10. THE DUAL SIMPLEX METHOD.

In Section 5, we have observed that solving an LP problem by the simplex method, we obtain a solution of its dual as a by-product. Vice versa, solving the dual we also solve the primal. This observation is useful for solving problems such as

$$\begin{array}{llllllll} \text{maximize} & -4x_1 & - & 8x_2 & - & 9x_3 & & \\ \text{subject to} & 2x_1 & - & x_2 & - & x_3 & \leq & 1 \\ & 3x_1 & - & 4x_2 & + & x_3 & \leq & 3 \\ & -5x_1 & & & - & 2x_3 & \leq & -8 \\ & & & & & x_1, x_2, x_3 & \geq & 0. \end{array} \tag{1}$$

Since this problem does not have feasible origin, the routine approach calls for the two-phase method. Nevertheless, we can avoid the two-phase method as soon as we realize that the dual of (1),

$$\begin{array}{llllllll} \text{minimize} & y_1 & + & 3y_2 & - & 8y_3 & & \\ \text{subject to} & 2y_1 & + & 3y_2 & - & 5y_3 & \geq & -4 \\ & -y_1 & - & 4y_2 & & & \geq & -8 \\ & -y_1 & + & y_2 & - & 2y_3 & \geq & -9 \\ & & & & & y_1, y_2, y_3 & \geq & 0. \end{array} \tag{2}$$

does have feasible origin. Hence we may simply solve the dual and then read the optimal primal solution off the final table for the dual. In this section, we shall discuss a way of solving the dual without actually saying so. That is accomplished by a method due to C. E. Lemke [ ] which is ucually called the dual simplex method. We shall first describe it as a mirror image of the simplex method and then we shall illustrate it on the example (1). Only then we shall note (without proof) that the dual simplex method is nothing but a disguised simplex method working on the dual. In closing, we shall mention a context in which the dual simplex method is particularly usuful.

To begin with, we need some new terminology. So far, we have called tables "feasible" if they described feasible solutions; from now on, we shall call such tables primal feasible. On the other hand, we shall call a table dual feasible uf in its formula for the objective function, every variable has a nonpositive coefficient. Note that the simplex method produces a sequence of promal feasible tables; as soon as it finds one which is also dual feasible, the method terminates. On the other hand, the dual simplex method produces a sequence of dual feasible tables; as soon as it finds one which is also promal feasible, the method terminates. In each iteration of the simplex method, we first choose the entering variable and then determine the leaving variable. For the entering variable, we may choose any nonbasic variable with a positive coeeficient in the z -row; as rule, we choose the variable with the largest positive coeeficient. Then we determine the leaving variable so as to preserve promal feasibility in our next table. On the other hand, in each iteration of the dual simplex method, we first choose the leaving variable and then determine the entering variable. For the leaving variable, we may choose any basic variable whose current value is negative; as a rule, we shall choose the variable with the largest absolute value. Then we shall determine the entering variable so as to preserve dual feasibiity in our next table; this point deserves a more detailed explanation. For definiteness, let the row that describes the leaving variable x_i read

$$x_i = -b + \sum_{j \in N} a_j x_j \tag{3}$$

and let the last row read

$$z = v + \sum_{j \in N} d_j x_j.$$

If $a_j \leq 0$ for every $j \in N$ then our problem has no feasible solution: indeed, (3) implies that $x_i \leq -b < 0$ whenever $x_j \geq 0$ for all $j \in N$. On the other hand, if $a_j > 0$ for at least one $j \in N$ then we choose for the entering variable that x_k for which $a_k > 0$ and which minimizes the ratio d_j/a_j. Let us verify that this choice does indeed preserve dual feasibility in our next table. Since we have

$$x_k = \frac{b}{a_k} + \frac{x_i}{a_k} - \sum_{j \neq k} \frac{a_j}{a_k} x_k,$$

the last row of our next table reads

$$z = v - \sum_{j \neq k} d_j x_j - d_k \left(\frac{b}{a_k} + \frac{x_i}{a_k} - \sum_{j \neq k} \frac{a_j}{a_k} x_k \right)$$

or, after simplifications,

$$z = \left(v - \frac{d_k b}{a_k}\right) - \frac{d_k}{a_k} x_i - \sum_{j \neq k} \left(d_j - \frac{d_k a_j}{a_k}\right) x_j. \tag{4}$$

We have, of cource, $a_k > 0$ and $d_j \geq 0$ for every j; in addition, $a_j > 0$ implies $dj/a_j \geq d_k/a_k$. Hence the coefficient at each variable in (4) is negatibe or zero; our new table is dual feasible. Finally, let us recall that in absence of degeneracy, each iteration of the simplex method increases the value of z (and so cycling cannot occur). By dual degeneracy, we mean the phenomenon of at least one nonbasic variable having the coefficient zero in the z-row of a dual feasible table. It follows directly from (4) that in absense of dual degeneracy, each iteration of the dual simplex method decreases the value of z (and so cycling cannot occur). In Section 3, we have proved that degeneray can be prevented by the perturbation technique. Similarly, dual degeneracy can be prebented when, for a hypothetial small ϵ, the objective function $\sum c_j x_j$ is replacced by

$$\sum_{j=1}^{n} (c_j + \epsilon^j) x_j.$$

Next, we shall illustrate the dual simplex method on the example (1). Writing down the formulas for the slack variables and for the objective function, we obtain the table

$$\begin{array}{rcrcrcrcr} x_4 & = & 1 & - & 2x_1 & + & x_2 & + & x_3 \\ x_5 & = & 3 & - & 3x_1 & + & 4x_2 & - & x_3 \\ x_6 & = & -8 & + & 5x_1 & & & + & 2x_3 \\ \hline z & = & & - & 4x_1 & - & 8x_2 & - & 9x_3. \end{array}$$

Since this table is dual feasible, we may use it to initialize the dual simplex method. Next, we have to choose the leaving variable. Since only one bariable has a negative value, the choice is unique: x_6 will leave. In order to determine the entering variable, we compare the ratios $4/5$ and $9/2$; since the first is smaller, x_1 will enter. Pivoting as usual, we arrive at the table

$$\begin{array}{rcrcrcrcr} x_1 & = & \frac{8}{5} & & & - & \frac{2}{5}x_3 & + & \frac{1}{5}x_6 \\ x_4 & = & -\frac{11}{5} & + & x_2 & + & \frac{9}{5}x_3 & - & \frac{2}{5}x_6 \\ x_5 & = & -\frac{9}{5} & + & 4x_2 & + & \frac{1}{5}x_3 & - & \frac{3}{5}x_6 \\ \hline z & = & -\frac{32}{5} & - & 8x_2 & - & \frac{37}{5}x_3 & - & \frac{4}{5}x_6. \end{array}$$

Note that the value of z has decreased. Now there are two negative variables; since x_4 has the larger absolute value, we shall make it leaving. In order to determine the entering variable, we compare the ratios $8/1$ and $37/9$ (since $-2/5$ is negative, the ratio $4/2$ is ignored); since the second is smaller, x_3 will enter. Our next table reads

$$\begin{array}{rcrcrcrcr} x_3 & = & \frac{11}{9} & - & \frac{5}{9}x_2 & + & \frac{2}{9}x_6 & + & \frac{5}{9}x_4 \\ x_1 & = & \frac{10}{9} & + & \frac{2}{9}x_2 & + & \frac{1}{9}x_6 & - & \frac{2}{9}x_4 \\ x_5 & = & -\frac{14}{9} & + & \frac{35}{9}x_2 & - & \frac{5}{9}x_6 & + & \frac{1}{9}x_4 \\ \hline z & = & -\frac{139}{9} & - & \frac{35}{9}x_2 & - & \frac{22}{9}x_6 & - & \frac{37}{9}x_4. \end{array}$$

Next, x_5 leaves and x_2 enters:

$$\begin{array}{rcrcrcrcr} x_2 & = & \frac{2}{5} & + & \frac{1}{7}x_6 & - & \frac{1}{35}x_4 & + & \frac{9}{35}x_5 \\ x_3 & = & 1 & + & \frac{1}{7}x_6 & + & \frac{4}{7}x_4 & - & \frac{1}{7}x_5 \\ x_1 & = & \frac{6}{5} & + & \frac{1}{7}x_6 & - & \frac{8}{35}x_4 & + & \frac{2}{35}x_5 \\ \hline z & = & -17 & - & 3x_6 & - & 4x_4 & - & x_5. \end{array}$$

The last table, being both dual feasible and primal feasible, is the final table for our problem: the optimal solution of (1) is $x_1 = 6/5$, $x_2 = 2/5$, $x_3 = 1$.

We have accused the dual simplex method of being "nothing but a disguised simpex method working on the dual". In order to examine this accusation, we shall now solve the dual (2) of (1). In the canonical form, (2) reads

$$\begin{array}{lrcrcrcl}
\text{maximize} & -y_1 & - & 3y_2 & + & 8y_3 & & \\
\text{subject to} & 2y_1 & - & 3y_2 & + & 5y_3 & \le & 4 \\
 & y_1 & + & 4y_2 & + & & \le & 8 \\
 & y_1 & - & y_2 & + & 2y_3 & \le & 9 \\
 & & & & & y_1, y_2, y_3 & \ge & 0.
\end{array}$$

Applying the simplex method, we construct the following sequence of tables:

First table:

$$\begin{array}{rcrcrcrcr}
y_6 & = & 9 & - & 2y_3 & + & y_2 & - & y_1 \\
y_5 & = & 8 & & & - & 4y_2 & - & y_1 \\
y_4 & = & 4 & - & 5y_3 & + & 3y_2 & + & 2y_1 \\
\hline
z & = & & & 8y_3 & - & 3y_2 & - & y_1.
\end{array}$$

Second table:

$$\begin{array}{rcrcrcrcr}
y_3 & = & \frac{4}{5} & + & \frac{3}{5}y_2 & + & \frac{2}{5}y_1 & - & \frac{1}{5}y_4 \\
y_6 & = & \frac{37}{5} & - & \frac{1}{5}y_2 & - & \frac{9}{5}y_1 & + & \frac{2}{5}y_4 \\
y_5 & = & 8 & - & 4y_2 & - & y_1 & & \\
\hline
z & = & \frac{32}{5} & + & \frac{9}{5}y_2 & + & \frac{11}{5}y_1 & - & \frac{8}{5}y_4.
\end{array}$$

Third table:

$$\begin{array}{rcrcrcrcr}
y_1 & = & \frac{37}{9} & - & \frac{1}{9}y_2 & + & \frac{2}{9}y_4 & - & \frac{5}{9}y_6 \\
y_3 & = & \frac{22}{9} & + & \frac{5}{9}y_2 & - & \frac{1}{9}y_4 & - & \frac{2}{9}y_6 \\
y_5 & = & \frac{35}{9} & - & \frac{35}{9}y_2 & - & \frac{2}{9}y_4 & + & \frac{5}{9}y_6 \\
\hline
z & = & \frac{139}{9} & + & \frac{14}{9}y_2 & - & \frac{10}{9}y_4 & - & \frac{11}{9}y_6.
\end{array}$$

Forth table:

$$
\begin{array}{rclllllll}
y_2 & = & 1 & - & \frac{2}{35}y_4 & + & \frac{1}{7}y_6 & - & \frac{9}{35}y_5 \\
y_1 & = & 4 & + & \frac{8}{35}y_4 & - & \frac{4}{7}y_6 & + & \frac{1}{35}y_5 \\
y_3 & = & 3 & - & \frac{1}{7}y_4 & - & \frac{1}{7}y_6 & - & \frac{1}{7}y_5 \\
\hline
z & = & 17 & - & \frac{6}{5}y_4 & - & y_6 & - & -\frac{2}{5}y_5.
\end{array}
$$

Comparing this sequence of four tables with the sequence of four tables produced by the dual simplex method, we shall uncover an interesting correspondence. To begin with, let us forget all about the actual coefficients in those tables; instead, let us concentrate on the basic-nonbasic status of variables, as recorded below.

	The dual simlex method		The simplex method on the dual	
	Basic	Nonbasic	Basic	Nonbasic
First table	x_4, x_5, x_6	x_1, x_2, x_3	y_6, y_5, y_4	y_3, y_2, y_1
Second table	x_1, x_4, x_5	x_2, x_3, x_6	y_3, y_6, y_5	y_2, y_1, y_4
Third table	x_3, x_1, x_5	x_2, x_6, x_4	y_1, y_3, y_5	y_2, y_4, y_6
Forth table	x_2, x_3, x_1	x_6, x_4, x_5	y_2, y_1, y_3	y_4, y_6, y_5

In order to discern the pattern of this table, we shall note that the variables $x_1, x_2, \cdots, x_6$ can be matched up with the variables $y_1, y_2, \cdots, y_6$ in a rather natural way. For example, both x_4 and y_1 are associated with the first primal constraint: x_4 is its slack and y_1 is its multiplier. In the same way, every constraint, primal or dual, associates with a pair of variables x_i, y_j:

the first primal constraint	$\cdots$	x_4, y_1
the second primal constraint	$\cdots$	x_5, y_2
the third primal constraint	$\cdots$	x_6, y_3
the first dual constraint	$\cdots$	y_4, x_1
the second dual constraint	$\cdots$	y_5, x_2
the third dual constraint	$\cdots$	y_6, x_3.

Now we may observe that at each stage of the computations, from the first table to the fourth, our correspondence carries the nonbasic (resp. basic) variables x_i onto the basic (resp. nonbasic) variables y_j. Next, bringing in the numerical values of the coefficients, we shall make a startling discovery. For example, let us compare the third tables in each sequence:

$$
\begin{array}{rcrcrcrcr}
x_3 & = & \dfrac{11}{9} & - & \dfrac{5}{9}x_2 & + & \dfrac{2}{9}x_6 & + & \dfrac{5}{9}x_4 \\
x_1 & = & \dfrac{10}{9} & + & \dfrac{2}{9}x_2 & + & \dfrac{1}{9}x_6 & - & \dfrac{2}{9}x_4 \\
x_5 & = & -\dfrac{14}{9} & + & \dfrac{35}{9}x_2 & - & \dfrac{5}{9}x_6 & + & \dfrac{1}{9}x_4 \\
\hline
z & = & -\dfrac{139}{9} & - & \dfrac{35}{9}x_2 & - & \dfrac{22}{9}x_6 & - & \dfrac{37}{9}x_4
\end{array}
\tag{5}
$$

and

$$
\begin{array}{rcrcrcrcr}
y_1 & = & \dfrac{37}{9} & - & \dfrac{1}{9}y_2 & + & \dfrac{2}{9}y_4 & - & \dfrac{5}{9}y_6 \\
y_3 & = & \dfrac{22}{9} & + & \dfrac{5}{9}y_2 & - & \dfrac{1}{9}y_4 & - & \dfrac{2}{9}y_6 \\
y_5 & = & \dfrac{35}{9} & - & \dfrac{35}{9}y_2 & - & \dfrac{2}{9}y_4 & + & \dfrac{5}{9}y_6 \\
\hline
z & = & \dfrac{139}{9} & + & \dfrac{14}{9}y_2 & - & \dfrac{10}{9}y_4 & - & \dfrac{11}{9}y_6.
\end{array}
\tag{6}
$$

The two tables (5) and (6) look dangerously alike. For example, the numbers in the last row of (5) are, from left to right, $-139/9$, $35/9$, $22/9$, $7/9$. Similarly, the numbers in the x_1 -row of (5) are, from left to right, $10/9$, $2/9$, $1/9$, $-2/9$ whereas the numbers in the y_4 -column of (6) are, from bottom to top, $-10/9$, $-2/9$, $-1/9$, $2/9$. And so on. The entire table (6) can be reconstructed from (5) and vice versa. The same correspondence exists between the first tables in each sequence, between the second tables in each sequence and between the fourth tables in each sequence. In fact, that correspondence is quite general: given a table for some problem, we may readily construct its mirror image for the dual. Pivoting from one primal table to another amounts to pivoting from the first mirror image to the second. The correspondence can be described, and its validity established, without much difficulty. However, the argument involves a fair amount of formal plugging and griding which is not our cup of tea. We simply wanted to point out the close parallelism between the two sequences.

Finally, we shall discuss a use of the dual simplex method which often comes up in applications. For example, let us return to Nikki's nutrition problem from Section 1. With a little less forethought, she might bave formulated her problem as

$$
\begin{array}{lrcrcrcrcrcrcr}
\text{maximize} & 3x_1 & + & 24x_2 & + & 13x_3 & + & 9x_4 & + & 20x_5 & + & 19x_6 & & \\
\text{subject to} & 110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \geq & 2000 \\
 & 4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \geq & 55 \\
 & 2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \geq & 800 \\
 & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0.
\end{array}
\tag{7}
$$

Solving (7), she would arrive at the optimal solution $x_1 = 14.24, x_2 = x_3 = 0, x_4 = 2.71, x_5 = x_6 = 0$. That menu, involving more than fourteen servings of oatmeal is clearly unacceptable to her. It would be only now that she would recognize the imperative of imposing an upper bound on the amount of oatmeal to be devoured each day. Thus she might add the constraint $x_1 \leq 4$ to (7) and solve the new problem from scratch. Doing so, she would waste all her calculations which led to solving (7); that could be avoided by appropriate use of the dual simplex method.

To explain how the dual simplex method is used in such a situation, we shall consider an example which is numerically simpler,

$$\begin{array}{lrcrcrcl} \text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\ & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\ & & & & & x_1, x_2, x_3 & \geq & 0. \end{array} \tag{8}$$

In fact, this is the first LP problem we have ever solved; the final table reads

$$\begin{array}{rcrcrcrcr} x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\ x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\ x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\ \hline z & = & 13 & - & 3x_2 & - & x_4 & - & x_6. \end{array} \tag{9}$$

For some reason, we decide to add a new constraint, $x_1 + x_2 + x_3 \leq 1$, to the old constraints of (8). That constraint makes the optimal solution $x_1 = 2, x_2 = 0, x_3 = 1$ infeasible in the new problem,

$$\begin{array}{lrcrcrcl} \text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\ & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\ & x_1 & + & x_2 & + & x_3 & \leq & 1 \\ & & & & & x_1, x_2, x_3 & \geq & 0. \end{array} \tag{10}$$

In order to solve (10), we may simply start from scratch; an alternative is to apply the dual simplex method to an enlarged version of (9). Pursing that line we have to express the new slack variable

$$x_7 = 1 - x_1 - x_2 - x_3$$

in terms of the nonbasic variables x_2, x_4, x_6 of (9). The desired expression is obtained simply by substituting for x_1 and x_3 from (9):

$$\begin{aligned} x_7 &= 1 - (2 - 2x_2 - 2x_4 + x_6) - x_2 - (1 + x_2 + 3x_4 - 2x_6) \\ &= -2 - x_4 + x_6. \end{aligned}$$

Adding this formula to (9) we obtain the table

$$
\begin{array}{rcrcrcrcr}
x_7 & = & -2 & & & - & x_4 & + & x_6 \\
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6
\end{array}
$$

which, being dual feasible, initializes the dual simlex method. Leave x_7, enter x_6:

$$
\begin{array}{rcrcrcrcr}
x_7 & = & -2 & & & - & x_4 & + & x_6 \\
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6.
\end{array}
$$

Leave x_3, enter x_4:

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_2 & + & 2x_7 & + & x_3 \\
x_6 & = & 5 & - & x_2 & + & 3x_7 & + & x_3 \\
x_1 & = & 1 & - & x_2 & - & x_7 & - & x_3 \\
x_5 & = & 7 & + & 3x_2 & + & 4x_7 & + & 2x_3 \\
\hline
z & = & 5 & - & x_2 & - & 5x_7 & - & 2x_3.
\end{array}
$$

The last table, being both primal feasible and dual feasible, represents an optimal solution of (10).

In this example, the attack from scratch would bring us to the optimal solution in only one iteration whereas our strategy took two iterations. However, the dual simplex method often turns out to be the more economical of the two. Used in this context, the dual simplex method constitutes an important subroutine of an algorithm we shall discuss in the next section.