Lectures on Semidefinite Programming for Combinatorial Optimization Problems at the School of Computer Science, McGill University

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1 Introduction to Semidefinite Programming (SDP)

The standard formulation of SDP is as follows, where (P) is the primal problem and (D) is its dual.

$$\begin{array}{lll|lll} (P) & \max & C \cdot X & \text{(D)} & \min & b^T y \\ & \text{s.t.} & A_i \cdot X = b_i \text{ for } i = 1 \ldots m, & & \text{s.t.} & Z = \sum_{i=1}^m y_i A_i - C, \\ & & X \succeq 0. & & & Z \succeq 0. \end{array}$$

where $A \cdot X = \sum_{i,j} A_{i,j} B_{i,j} = \text{trace}\,(B^T A)$, b and y (as well as other small alphabets) denote column vectors, while y_i's denote y's i-th component.

Notes

1. All matrices of interest, such as X and Z but not necessarily A_i or C [1] in the above formulation, are symmetric and real.
2. SDP is a linear problem except for the positive semidefinite (psd) contraint.
3. The set of psd matrices is a closed convex cone (ccc).
4. SDP is a convex optimization problem.

[1]However, we can construct A_i and C to be symmetric. Therefore, it is safe to assume that all matrices are symmetric.

5. As an example, let us write down the SDP relaxation for the MAX-CUT problem.

$$\begin{aligned} \max \quad & \sum_{i,j} \left(\frac{1 - X_{i,j}}{2} \right) w_{i,j} \\ \text{s.t.} \quad & X_{i,i} = 1 \ \forall i, \\ & X \succeq 0, \\ & \text{rank}(X) = 1. \end{aligned}$$

Roughly speaking, the diagonal-ones and rank constraints require that all elements of X be in $\{-1, 1\}$, while the psd constraint eliminates non-cut integer solutions. Removing the rank constraint gives the SDP relaxation.

2 PSD Matrices

Definition 1 *A matrix $A \in S^n$, where S^n denotes the set of $n \times n$ real symmetric matrices, is said to be psd if for all $y \in R^n$:*

$$y^T A y = \sum_{i,j} A_{i,j} y_i y_j \geq 0.$$

When the condition holds with strict inequality, A is said to be positive definite (pd).

Example The matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is a rank-1 feasible solution for MAX-CUT.

Notes

1. A non-negative linear combination of psd matrices is psd:

$$y^T \left(\sum_j \lambda_j X_j \right) y \quad = \quad \sum_j \lambda_j \left(y^T X_j y \right) \geq 0$$

for $\lambda_j \geq 0, X_j \succeq 0$.

2. Definition of cone: If X is an element of the cone, then λX is also in the cone for all $\lambda \geq 0$.

   Therefore, the psd matrices form a convex cone.

3. The closedness of the cone means that the boundary is included. In fact, the boundary is precisely the collection of the singular psd matrices.

4. We denote the cone of psd matrices by S^n_+. The relative interior of the cone are the pd matrices denoted by S^n_{++}.

5. The partial order over the psd cone is called the Löwner partial order: $A \succeq B \iff A - B \succeq 0$. To verify that it is a partial order, note that:

   - $A \succeq A$,
   - $A \succeq B$ and $B \succeq A \implies A = B$.

- $A \succeq B$ and $B \succeq C \implies A \succeq C$.

It is not a total order because $\exists\ A, B$ s.t. $A \not\succeq B$ and $B \not\succeq A$. See the exercise.

Theorem 1 *$A \in S^n \iff A$ is orthogonally diagonalizable, i.e. $\exists$ orthogonal matrix U s.t.*

$$A = UDU^T$$

with $U^TU = UU^T = I$ and D a diagonal matrix of the eigenvalues of A.

Corollary 1 *All the eigenvalues of A are real.*

From the above facts, we have the first characterization of psd matrices.

Theorem 2 *For $A \in S^n$, $A \succeq 0 \iff$ all the eigenvalues of A are non-negative.*

Corollary 2 *The trace and determinant of a psd matrix are non-negative.*

Definition 2 *Given $A \in S^n$ and a subset $I \subseteq \{1 \dots n\}$, the principal submatrix of A corresponding to I is the submatrix with rows and columns indexed by I. Its determinant is called the principal minor.*

Example For $I = \{1, 3\}$ and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix},$$

the principal submatrix is

$$A(I) = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix}$$

and the principal minor is $\det A(I) = -3$.

Theorem 3 *$A \succeq 0 \iff$ all principal minors are non-negative.*

Another characterization is the fact that psd matrices are Gram matrices.

Theorem 4 *$A \succeq 0 \iff \exists$ set of vectors $\{w_1, \dots, w_n\}$ s.t. $A_{i,j} = w_i^T w_j$, i.e., $A = W^T W$ where $W = (w_1, \dots, w_n)$.*

Cholesky decomposition: $A \succeq 0 \iff A = LL^T$ where L is a lower triangular matrix.
If A is pd, then L is nonsingular with positive entries on the diagonal.
Note that it is easier to solve $Ax = b$: first solving $Ly = b$ and then $L^T x = y$.

Theorem 5 (Schur Complement) *If*

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

and A is pd, then $M \succeq 0 \iff C - B^T A^{-1} B \succeq 0$.

Theorem 6 (Fejer's Theorem) *$A \succeq 0 \iff A \cdot B \geq 0$ for all $B \succeq 0$.*

Since S^n is a finite dimensional Euclidean space, with the scalar product $A \cdot B = \text{trace}\,(B^T A)$, with the cone S^n_+ is associated the dual cone:

$$\begin{aligned}(S^n_+)^* &= \{Y \in S^n | X \cdot Y \geq 0 \ \forall X \in S^n_+\} \\ &= S^n_+ \ \text{ by Fejer's Theorem.}\end{aligned}$$

That is, S^n_+ is self-dual. The same holds for R^n_+ with the usual scalar product. Another example of self-duality is the second order cone defined as $x_0 \geq \sqrt{x_1^2 + \ldots + x_n^2}$.

3 Duality in SDP

Recall the SDP formulation:

$$\begin{aligned}(P) \qquad & \max \ C \cdot X && (1)\\ & \text{s.t. } \ A_i \cdot X = b_i \text{ for } i = 1 \ldots m && (2)\\ & X \succeq 0. && (3)\end{aligned}$$

$$\begin{aligned}(D) \qquad & \min \ b^T y && (4)\\ & \text{s.t. } \ Z = \sum_{i=1}^{m} y_i A_i - C && (5)\\ & Z \succeq 0. && (6)\end{aligned}$$

Theorem 7 (Weak Duality) *If $\tilde{X}$ is feasible for (P) and $\tilde{y}, \tilde{Z}$ for (D), then $C \cdot \tilde{X} \leq b^T \tilde{y}$.*

Proof. We have:

$$\begin{aligned}C \cdot \tilde{X} - \tilde{b}^T \tilde{y} &= \left(\sum_{i=1}^{m} \tilde{y}_i A_i - \tilde{Z}\right) \cdot \tilde{X} - \sum_i \tilde{y}_i A_i \cdot \tilde{X} \\ &= -\tilde{Z} \cdot \tilde{X} \leq 0.\end{aligned}$$

□

However, because of the non-linear psd constraint, the strong duality analogy with Linear Programming (LP) fails in SDP. The following examples are from [22, pp. 71-72].

1. In LP, if both (P) and (D) are feasible, then there is no duality gap. This may fail for SDP. For example,

$$\begin{aligned}(P) \max \quad & -a x_{11} \\ \text{s.t.} \quad & x_{11} + 2x_{23} = 1, \\ & x_{22} = 0, \\ & X \succeq 0.\end{aligned}$$

$$
\begin{array}{lll}
(D) \text{ min} & -y_2 & \\
\text{s.t.} & Y = \begin{pmatrix} y_2 - a & 0 & 0 \\ 0 & y_1 & y_2 \\ 0 & y_2 & 0 \end{pmatrix} \preceq 0. &
\end{array}
$$

It is easy to see that (P) has the optimal objective value $-a$, while (D) has 0.

2. Even if there is no duality gap at optimality, the optimal value may not be attained for (P) and (D). Consider

$$
\begin{array}{ll}
(P) \text{ sup} & 2x_{12} \\
\text{s.t.} & \begin{pmatrix} 1 & x_{12} \\ x_{12} & 0 \end{pmatrix} \succeq 0.
\end{array}
$$

$$
\begin{array}{ll}
(D) \text{ inf} & y_1 \\
\text{s.t.} & \begin{pmatrix} y_1 & -1 \\ -1 & y_2 \end{pmatrix} \succeq 0.
\end{array}
$$

(P) has the objective value 0 which is attained, but $\inf y_1 = 0$ is not.

To avoid these difficulties, we require that the SDP pair satisfy a constraint qualification (CQ). We will use Slater's CQ which imposes the existence of a feasible pd solution for the SDP. The main purpose of requiring a constraint qualification is to ensure the existence of Lagrange multipliers at optimality. These multipliers are an optimal solution to the dual problem and ensure that there is no duality gap.

4 Primal-Dual Interior Point Methods for SDP

For linear programming (LP), consider the primal-dual pair

$$
\begin{array}{ll}
\max & c^T x \qquad (7) \\
\text{s.t.} & Ax = b, \qquad (8) \\
 & x \geq 0. \qquad (9)
\end{array}
$$

$$
\begin{array}{ll}
\min & b^T y \qquad (10) \\
\text{s.t.} & z = A^T y - c, \qquad (11) \\
 & z \geq 0. \qquad (12)
\end{array}
$$

The central path equations are:

$$
\begin{array}{rcll}
Ax & = & b & \text{(primal feasibility)} \qquad (13) \\
A^T y - z & = & c & \text{(dual feasibility)} \qquad (14) \\
x \circ z & = & \mu e & \text{(perturbed complementary slackness)} \qquad (15)
\end{array}
$$

We also require (separately) that x and z are interior. The symbol $\circ$ denotes component-wise product, so that $x \circ z$ is a column vector whose i-th element is $x_i z_i$ and e denotes the column vector whose elements are all 1. Applying Newton's method we obtain the linearized version of this system:

$$
\begin{aligned}
A\Delta x &= 0 && (16)\\
A^T\Delta y - \Delta z &= 0 && (17)\\
z \circ \Delta x + x \circ \Delta z &= \mu e - (x \circ z). && (18)
\end{aligned}
$$

There are $m + 2n$ equations in total: m equations in Eq. (16), n equations in Eq. (17), and n equations in Eq. (18); and the number of variables from $\Delta x, \Delta y$ and Δz is also $m + 2n$.

Generic Primal-dual Interior Point Algorithm for LP (and SDP)

Step 0. Start with an initial point (x, y, z) and the corresponding μ "near" the central path, satisfying Eq. (13), (14), and (15).

Step 1. Reduce μ.

Step 2. Solve Eq. (16,17,18) for $(\Delta x, \Delta y, \Delta z)$ and take a step $\alpha \in (0, 1]$ in that direction, i.e.

$$x \leftarrow x + \alpha\Delta x, \quad y \leftarrow y + \alpha\Delta y, \quad z \leftarrow z + \alpha\Delta z,$$

such that $x + \alpha\Delta x > 0$ and $z + \alpha\Delta z > 0$ (ensuring that the next iteration is away from the boundary).

Step 3. If $\sum_i x_i z_i$ is sufficiently small, STOP (because we are sufficiently close to the optimum).

Otherwise, if the current point is sufficiently close to the central path, go to Step 1.

Else, go to Step 2 to obtain a point for the current value of μ which is closer to the central path.

For SDP, the central path equations are: (Notice the similarity with those for LP)

$$
\begin{aligned}
A_i \cdot X &= b_i \text{ for } i = 1, \ldots, m \quad \text{(primal feasibility)} && (19)\\
\sum_i y_i A_i - Z &= C \quad \text{(dual feasibility)} && (20)\\
XZ &= \mu I \quad \text{(perturbed complementary slackness)} && (21)
\end{aligned}
$$

Note that $XZ = 0$ at optimality.

The Newton system for SDP is as follows. (Again, it is similar to the LP case.)

$$
\begin{aligned}
A_i \cdot \Delta X &= 0 \text{ for } i = 1, \ldots, m && (22)\\
\sum_i \Delta y_i A_i - \Delta Z &= 0 && (23)\\
\Delta X\ Z + X\ \Delta Z &= \mu I - XZ && (24)
\end{aligned}
$$

However, unlike the LP case, there are $m + \binom{n+1}{2} + n^2$ equations (the first term from Eq. (22), the second term from Eq. (23) by noting that the left hand side is symmetric, and the last term from Eq. (24)) while there are $m + 2\binom{n+1}{2}$ variables which implies an overdetermined system.

The generic algorithm for SDP is similar to that for LP, replacing the corresponding variables. Note that ΔX may be non-symmetric and thus additional steps are needed to ensure the validity of the next iteration $X + \alpha \Delta X$.

Here are three approaches that have been proposed.

1. **(AHO direction)** Symmetrize the equation $XZ = \mu I$ to $\frac{XZ+ZX}{2} = \mu I$. The symmetric linearized version is:

$$\frac{1}{2}\left(\Delta X\ Z + Z\ \Delta X + X\ \Delta Z + \Delta Z\ X\right) = \mu I - \frac{1}{2}\left(XZ + ZX\right). \tag{25}$$

   This technique essentially reduces the number of equations to the number of variables. One important drawback is that it is slower than several other methods in practice.

2. **(HRVW/KSH/M or simply HKM direction)** Solve for unsymmetric ΔX, then symmetrize:

$$\Delta X \leftarrow \frac{\Delta X + (\Delta X)^T}{2}. \tag{26}$$

   It is fast, and arguably the most popular method.

3. **(GN direction)** Find the least squares solution for the overdetermined system (22-24). This is motivated by the application of a technique called Gauss-Newton (GN) to find the minimum norm solution of the system of non-linear equations (19-21). It is a very robust method, though again more expensive computationally.

5 Application of SDP to MAX-CUT

Let $G = (V, E)$ denote an edge weighted undirected graph without loops or multiple edges. For $S \subseteq V$, let $\delta(S)$ denote the set of edges with one end in S and the other in $V \setminus S$. The weight of this cut is $w(\delta(S)) = \sum_{(i,j)\in\delta(S)} w_{ij}$. The MAX-CUT problem is to maximize $w(\delta(S))$ over all $S \subseteq V$.

Without loss of generality, we assume that G is a complete graph (set $w_{ij} = 0$ for all non-edges). Let $W = (w_{ij})$ be the weighted adjacency matrix of G (note that $w_{ij} = w_{ji}$). The following is our first formulation for MAX-CUT.

$$\begin{array}{ll} \max & \displaystyle\sum_{i<j} \left(\frac{1 - v_i v_j}{2}\right) w_{ij} \\ \text{s.t.} & v_i^2 = 1,\ i = 1, \ldots, n = |V| \\ \text{where} & v_i = \begin{cases} +1 & \text{if node } i \in S \\ -1 & \text{otherwise} \end{cases} \end{array} \tag{27}$$

Since $w_{ij} = w_{ji}$, the formulation can be written in the full sum as follows.

$$\begin{array}{ll} \max & \displaystyle\sum_{i,j} \left(\frac{1 - v_i v_j}{4}\right) w_{ij} \\ \text{s.t.} & v_i^2 = 1,\ i = 1, \ldots, n. \end{array} \tag{28}$$

The objective function can be expressed as:

$$
\begin{aligned}
\frac{1}{4}\sum_{ij}(1-v_iv_j)w_{ij} &= \frac{1}{4}\sum_i\left(\sum_j w_{ij}v_i^2 - \sum_j w_{ij}v_iv_j\right) \\
&= \frac{1}{4}\left(\sum_i(\sum_j w_{ij})v_iv_i - \sum_{ij} v_iv_j\right) \\
&= \frac{1}{4}(v^T\mathrm{Diag}(We)v - v^TWv) \\
&= \frac{1}{4}v^TLv, \qquad (29)
\end{aligned}
$$

where $L = \mathrm{Diag}(We) - W$ is the Laplacian of the graph G, and $\mathrm{Diag}(x)$ is the diagonal matrix with diagonal elements defined by x. Thus, we have our second formulation for MAX-CUT as a problem with quadratic objective function and quadratic constraints.

$$
\begin{array}{ll}
\max & v^TQv \\
\text{s.t.} & v_i^2 = 1 \text{ for all } i
\end{array}
\qquad (30)
$$

where $Q = \frac{1}{4}L$.

We use this last formulation to get a matrix formulation. Define the **cut matrices** $X = vv^T$ for $v \in \{-1,1\}^n$. These matrices are real, symmetric, psd, rank-1, and satisfy $\mathrm{diag}(X) = e$. In fact, MAX-CUT is equivalent to:

$$
\begin{array}{ll}
\max & Q \cdot X \\
\text{s.t.} & \mathrm{diag}(X) = e \\
 & X \succeq 0 \\
 & \mathrm{rank}(X) = 1
\end{array}
\qquad (31)
$$

Dropping the rank constraint (which is not convex), we have the following primal-dual pair of SDP relaxations for MAX-CUT.

$$
\begin{array}{lll|lll}
(P) & \max & Q \cdot X & (\mathrm{D}) & \min & e^Ty \\
 & \text{s.t.} & \mathrm{diag}(X) = e & & \text{s.t.} & Z = \mathrm{Diag}(y) - Q \\
 & & X \succeq 0. & & & Z \succeq 0.
\end{array}
$$

Notes

1. The feasible set of (P) is called the elliptope:

$$\mathcal{E}_n = \{X \in S^n \mid \mathrm{diag}(X) = e,\ X \succeq 0\}.$$

For $n = 3$,

$$
X = \begin{pmatrix} 1 & X_{12} & X_{13} \\ X_{12} & 1 & X_{23} \\ X_{13} & X_{23} & 1 \end{pmatrix}
$$

and $\mathcal{E}_3$ can be graphed in three-dimensional space (see [9, p.518]). $\mathcal{E}_n$ is a non-polyhedral relaxation of MAX-CUT.

The "ideal" relaxation is the cut polytope,

$$C_n = \text{conv}\{X = vv^T, \text{ for } v \in \{-1,1\} \}.$$

2. Both (P) and (D) satisfy Slater's constraint qualification.

3. The SDP (D) is actually the Lagrangian dual of Formulation 2 [18, 16].

4. We can consider the triangle inequalities for elements of $\mathcal{E}_n$. Define the metric polytope:

$$M_n = \{X \in S^n \mid \text{diag}(X) = e, X_{ij} + X_{ik} + X_{jk} \geq -1, X_{ij} - X_{ik} - X_{jk} \geq -1 \ \forall i,j,k \}.$$

(Obtain these by substituting $(1 - X_{ij})/2$ for X_{ij} in the 0/1-space triangle inequalities.)

6 The Goemans-Williamson Approximation Algorithm for MAX-CUT

Håstad proved that, unless P=NP, MAX-CUT cannot be approximated better than $16/17 \approx 0.94117$. Until the algorithm of Goemans and Williamson (GW), the best approximation guarantee was 1/2, from the algorithm that assigns each node in G to either side of the cut with equal probability. The GW algorithm gives a 0.87856 approximation under the assumption that $w_{ij} \geq 0$. Here is the outline of the GW algorithm.

Algorithm: Given G with $w_{ij} \geq 0$.

Step 1. Solve (P) and compute vectors $v_1, v_2, \ldots, v_n$ such that $X^*_{ij} = v_i^T v_j$ for the optimal solution X^*.

Step 2. Generate a random unit vector r and set $S = \{i \mid v_i^T r \geq 0\}$.

Then, $\delta(S)$ has expected weight satisfying the approximation ratio as stated in the following theorem.

Theorem 8

$$(\textit{expected weight of randomized cut}) \geq \alpha_0 \ (\textit{optimal value of SDP}),$$

where $\alpha_0 := \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1-\cos\theta} > 0.87856$.

Since the weight of the maximum cut is at least the expected weight of the randomized cut, we get the approximation guarantee.

Sketch of proof:

1. Prob[edge(ij) is cut] $= \frac{\arccos(v_i^T v_j)}{\pi} \in [0,1]$.

2. The expected weight of the cut is

$$\begin{aligned} E &= \sum_{i<j} w_{ij} \frac{\arccos(v_i^T v_j)}{\pi} \\ &= \sum_{i<j} w_{ij} \left(\frac{1 - v_i^T v_j}{2}\right) \left(\frac{2}{\pi} \frac{\arccos(v_i^T v_j)}{1 - v_i^T v_j}\right) \\ &\geq \alpha_0 \text{ (opt. value SDP)}. \end{aligned}$$

6.1 Higher Liftings

When we go from v_i's to cut matrices, we are lifting the problem from R^n to S^n. One way to see this is to view the X matrices as being indexed by v_i's, i.e. $X = (X_{ij})$ such that $X_{ij} = v_i v_j$. If we call this a first lifting, what is a second lifting?

Idea: Construct a matrix in $S^{\binom{n}{2}}$ indexed by the edges. For $n = 3$, the matrix X is as follows.

$$X = \begin{pmatrix} 1 & X_{12} & X_{13} & X_{23} \\ X_{12} & 1 & X_{23} & X_{13} \\ X_{13} & X_{23} & 1 & X_{12} \\ X_{23} & X_{13} & X_{12} & 1 \end{pmatrix}. \tag{32}$$

Theorem 9 *All the triangle inequalities hold for the second lifting.*

Proof. The theorem holds for any dimension. For brevity, only $n = 3$ is presented. From the psd condition for the matrix X in Eq. (32),

$$e^T X e = 4 + 4X_{12} + 4X_{13} + 4X_{23} \geq 0$$

This yields the triangle inequality $X_{12} + X_{13} + X_{23} \geq -1$. Similarly, other triangle inequalities can be obtained. $\square$

If F_n is the projection of the second lifting back to S_n, then $C_n \subseteq F_n \subseteq \mathcal{E}_n \cap M_n$ with strict containments for $n \geq 5$. Further results are given in [1, 5, 3, 14, 2].

7 Conclusion

Semidefinite programming is currently one of the most active research areas in optimization. I hope that these lectures wet your appetite to learn more about it, and I will conclude with some references for further reading.

The Handbook [22] contains a wealth of information on SDP and applications. It also contains an extensive bibliography up to the time of publication.

Excellent additional sources on the theoretical aspects underlying SDP are [9, 8, 19, 11]

The approximation algorithm of Goemans and Williamson is published in [10]. Two recent surveys covering many applications of SDP to combinatorial optimization problems are [4] and [15].

Research papers relevant to parts of these lectures are [18, 16, 21, 12, 5, 3, 13, 14]. Other lifting schemes, which I did not have time to mention, have been proposed in the literature. Important references include [6, 20, 17, 7].

Finally, if you have any questions or want more information about SDP, please contact me at anjos@stanfordalumni.org.

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