail during the last 30 years (see [1–3,6–10,12–16], among others).

Throughout the note, we will assume that the set N to be considered is ordered such that $i < j \Rightarrow a_i \geqslant a_j$. Let S_h denote the set of the first h elements of S.

It is well known that facets and other valid inequalities of lower-dimensional polytopes can be lifted to generate facet-defining inequalities of greater-dimensional polytopes. When the lifting is applied to the minimal cover (2), the following type of tighter inequality can be generated (the so-called *Lifted Cover Inequality* (LCI)):

$$\sum_{j\in S} x_j + \sum_{j\in N-S} \beta_j x_j \leqslant |S| - 1, \tag{4}$$

where $\beta_j > 0$ are the so-called *lifting coefficients*.

The coefficients of a facet-defining inequality that are obtained one by one depend upon the sequence in which they are computed. In general, there can be an exponential number of sequences yielding distinct facets of P. The facets obtained by such type of procedures are called *sequentially lifted facets* (see [12,13]).

Balas [1] and Balas and Zemel [3] characterize the lifting coefficients $\beta_j\ \forall j \in N - S$ for the sequentially lifted facets (4) by using explicit expressions. However, these are still a subset of indices in $N - S$ whose coefficients are not determined in a precise way, but two potential values are considered.

In this note we present an $O(n \log n)$ procedure for identifying the set of facets derived from strong minimal covers, such that the lifting coefficients have integer values and do not depend upon the ordering of the indices in $N - S$. Our approach takes benefit from the B–Z results. However, the information from the maximal cliques implied by the given knapsack constraint is used for structuring the LCIs.

The rest of the note is organized as follows. Section 2 presents the computational procedure for obtaining tight inequalities. Section 3 analyzes the relation between the new inequalities and the sequentially lifted facets. Finally, Section 4 draws some conclusions from the work.

2. Obtaining tight inequalities

See [5] for two procedures with complexity $O(n \log n)$ for lifting cover inequalities in 0–1 programs, called efficient cover inequality tightening (ECIT) and probing-based cover inequality tightening (PCIT).

In [1] it was proved that the maximal clique inequalities are facet defining of P. See [4] for an $O(n)$ procedure for identifying all such inequalities implied by a given knapsack constraint. Furthermore, all the covers to be obtained as the non-dominated extensions of alternate sets of minimal covers with consecutive coefficients can be identified with $O(n)$ complexity as well (see also [4]). Given the characterization of a strong minimal cover in Proposition 1.1, and the construction of the procedure introduced in [4] for identifying the above class of minimal covers, it is easy to see that they are strong minimal covers.

Let the induced inequality of a minimal cover be S,

$$\sum_{j\in E(S)} x_j \leqslant |S|-1, \qquad |S|\geqslant 3. \tag{5}$$

The basic scheme of the procedures ECIT and PCIT for lifting cover inequalities, consists of increasing the coefficients of the variables in the inequalities while keeping the right-hand side (rhs) fixed. The problem to solve for increasing the coefficient c_i, $i\in N$, in (5) can be written as

$$z_i = \max \sum_{j\in N} c_j x_j,$$

s.t.

$$\sum_{j\in N} a_j x_j \leqslant a_0,$$

$$x_i = 1, \quad x_j\in\{0,1\}, \quad \forall j\in N-\{i\},$$

$$c_j = 1, \quad j\in E(S), \quad c_j = 0, \quad j\in N-E(S). \tag{6}$$

If $z_i < |S|-1$, the lifting coefficient, say $\bar{c}_i$, can be expressed as

$$\bar{c}_i = c_i + |S| - 1 - z_i. \tag{7}$$

The new inequality

$$\sum_{j\in N} \bar{c}_j x_j \leqslant |S|-1 \tag{8}$$

is obtained, where $\bar{c}_j = c_j$ if $j\neq i$ and $\bar{c}_i$ takes the value given in (7). In this way, n new inequalities with lifting coefficients can be obtained from (5). The new system of inequalities is 0–1 equivalent and tighter than the original one, provided that there exists $i\in N$ such that $\bar{c}_i > c_i$.

However, it is well known, see [1,2], among others, that if S is a minimal cover implied by (1), then for each inequality (8) $\bar{c}_i = 1$, if $i\in S$. Additionally, see [5], if there exists some clique $C\subseteq N$ implied by the original 0–1 program, such that $H\subseteq C$ for $|H|\geqslant 2$, where $H=\{i\in N : \bar{c}_i > c_i\}$, the system of the $|H|$ inequalities (8) corresponding to each individual coefficient increment is 0–1 equivalent and dominated by the inequality

$$\sum_{i\in H} \bar{c}_i x_i + \sum_{i\in N-H} c_i x_i \leqslant |S|-1.$$

Next consider the following result.

Proposition 2.1. *If S is a strong minimal cover implied by* (1), *then for each inequality* (8), $\bar{c}_i = 0, i\in N-E(S)$.

Proof. Let i_1 be the first index in $N-E(S)$ (i.e., $i1 = \operatorname{argmax}\{a_j : j\in N-E(S)\}$). Similar to the above, for increasing the coefficient c_{i1}, consider the following problem to solve:

$$z_{i1} = \max \sum_{j\in E(S)} x_j,$$

s.t.

$$\sum_{j\in N} a_j x_j \leqslant a_0,$$

$$x_{i1} = 1, \quad x_j\in\{0,1\}, \quad \forall j\in N-\{i_1\}. \tag{9}$$

It is easy to see that the optimal solution of (9) is as follows:

$$x_{i1}=1, \quad x_j = \begin{cases} 1, & j=|S|,|S|-1,\ldots,|S|-\alpha+1, \\ 0, & j=1,\ldots,|S|-\alpha,\ j\neq i1, \end{cases}$$

where α is the first integer from 1 to $|S|$ such that $A_{\alpha-1}\leqslant a_0 - a_{i1} < A_\alpha$ or equivalently $A_{\alpha-1} + a_{i1} \leqslant a_0 < A_\alpha + a_{i1}$, where A_α is the sum of the αth smallest knapsack coefficients with indices in S. The optimal objective function value in (9) is $z_{i1}=\alpha-1$. Let $j1\in S$ as above. So, if $\alpha<|S|$, then there exists a minimal cover of the same size as S, say $S'=S-\{j_1\}\cup\{i_1\}$ (note that S' has consecutive coefficients, if S has consecutive coefficients), such that $E(S)\subset E(S')$, which is a contradiction, since by hypothesis, S is a strong minimal cover. (Note that S' is a minimal cover since $\sum_{j\in S'-\{i\}} a_j \leqslant \sum_{j\in S-\{j1\}} a_j \leqslant a_0$ for $i\in S'$ given S is a minimal cover, and $\sum_{j\in S'} a_j > a_0$, given $A_\alpha + a_{i1} > a_0$ and assuming $\alpha<|S|$). Then $z_{i1}=|S|-1$ and $\bar{c}_{i1}=c_{i1}=0$. If $k>i_1$, $k\in N-E(S)$, $z_k\geqslant z_{i1}$, and thus, $\bar{c}_k = c_k = 0\ \forall k\in N-E(S)$. $\square$

Proposition 2.1 has important consequences for the computational effort required to obtain the facets of the knapsack polytope derived from a strong minimal cover. As a result, the new inequalities to generate can be expressed as

$$\sum_{i\in S} x_i + \sum_{i\in E(S)-S} \beta_i x_i \leqslant |S|-1, \tag{10}$$

where the coefficients $\beta_i\in Z^+$ can be determined in function of $\bar{c}_i$, given by (7) see the next section. It is also shown that these inequalities are facets of P.

3. Obtaining facet-defining inequalities

Next we present an $O(n \log n)$ procedure for obtaining the LCIs (10), which define facets of P. According to the concepts and notation introduced in [1], see also [3], if S is a minimal cover for (1), $E(S)$ its extension to N, and S_h the set of the first h elements in S, $h=1,\dots,|S|$, then the set N can be partitioned into the subsets $N_0, N_1, \dots, N_q$, $q=|S|-1$, where

$$N_0 = N - E(S), \quad N_1 = E(S) - \bigcup_{h=2}^{q} N_h,$$

$$N_h = \left\{ i \in E(S) : \sum_{j \in S_h} a_j \leqslant a_i < \sum_{j \in S_{h+1}} a_j \right\}, \quad h = 2,\dots,q. \tag{11}$$

Let also the coefficients π_j, $j \in N$, and the sets $I, J \subseteq N - S$, $N - S = I \cup J$, such that

$$\pi_j = h \quad \forall j \in N_h, \quad h = 0,1,\dots,q, \tag{12}$$

$$I = \left\{ i \in N - S : \sum_{j \in S - S_{\pi_i+1}} a_j \leqslant a_0 - a_i \right\},$$

$$J = (N - S) - I. \tag{13}$$

Remark 3.1. Since $\pi_i = 0$ for $i \in N_0 = N - E(S)$, it results that $N - E(S) \subseteq I$, provided that S is a strong minimal cover (see [2]).

Remark 3.2. $N - S = I$ (i.e., $J = \emptyset$) implies that S is a strong minimal cover (see [2]).

Moreover, the next proposition shows the relationship between the coefficients $\bar{c}_i$ (7) and π_i (12), $i \in N - S$.

Proposition 3.1. *If S is a minimal cover implied by* (1) *then*

$$\bar{c}_i = \begin{cases} \pi_i, & i \in I, \\ \pi_i + 1, & i \in J. \end{cases} \tag{14}$$

Proof. (1) *Case for $i \in I$*: From the definition of π_i, we have

$$\sum_{j \in S_{\pi_i}} a_j \leqslant a_i$$

and since S is a cover implied by (1), it results

$$a_0 < \sum_{j \in S} a_j = \sum_{j \in S_{\pi_i}} a_j + \sum_{j \in S - S_{\pi_i}} a_j \leqslant a_i + \sum_{j \in S - S_{\pi_i}} a_j$$

and

$$\sum_{j \in S - S_{\pi_i}} a_j > a_0 - a_i.$$

Furthermore, since $i \in I$, the following condition is satisfied:

$$\sum_{j \in S - S_{\pi_i+1}} a_j \leqslant a_0 - a_i < \sum_{j \in S - S_{\pi_i}} a_j,$$

i.e.,

$$A_{|S|-(\pi_i+1)} \leqslant a_0 - a_i < A_{|S|-\pi_i}.$$

Since $\alpha = |S| - \pi_i$ (see the proof of Proposition 2.1) the solution value of (6) can be written as

$$z_i = \begin{cases} |S| - \pi_i & \text{for } i \in I \cap (E(S) - S), \\ |S| - \pi_i - 1 & \text{for } i \in I \cap (N - E(S)), \end{cases}$$

and, in any case, from the definition of $\bar{c}_i$ (7), it results $\bar{c}_i = \pi_i$ since $c_i = 1$ for $i \in E(S) - S$ and $c_i = 0$ for $i \in N - E(S)$.

(2) *Case for $i \in J$*: From the definition of π_i, we have

$$\sum_{j \in S_{\pi_i+1}} a_j > a_i$$

and since S is a minimal cover implied by (1), it results

$$a_i + \sum_{S - S_{\pi_i+2}} a_j < \sum_{j \in S_{\pi_i+1}} a_j + \sum_{S - S_{\pi_i+2}} a_j \leqslant a_0,$$

and, then,

$$\sum_{S - S_{\pi_i+2}} a_j < a_0 - a_i. \tag{15}$$

Furthermore, since $i \in J$, the following condition is satisfied:

$$a_0 - a_i < \sum_{j \in S - S_{\pi_i+1}} a_j. \tag{16}$$

From (15) and (16), we have $\alpha = |S| - \pi_i - 1$ and, so,

$$z_i = \begin{cases} |S| - \pi_i - 1 & \text{for } i \in J \cap (E(S) - S), \\ |S| - \pi_i - 2 & \text{for } i \in J \cap (N - E(S)), \end{cases}$$

and, in any case, from the definition of $\bar{c}_i$ (7), it results $\bar{c}_i = \pi_i + 1$ since $c_i = 1$ for $i \in E(S) - S$ and $c_i = 0$ for $i \in N - E(S)$. $\square$

Consider the following partial result from Theorem 3 in [3].

Proposition 3.2. *For any sequentially lifted facet* (4) *of P*,

$$\beta_i = \begin{cases} \pi_i, & i \in I, \\ \pi_i \text{ or } \pi_i + 1, & i \in J. \end{cases}$$

Proof. See [3]. □

In addition, as it is also stated in [3], given a sequentially lifted facet F of P obtained from a minimal cover S, the set of sequentially lifted facets that can be obtained from S requires changing the positions in F of the indices j from J, such that $\beta_j = \pi_j + 1$.

Theorem 3.1. *If S is a minimal cover implied by* (1) *and* $N - S = I$, $J = \emptyset$, *then the inequality*

$$\sum_{i \in S} x_i + \sum_{i \in N-S} \pi_i x_i \leqslant |S| - 1,$$

where the coefficients π_i, $i \in N - S$ *are given by* (12), *is the unique facet of P which can be obtained from S. Furthermore, it is the unique facet of P having coefficients equal to 1 for all* $j \in S$ *and a rhs of* $|S| - 1$.

Proof. See [3]. □

Corollary 3.1. *As it has been shown in Proposition* 3.1, $\bar{c}_i = \pi_i \ \forall i \in I$, *then inequality* (10) *for* $J = \emptyset$ *defines a facet of P for* $\beta_i = \bar{c}_i \ \forall i \in I$. *We recall that* $J = \emptyset$ *implies that S is a strong minimal cover.*

In Theorem 3.2 below, we show what indices from J must have the coefficient $\beta_j = \pi_j$, and what indices must have $\beta_j = \pi_j + 1$ for each facet. This information is provided by the maximal cliques implied by the given knapsack constraint (1). This crucial observation implies that the coefficients β_j can be directly obtained without requiring any sequential lifting of the cover inequality.

Given the knapsack inequality (1) and a minimal cover S implied by (1), let

$$\mathscr{M}(N - S) = \left\{ M \subseteq (N - S) : \sum_{j \in M} a_j \leqslant a_0 \right\}.$$

Note that the elements of $\mathscr{M}(N - S)$ are subsets of indices whose variables can take simultaneously the value 1 in a feasible solution of (1), i.e.,

$$x_j = \begin{cases} 1, & j \in M, \\ 0, & j \in (N - S) - M \end{cases}$$

for any $M \in \mathscr{M}(N - S)$. Furthermore, associated with each subset M, we can define the 0–1 variable x_M, with coefficient $a_M = \sum_{j \in M} a_j$. The introduction of this class of sets, see [3], allows for the generalization of the definition of the lifting coefficients such that the related coefficient, say, $\bar{c}_M$ of the new variable x_M has the expression $\bar{c}_M = c_M + |S| - 1 - z_M$, where c_M gives the number of variables from $E(S) - S$ in set M (i.e., $c_M = |E(S) - S \cap M|$), and z_M is the optimal objective function value of the knapsack problem

$$z_M = \max \sum_{j \in E(S)} x_j,$$

s.t.

$$\sum_{j \in N-M} a_j x_j \leqslant a_0 - a_M,$$

$$x_M = 1,$$

$$x_j \in \{0, 1\} \quad \forall j \in N - M. \tag{17}$$

The optimal objective function value in (17) is

$$z_M = c_M + (\alpha - 1), \tag{18}$$

where α is, as above, the first integer from 1 to |S| such that $A_{\alpha-1} \leqslant a_0 - a_M < A_\alpha$. Similarly, π_M is the value given in (12), where j is replaced by M.

By using the above notation, consider the following result.

Theorem 3.2. *Let S be a strong minimal cover implied by* (1) *and* $N - S = I \cup J$, $J \neq \emptyset$. *Then for each maximal clique C implied by* (1), *such that* $J \cap C \neq \emptyset$, *the LCI* (19) *is a facet of P*:

$$\sum_{j \in S} x_j + \sum_{j \in E(S)-S} \beta_j x_j \leqslant |S| - 1, \tag{19}$$

where

$$\beta_j = \begin{cases} \bar{c}_j, & j \in I, \\ \bar{c}_j, & j \in J \cap C, \\ \bar{c}_j - 1 & \text{for any other } j \in J. \end{cases} \tag{20}$$

Proof. Consider the characterization of the class of facets of P associated with minimal covers given in [3]. In this case, inequality (19) defines a facet of P if and only if

$$\beta_j = \begin{cases} \pi_j, & j \in I, \\ \pi_j + \delta_j, & j \in J, \end{cases} \tag{21}$$

with $0 \leqslant \delta_j \leqslant 1$, $j \in J$, where $\delta \in R^{|J|}$ is a vertex of the set

$$T = \left\{ \delta \in R^{|J|} : \sum_{j\in M} \delta_j \leqslant \bar{c}_M - \sum_{j\in M} \pi_j, M \in \mathscr{M}(J) \right\}.$$

Note that expression (21) holds from Proposition 3.2, where δ_j for $j \in J$ is chosen as

$$\delta_j = \begin{cases} 1, & j \in J \cap C, \\ 0, & j \in J - C \end{cases} \tag{22}$$

and, then, expression (20) follows, provided that the vector $\delta \in R^{|J|}$ as defined in (22) belongs to the set T. Next, we prove it.

For $M \in \mathscr{M}(J)$ and each maximal clique C, see that $|M \cap C| = 1$, from the definition of the set $\mathscr{M}(J)$ and since C is a clique. That is, $\sum_{j\in M} \delta_j = 1$. Now, let $a_{j_1} \geqslant \cdots \geqslant a_{j_M}$ and $h_1 = \pi_{j_1}, \ldots, h_M = \pi_{j_M}$ for the variables x_{j_i}, $j_i \in M$. Then, from (11) and (12), we have

$$\begin{aligned} a_{M-\{j_1\}} = a_M - a_{j_1} &\geqslant \sum_{j\in S_{h_2}} a_j + \cdots + \sum_{j\in S_{h_M}} a_j \\ &\geqslant \sum_{j\in S_{h_2+\cdots+h_M}} a_j. \end{aligned} \tag{23}$$

From (12) and (16), since $j_1 \in J$, it results the first inequality in (24). From expression (23) it results the second inequality, and the last one follows:

$$\begin{aligned} a_0 - a_M = a_0 - a_{j_1} - a_{M-\{j_1\}} &< \sum_{j\in S-S_{h_1+1}} a_j - a_{M-\{j_1\}} \\ &\leqslant \sum_{j\in S-S_{h_1+1}} a_j - \sum_{j\in S_{h_2+\cdots+h_M}} a_j \\ &= \sum_{j\in S} a_j - \left(\sum_{j\in S_{h_1+1}} a_j + \sum_{j\in S_{h_2+\cdots+h_M}} a_j \right) \\ &\leqslant \sum_{j\in S-S_{h_1+h_2+\cdots+h_M+1}} a_j. \end{aligned} \tag{24}$$

Hence, the value of α in (18) satisfies

$$\alpha \leqslant |S| - \sum_{j=1}^{M} h_j - 1,$$

and, then, from the definition of the coefficient $\bar{c}_M$, it results

$$\bar{c}_M \geqslant 1 + \sum_{j=1}^{M} h_j = \sum_{j\in M} \delta_j + \sum_{j\in M} \pi_j.$$

Thus, δ satisfies the inequalities defining T, i.e., $\delta \in T$. Therefore, δ is a vertex of T if and only if condition (25) is satisfied for $|J|$ linearly independent sets M_i:

$$\sum_{j\in M_i} \delta_j = \bar{c}_{M_i} - \sum_{j\in M_i} \pi_j. \tag{25}$$

Note that the sets M_i are linearly independent sets provided that the $|J|$ vectors $u(M_i) \in R^{|J|}$, $i \in J$ are linearly independent, where

$$u_j(M_i) = \begin{cases} 1, & j \in M_i, \\ 0, & \text{otherwise.} \end{cases}$$

By choosing $M_j = \{j\}$, for $j \in J$, it is easy to see that (25) holds from Proposition 3.1, since $\bar{c}_j = 1 + \pi_j$, $j \in J$. $\square$

Corollary 3.2. *Inequality* (19) *is a sequentially lifted facet of P, provided that* $\delta_j \in \{0, 1\}$ *in Theorem* 3.2.

Proposition 3.3. *The set J of indices defined in* (13) *can be identified with complexity* $O(\bar{s} \log s)$, *where* $\bar{s} = |E(S) - S|$ *and* $s = |S|$.

Proof. Let A_i and B_i, $i = 0, \ldots, s$ be the sum of the i smallest and largest coefficients a_j, $j \in S$, respectively. For any coefficient $0 \leqslant a \leqslant a_0$, let α_a be the smallest integer such that

$$A_{|S|-1-\alpha_a} \leqslant a_0 - a < A_{|S|-\alpha_a}$$

and let γ_a be the largest integer such that

$$B_{\gamma_a} \leqslant a < B_{\gamma_a+1}.$$

The integers α_a and γ_a for a given a can be obtained through a binary search and, so, the complexity is $O(\log s)$. Note also that there are only $\bar{s}$ iterations. $\square$

Example 3.1. Consider the inequality with 0–1 variables

$$5x_1 + 3x_2 + 3x_3 + 3x_4 + 2x_5 + 2x_6 + 2x_7 + 2x_8 \leqslant 6, \tag{26}$$

taken from [3]. Let the minimal cover $S = \{5, 6, 7, 8\}$ and its extension $E(S) = N = \{1, 2, \ldots, 8\}$, hence $N - S = \{1, 2, 3, 4\}$. So, S is a strong minimal cover. The sets N_h (11) are obtained as follows: $N_0 = \{\emptyset\}$, $N_2 = \{1\}$ and then $\pi_1 = 2$, $N_1 = \{2, \ldots, 8\}$, and then $\pi_i = 1$, $i = 2, \ldots, 8$. So, $I = \emptyset$ and $J = \{1, 2, 3, 4\}$. We can obtain the lifting coefficients $\bar{c}_i$ $\forall i \in N - S$ either by using (7) or (14):

$$\bar{c}_1 = 3, \quad \bar{c}_2 = 2, \quad \bar{c}_3 = 2, \quad \bar{c}_4 = 2.$$

Note that (26) implies the maximal cliques $\{1, 2\}$, $\{1, 3\}$ and $\{1, 4\}$. The following inequalities can be obtained by using (19):

$$3x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leqslant 3,$$

$$3x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leqslant 3,$$

$$3x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 \leqslant 3. \tag{27}$$

It is easy to see that (27) is a set of sequentially lifted facets. Moreover, these facets are the only distinct facets with integer coefficients that can be obtained from the minimal cover S. Note that system (27) can be obtained by the procedures ECIT and PCIT [5].

Example 3.2. Consider the inequality with 0–1 variables

$$43x_1 + 41x_2 + 40x_3 + 21x_4 + 20x_5 + 20x_6 + 20x_7 + 20x_8 \leqslant 93. \tag{28}$$

Let the minimal cover $S = \{4, 5, 6, 7, 8\}$ and its extension $E(S) = N = \{1, 2, \ldots, 8\}$, hence $N - S = \{1, 2, 3\}$. So, S is a strong minimal cover. The sets N_h (11) are obtained as follows: $N_0 = \{\emptyset\}$, $N_2 = \{1, 2\}$ and $N_1 = \{3, \ldots, 8\}$, and then $\pi_1 = \pi_2 = 2$, and $\pi_i = 1, i = 3, \ldots, 8$. So, $I = \{1, 2\}$ and $J = \{3\}$. We can obtain the lifting coefficients $\bar{c}_i$ $\forall i \in N - S$ either by using (7) or (14):

$$\bar{c}_1 = 2, \quad \bar{c}_2 = 2, \quad \bar{c}_3 = 2.$$

Note that (28) implies only trivial cliques. The following inequality can be obtained by using (19):

$$2x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leqslant 4. \tag{29}$$

It is easy to see that (29) is a sequentially lifted facet. It is the unique facet with integer coefficients that can be obtained from the minimal cover S. Note that the facet (29) cannot be obtained by the procedures ECIT and PCIT; instead three tightening cover-induced inequalities are identified. Obviously, those inequalities are dominated by (29).

4. Conclusions

We have shown that the complexity for identifying the facets with integer coefficients to be obtained from a strong minimal cover, say S, is $O(n \log n)$ for $n \equiv |N|$, where N is the set of indices in the given knapsack constraint. These facets are a subset of the class of facets that are characterized in Theorem 9 of [1]. An important consideration is that the lifting coefficients do not depend upon the ordering of the indices in $N - S$ and, in any case, the indices belong to the subset $E(S) - S$. Additionally, this subset can be split into the sets, say, I and J, such that the coefficients whose indices belong to the set I can also be obtained according to the Balas–Zemel expression. On the other hand, explicit expressions are given for the coefficients whose indices belong to the sets $J \cap C$ and $J - C$, where C is a maximal clique implied by the knapsack constraint.

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