pointingly small size. In Chapter 15, in fact, we shall introduce the notion of *NP-completeness* in order to characterize those problems which, like ILP, appear to be intrinsically difficult exactly because of their generality.

13.2 Total Unimodularity

Recall that in the max-flow and weighted bipartite matching problems, for example, solutions to the linear program without special integer constraints were, nevertheless, always integer. It is natural to ask what is at the root of our good fortune in such cases, so that we can take full advantage of such a mechanism. To answer this question, we first need the following definition of a central concept.

Definition 13.1

A square, integer matrix B is called *unimodular* (UM) if its determinant $\det(B) = \pm 1$. An integer matrix A is called *totally unimodular* (TUM) if every square, nonsingular submatrix of A is UM. □

If B is formed from a subset of m linearly independent columns of A, it determines the basic solution

$$x = B^{-1}b = \frac{B^{adj}b}{\det(B)}$$

where B^{adj} is the adjoint of B, and so if B is UM and b is integer (which we always assume), x is integer. If we define the polytope

$$R_1(A) = \{x: Ax = b, \quad x \geq 0\}$$

to be the usual feasible set for the standard form LP, we have proved the following theorem.

Theorem 13.1 *If A is TUM, then all the vertices of $R_1(A)$ are integer for any integer vector b.*

Thus a standard form LP with TUM matrix will always lead to an integer optimum when solved by the simplex algorithm.

When an LP is formulated with inequality constraints, the same result holds. Let the corresponding polytope be

$$R_2(A) = \{x: Ax \leq b, \quad x \geq 0\}$$

Then we have the next theorem.

Theorem 13.2 *If A is TUM, then all the vertices of $R_2(A)$ are integer for any integer vector b.*

Proof This amounts to showing that if A is TUM, so is $(A\,|\,I)$, for then we can add slack variables and apply Theorem 13.1. Let C be a square, nonsingular submatrix of $(A\,|\,I)$. The rows of C can be permuted so that it can be written

$$C = \left(\begin{array}{c|c} B & O \\ \hline D & I_k \end{array}\right)$$

where I_k is an identity matrix of size k and B is a square submatrix of A, possibly with its rows permuted. Therefore

$$\det(C) = \det(B) = \pm 1$$

because A is TUM and C is nonsingular. □

We shall now show that the cases we have observed in previous chapters where integer solutions were automatic were in fact cases where the constraint matrix was TUM. The convenient sufficient (but not necessary) condition is given by Theorem 13.3.

Theorem 13.3 *An integer matrix A with $a_{ij} = 0, \pm 1$ is TUM if no more than two nonzero entries appear in any column, and if the rows of A can be partitioned into two sets I_1 and I_2 such that:*

1. *If a column has two entries of the same sign, their rows are in different sets;*
2. *If a column has two entries of different signs, their rows are in the same set.*

Proof The proof is by induction on the size of submatrices. For the basis, we need only observe that any submatrix of one element is TUM. Let C be any submatrix of size k. If C has a column of all zeros, it is singular. If C has a column with one nonzero entry, we can expand its determinant along that column, and the result follows from the induction hypothesis.

The last case occurs when C has two nonzero entries in every column. Then Conditions 1 and 2 of the theorem imply that

$$\sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij} \qquad \text{for every } j$$

That is, a linear combination of rows is zero, and hence $\det(C) = 0$. □

We then have the desired result.