COMP 567 Assign. 2 Solutions

Question 1
In this question you were asked to show that minimum covering by cliques is dual to maximum stable set.

For any stable set $S = \{v_1, ..., v_k\}$ and covering by cliques $C = \{C_1, ..., C_l\}$ at most one element of S can be in some clique of C, since if $v_i \in S$ and $v_j \in S$ are both in some $C_k \in C$ they must have an edge between them contradicting S being a stable set. Therefore, by pigeon hole $|S| \leq |C|$ for any stable set S and clique cover C and in particular the size of the maximum stable set is less than or equal to the size of the minimum clique cover.

We can also formulate this as an IP, let $\mathcal{C}$ be the familly of cliques in a graph G, the solution to the following IP is the minimum clique cover:

$$\begin{aligned} min \quad & \sum_{C_i \in \mathcal{C}} x_{C_i} \\ s.t. \quad & \sum_{C_i \ni v_j} x_{C_i} \geq 1 \; for \; j = 1..n \\ & x_{C_i} \in \{0,1\} \qquad \forall C_i \in \mathcal{C} \end{aligned}$$

This has the dual:

$$\begin{aligned} max \quad & \sum_{v \in V(G)} y_v \\ s.t. \quad & \sum_{v \in C} y_v \leq 1 \qquad \forall C \in \mathcal{C} \\ & y_v \in \{0,1\} \qquad \forall v \in V(G) \end{aligned}$$

The solution to the dual IP is the size of the maximum clique.

(b) We just need to find a clique cover C, then we know that we can upper bound the size of a maximum stable set by $|C|$. In this graph $C = \{\{1,2\}, \{3,12,5\}, \{4,14\}, \{5,8\}, \{6,9\}, \{7\}, \{10,11,13\}\}$ forms a clique cover so we know that the maximum size of a stable set is at most $|C| = 7$.

Question 2
Since the objectives are the same for $(P1), (P2)$, and $(P3)$ we just need to show that the feasible regions are relaxations.

If we rewrite the constraint of $(P1)$ as:

$$\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + (2 - x_1 + x_2 - x_3 + x_4 - 3x_5)$$

Then we can let $w = 2 - x_1 + x_2 - x_3 + x_4 - 3x_5$, we know that $w \in \mathbb{Z}^1$ since $x \in \mathbb{Z}^5_+$ it just remains to show that $w \geq 0$ or equivalently $x_1 - x_2 + x_3 - x_4 + 3x_5 \geq 2$.

Taking the Gomory-Chvatal cut of the original constraint we get that:

$$\lfloor\frac{7}{4}x_1\rfloor + \lfloor\frac{-2}{3}x_2\rfloor + \lfloor\frac{5}{2}x_3\rfloor + \lfloor\frac{-5}{12}x_4\rfloor + \lfloor\frac{19}{6}x_5\rfloor \geq \lfloor\frac{8}{3}\rfloor$$

This gives: $x_1 - x_2 + x_3 - x_4 + 3x_5 \geq 2$ as desired. It follows that $(P2)$ is a relaxation of $(P1)$ since the w in $(P2)$ is less constrained.

(ii) This part was pretty straight forward, if we drop w from the equality constraint in $P2$, we get: $\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 \geq \frac{2}{3}$. Relaxing the integrality constraint so that $x \in \mathbb{R}^5_+$ we get $(P3)$.

Note that the constraint of $(P3)$ is the constraint we would get by applying the Gomory cutting plane procedure to the constraint in $(P1)$.

Question 3 We begin by adding the slack variable x_4 and switching the problem to a maximization so that it coincides with the cutting procedure from the book. Solving the LP we get the dictionary:

$$\begin{aligned} z &= -\frac{7337}{95} - \frac{3x_1}{19} - \frac{10x_2}{19} - \frac{23x_4}{95} \\ x_3 &= \frac{319}{95} - \frac{4x_1}{19} - \frac{7x_2}{19} + \frac{x_4}{95} \end{aligned}$$

Using the Gomory Cutting Plane procedure on the z row we get:

$$(\frac{3}{19} - \lfloor\frac{3}{19}\rfloor)x_1 + (\frac{10}{19} - \lfloor\frac{10}{19}\rfloor)x_2 + (\frac{23}{95} - \lfloor\frac{23}{95}\rfloor)x_4 \geq \frac{-7337}{95} - \lfloor\frac{-7337}{95}\rfloor$$

simplifying and adding the slack variable x_5 we get: $x_5 = \frac{-73}{95} + \frac{3x_1}{19} + \frac{10x_2}{19} + \frac{23x_4}{95}$ We then add this constraint to our dictionary and resolve the LP and pick a new constraint row to do the Gomory Cutting Plane procedure on. We repeat this process until the LP solution is integral for our original variables x_1, x_2, and x_3. See the maple output for this question for the complete solution to this question. Note that in practise it has been shown that adding all valid cut for your dictionary (do cuts on each row, add them, then solve the LP, instead of one at a time) has been shown to be computationally more efficient.

Question 4 By solving the original system, with a slack variable x_4 added, as an LP relaxation we see that the LP solution is $(x_1, x_2, x_3, x_4, s) = (0, 32/3, 0, 0, 0)$. So we can think of rewriting the inequality so that x_2 is basic, we get $\frac{2}{3}x_1 + x_2 + 3x_3 + \frac{1}{3}x_4 \leq \frac{32}{3} + \frac{s}{3}$. Using MIR on this inequality and the notation of the book we get that, $f = \frac{32}{3}\lfloor\frac{32}{3}\rfloor = \frac{2}{3}$, $f_1 = \frac{2}{3} - \lfloor\frac{2}{3}\rfloor = \frac{2}{3}$, $f_2 = 0$, $f_3 = 3 - \lfloor 3\rfloor$,so:

$$\lfloor\frac{2}{3}\rfloor x_1 + x_2 + 3x_3 \leq \lfloor\frac{32}{3}\rfloor + \frac{1}{1-\frac{2}{3}}(\frac{s}{3} - \frac{x_4}{3})$$

is a valid inequality. This gives:

$$x_2 + 3x_3 \leq 10 + s - x_4$$

Or $x_5 = 10 - x_2 - 3x_3 + s - x_4$ adding this constraint to our LP and solving we get an integral solution of $(x_1, x_2, x_3, s, x_4, x_5) = (1, 10, 0, 0, 0, 0)$ with has an objective value of: 80.