COMP 567 Assign. 3 Solutions

Question 1

If we let, x_1, x_2, and x_3 represent the acres of wheat, corn, and sugar beet planted respectively, we can write the SP in concise form as:

$$\begin{array}{rl} \min & 150x_1 + 230x_2 + 260x_3 - Exp_\xi[Q(x_1, x_2, x_3)] \\ \textit{subject to} & x_1 + x_2 + x_3 \leq 500 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Where $Exp_\xi[Q(x_1, x_2, x_3)]$ is the expected profit of the farmer purchasing/selling his commmodities with respect to the random variable ξ representing the weather. The weather is good, bad, or normal with a probability of 0.3,0.4, and 0.3 resectively. We let $\xi = 1$ for good weather, $\xi = 2$ for normal weather, and $\xi = 3$ for bad weather.

Let w_1 and w_2 be the amount of wheat and corn sold, y_1 and y_2 be the amount of wheat and corn bought, and w_3, w_4 be the amount of sugar beets sold below and above the quota, respectively. The second stage problem can be expressed as:

$$\begin{array}{rl} Q(x_1, x_2, x_3) = \max & 36w_3 + 10w_4 + q_{3,\xi}w_1 + q_{4,\xi}w_2 - q_{1,\xi}y_1 - q_{2,\xi}y_2 \\ \textit{subject to} & t_{1,\xi}x_1 + y_1 - w_1 \geq 200 \\ & t_{2,\xi}x_2 + y_2 - w_4 \geq 240 \\ & w_3 + w_4 \geq t_{3,\xi}x_3 \\ & w_3 \leq 6000 \\ & y_1, y_2, w_1, w_2, w_3, w_4 \geq 0 \end{array}$$

Where, $q = [214.2, 189, 153, 135]$ and $t = [3, 3.6, 24]$ if $\xi = 1$, $q = [238, 210, 170, 150]$ and $t = [2.5, 3.0, 20]$ if $\xi = 2$, and $q = [261.8, 231, 187, 165]$ and $t = [2, 2.4, 16]$ if $\xi = 3$.

Writing this in extensive form gives:

$$\begin{array}{rl} \min & 150x_1 + 230x_2 + 260x_3 \\ & -0.3(153w_{1,1} - 214.2y_{1,1} + 135w_{2,1} - 189y_{2,1} + 36w_{3,1} + 10w_{4,1}) \\ & -0.4(170w_{1,2} - 238y_{1,2} + 150w_{2,2} - 210y_{2,2} + 36w_{3,2} + 10w_{4,2}) \\ & -0.3(187w_{1,3} - 261.8y_{1,3} + 165w_{2,3} - 231y_{2,3} + 36w_{3,3} + 10w_{4,3}) \\ \textit{subject to} & x_1 + x_2 + x_3 \leq 500 \\ & 3x_1 + y_{1,1} - w_{1,1} \geq 200 \\ & 2.5x_1 + y_{1,2} - w_{1,2} \geq 200 \end{array}$$

$$
\begin{aligned}
&2x_1 + y_{1,3} - w_{1,3} \geq 200 \\
&3.6x_2 + y_{2,1} - w_{2,1} \geq 240 \\
&3x_2 + y_{2,2} - w_{2,2} \geq 240 \\
&2.4x_2 + y_{2,3} - w_{2,3} \geq 240 \\
&w_{3,i} \geq 6000 \qquad for\ i \in \{1,2,3\} \\
&w_{3,1} + w_{4,1} \leq 24x_3 \\
&w_{3,2} + w_{4,2} \leq 20x_3 \\
&w_{3,3} + w_{4,3} \leq 16x_3 \\
&x_j, y_{1,j}, y_{2,j}, y_{3,j}, w_{1,j}, w_{2,j}, w_{3,j}, w_{4,j} \geq 0 \forall j \in \{1,2,3\}
\end{aligned}
$$

Solving with CPLEX gives, $x_1 = 120, x_2 = 80, x_3 = 300$ with a profit of 107245.60 dollars.

Question 2

People took a few different approaches to discretizing the normal distribution. Depending on what they did, the answers varied. Since there was no real clear correct way of doing this, your answer may have varied from this one, but that doesn't mean it wasn't correct.

We would like to find probabilities that the demand is 40,60,80,100,120,140 or 160.

Let x be the number of papers we buy, y_i be the number of papers we sell, and w_i be the number of papers we return in scenario i (where we have 7 scenario's for the number of paper's sold). If ξ_i is the demand in scenario i, we can express the expected profit by the following IP:

$$
\begin{aligned}
\max \qquad & -0.5x + \sum_{i=1}^{7} p_i(y_i + 0.25w_i) \\
subject\ to \qquad & x \leq 200 \\
& y_k \leq \xi_k \forall k = 1, ..., 7 \\
& y_k + w_k \leq x \forall k = 1, ..., 7 \\
& x, y_k, w_k \in \mathbb{Z}^+ \forall k = 1, ..., 7
\end{aligned}
$$

Based on what values of p_i people had for the discretization, different values of x were obtained. If you chose to take the probability that a random variable with distribution $N(100, 20)$ was less than 40 to be p_1, between 40 and 60 to be p_2, between 60 and 80 to be p_3, between 80 and 100 to be p_4, between 100 and 120 to be p_5, between 120 and 140 to be p_6, and lastly greater than 140 to be p_7, then the p vector would be: $(0.006, 0.061, 0.242, 0.383, 0.242, 0.061, 0.006)$. Solving this IP gives an optimal solution of $x = 100$, this differs from the optimal solution in the continuous case is class that had an optimal integer solution of 109.

Question 3
In this question, step 2 could be skiped, since:
$w = \min v_1^+ + v_2^+ + v_3^+ + v_1^- + v_2^- + v_3^-$ subject to $Wy + Iv^+ + Iv^- = h^k - T^k x$, $y, v^+, v^- \geq 0$, can always be solved feasibly with $w = 0$ and $y \geq 0$ no matter what value x is. One can take v^+ and v^- to be the all zero vectors and if $h_k(1) - x$ is negative, $y = (0, 0, 0, -(h_k(1) - x), h_k(2), h_k(3))$ is a solution with $w = 0$ and if $h_k(1) - x$ is positive, $y = (h_k(1) - x, 0, 0, 0, h_k(2), h_k(3))$ is a solution with $w = 0$ for $k = 1, 2$.

So you could skip step 2 since no feasibility cuts will be added.

Iteration 1
Step 1: We begin with an optimal solution to the LP:

$$\begin{aligned} \min \quad & z = \theta \\ s.t. \quad & x \leq 20 \\ & x \geq -20 \\ & \theta \in \mathbb{R} \end{aligned}$$

Given $x^1 = -1$ we get $(x^1, \theta^1) = -1, -\infty)$.

Step 3:
We wish to solve:

$$\begin{aligned} \min \quad & w = q^1 y \\ s.t. \quad & Wy = h^1 - T^1 x^1 \\ & y \geq 0 \end{aligned}$$

pluging in q^1, W,h^1,T^1 and x^1 and solving returns $w = 0$ and $y = (0, 0, 0, 0, 2, 7)$. With simplex multiplier (or dual variables) $\pi_1^1 = (0, 0, 0)$.

For $k = 2$ we solve again:

$$\begin{aligned} \min \quad & w = q^2 y \\ s.t. \quad & Wy = h^2 - T^2 x^1 \\ & y \geq 0 \end{aligned}$$

We could solve the above LP, or solve the dual directly to get the simplex multipliers (equivalent). The dual is:

$$\begin{aligned} \max \quad & (h^2 - T^2 x^1)^T \pi_2^1 \\ s.t. \quad & W^T \pi_2^1 \leq q^2 \\ & \pi_2^1 \in \mathbb{R}^3 \end{aligned}$$

Plugging in the values for h^2,T^2,x^1, W, q^2 and solving the LP gives the optimal solution $\pi_2^1 = (3/2, 0, 0)$.

We can define

$$E_1 = \frac{1}{2}\pi_1^1(T^1)^T + \frac{1}{2}\pi_2^1(T^2)^T = \frac{1}{2}(0,0,0)(1,0,0)^T + \frac{1}{2}(3/2,0,0)(1,0,0)^T = \frac{3}{4}$$

and

$$e_1 = \frac{1}{2}(0,0,0)(-1,2,7)^T + \frac{1}{2}(3/2,0,0)(0,2,7)^T = 0$$

Now, $w^1 = e_1 - E_1 x^1 = \frac{3}{4} > -\infty = \theta^1$ so we add constraint: $\frac{3}{4}x + \theta \geq 0$ to the original problem.

Iteration 2
Step 1,

Solving,

$$\begin{array}{ll} \min & \theta \\ s.t. & -20 \leq x \leq 20 \\ & \frac{3}{4}x + \theta \geq 0 \\ & \theta \in \mathbb{R} \end{array}$$

Returns an optimal solutions of: $(x^2, \theta^2) = (20, -15)$.

Step 3,
Solving the dual:

$$\begin{array}{ll} \max & (h^k - T^k x^2)^T \pi_k^2 \\ s.t. & W^T \pi_k^2 \leq q^k \\ & \pi_k^2 \in \mathbb{R}^3 \end{array}$$

for $k = 1, 2$ we get the simplex multipliers, $\pi_1^2 = (0,0,0)$ and $\pi_2^2 = (-1,-1,-5/7)$.

Calculating e_2 and E_2 we get:

$$E_2 = \frac{1}{2}(0,0,0)(1,0,0)^T + \frac{1}{2}(-1,-1,-5/7)(1,0,0)^T = \frac{-1}{2}$$

$$e_2 = \frac{1}{2}(0,0,0)(-1,3,7)^T + \frac{1}{2}(-1,-1,-5/7)(0,2,7)^T = \frac{-7}{2}$$

Now, $w_2 = e_2 - E_2 x^2 = -7/2 + \frac{20}{2} = \frac{13}{2} \geq -15 = \theta^2$ so we add the cut $\frac{-1}{2}x + \theta \geq \frac{-7}{2}$.

Iteration 3
Step 1,

Solving,

$$\begin{array}{ll} \min & \theta \\ s.t. & -20 \leq x \leq 20 \\ & \dfrac{3}{4}x + \theta \geq 0 \\ & \dfrac{-1}{2}x + \theta \geq \dfrac{-7}{2} \\ & \theta \in \mathbb{R} \end{array}$$

Returns an optimal solutions of: $(x^3, \theta^3) = (14/5, -21/10)$.

Step 3,
Solving the dual,

$$\begin{array}{ll} \max & (h^k - T^k x^3)^T \pi_k^3 \\ s.t. & W^T \pi_k^3 \leq q^k \\ & \pi_k^3 \in \mathbb{R}^3 \end{array}$$

for $k = 1, 2$ we get the simplex multipliers, $\pi_1^3 = (0, 0, 0)$ and $\pi_2^3 = (\frac{-2}{7}, \frac{-2}{7}, 0)$.

Calculating e_3 and E_3 we get:

$$E_3 = \frac{1}{2}(0,0,0)(1,0,0)^T + \frac{1}{2}(\frac{-2}{7}, \frac{-2}{7}, 0)(1,0,0)^T = \frac{-1}{7}$$

$$e_3 = \frac{1}{2}(0,0,0)(-1,3,7)^T + \frac{1}{2}(\frac{-2}{7}, \frac{-2}{7}, 0)(0,2,7)^T = \frac{-2}{7}$$

Now, $w_3 = e_3 - E_3 x^3 = -2/7 + (\frac{1}{7})(\frac{14}{5}) = \frac{13}{2} \geq -21/20 = \theta^3$ so we add the cut $\frac{-1}{7}x + \theta \geq \frac{-2}{7}$ to our problem.