

Geometric Views of Linear Complementarity Algorithms and Their Complexity

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LCP - Definition

Given: $q \in \mathbf{R}^n$, $M \in \mathbf{R}^{n \times n}$

Find: $z \in \mathbf{R}^n$ so that

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz \geq \mathbf{0}$$

\perp means orthogonal:

$$\begin{aligned} z^T w &= 0 \\ \Leftrightarrow z_i w_i &= 0 \quad \text{all } i = 1, \dots, n. \end{aligned}$$

LP in inequality form

primal: **max** $c^T x$
 subject to $Ax \leq b$
 $x \geq 0$

dual: **min** $y^T b$
 subject to $y^T A \geq c^T$
 $y \geq 0$

Weak duality: x, y feasible (fulfilling constraints)
 \Rightarrow $c^T x \leq y^T A x \leq y^T b$

Strong duality: primal and dual are feasible
 $\Rightarrow \exists$ feasible x, y : $c^T x = y^T b$ (x, y optimal)

LCP generalizes LP

LCP encodes the complementary slackness of strong duality:

$$c^T x = y^T A x = y^T b$$

$$\Leftrightarrow (y^T A - c^T) x = 0, \quad y^T (b - A x) = 0.$$

$$\geq 0 \quad \geq 0 \quad \geq 0 \quad \geq 0$$

LP \Leftrightarrow LCP

$$\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array} \quad \perp \quad \begin{array}{|c|} \hline -c \\ \hline \end{array} \quad \begin{array}{|c|} \hline + A^T y \\ \hline \end{array} \geq 0$$

$$\begin{array}{|c|} \hline b \\ \hline \end{array} \quad \begin{array}{|c|} \hline -Ax \\ \hline \end{array} \geq 0$$

Symmetric equilibria of symmetric games

Given: $n \times n$ **payoff matrix** A for row player
 A^T for column player

mixed strategy x = probability distribution on $\{1, \dots, n\}$
 $\Leftrightarrow x \geq \mathbf{0}$, $\mathbf{1}^T x = 1$

equilibrium (x, x)

$\Leftrightarrow x$ **best response** to x

Remark: As general as $m \times n$ games (A, B) .

Best responses

Given: $n \times n$ **payoff matrix** A ,
mixed strategy y of column player

Ay = vector of **expected payoffs** against y ,
components $(Ay)_i$

x **best response** to y

$\Leftrightarrow x$ maximizes expected payoff $x^T Ay$

best response condition:

$\Leftrightarrow \forall i : x_i > 0 \Rightarrow (Ay)_i = u = \max_k (Ay)_k$

Symmetric equilibria as LCP solutions

equilibrium (x, x) of game with payoff matrix A

$\Leftrightarrow x$ best response to x

$$\Leftrightarrow \begin{array}{l} \mathbf{1}^T x = 1, \\ x \geq \mathbf{0} \quad \perp \quad Ax \leq \mathbf{1}u \end{array}$$

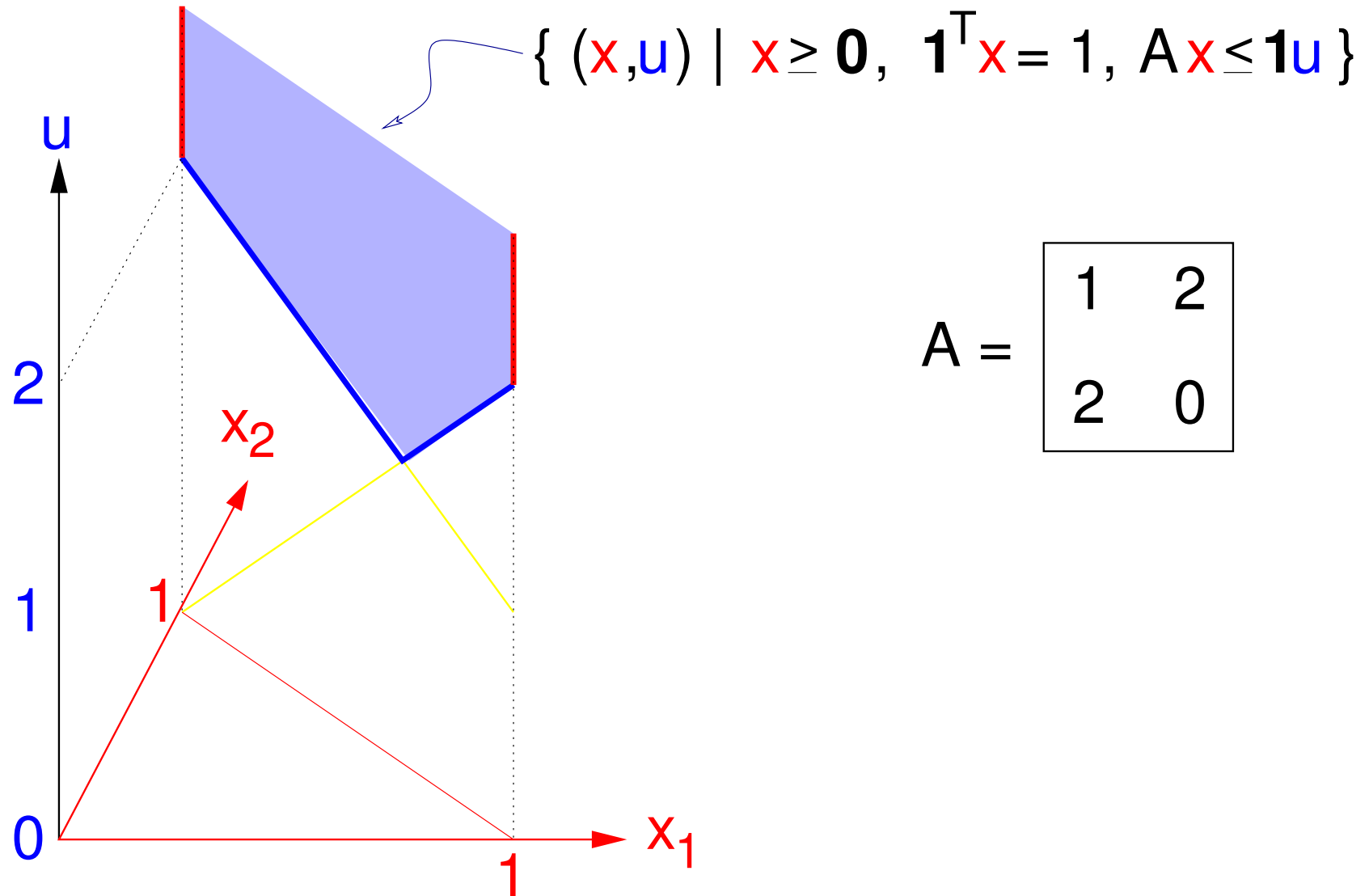
w.l.o.g. $A > 0 \Rightarrow u > 0,$

equilibrium (x, x)

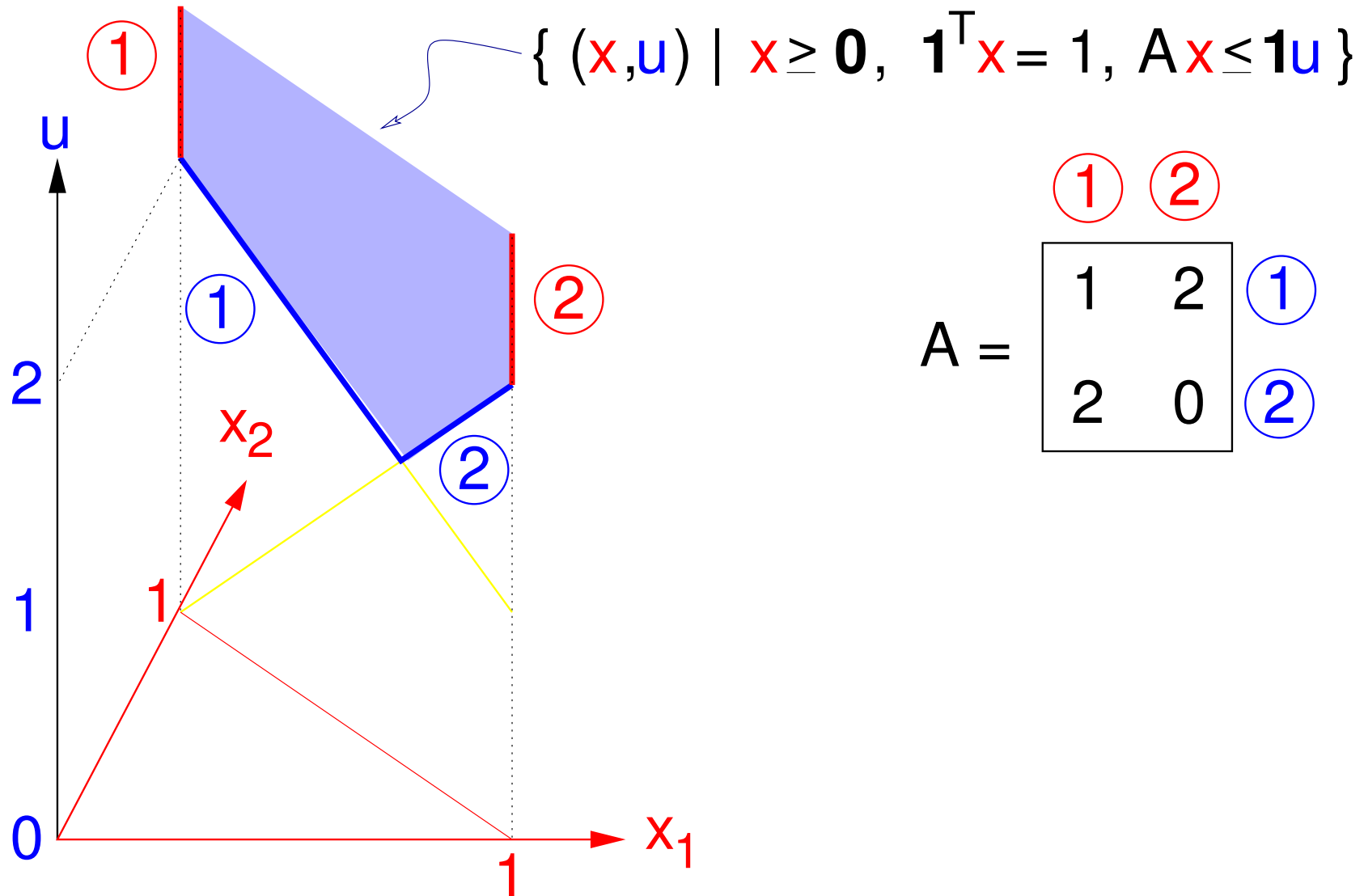
$$\Leftrightarrow z = (1/u) x \quad (1/u = \mathbf{1}^T z),$$

$z \geq \mathbf{0}$	\perp	$Az \leq \mathbf{1}$	"equilibrium z "
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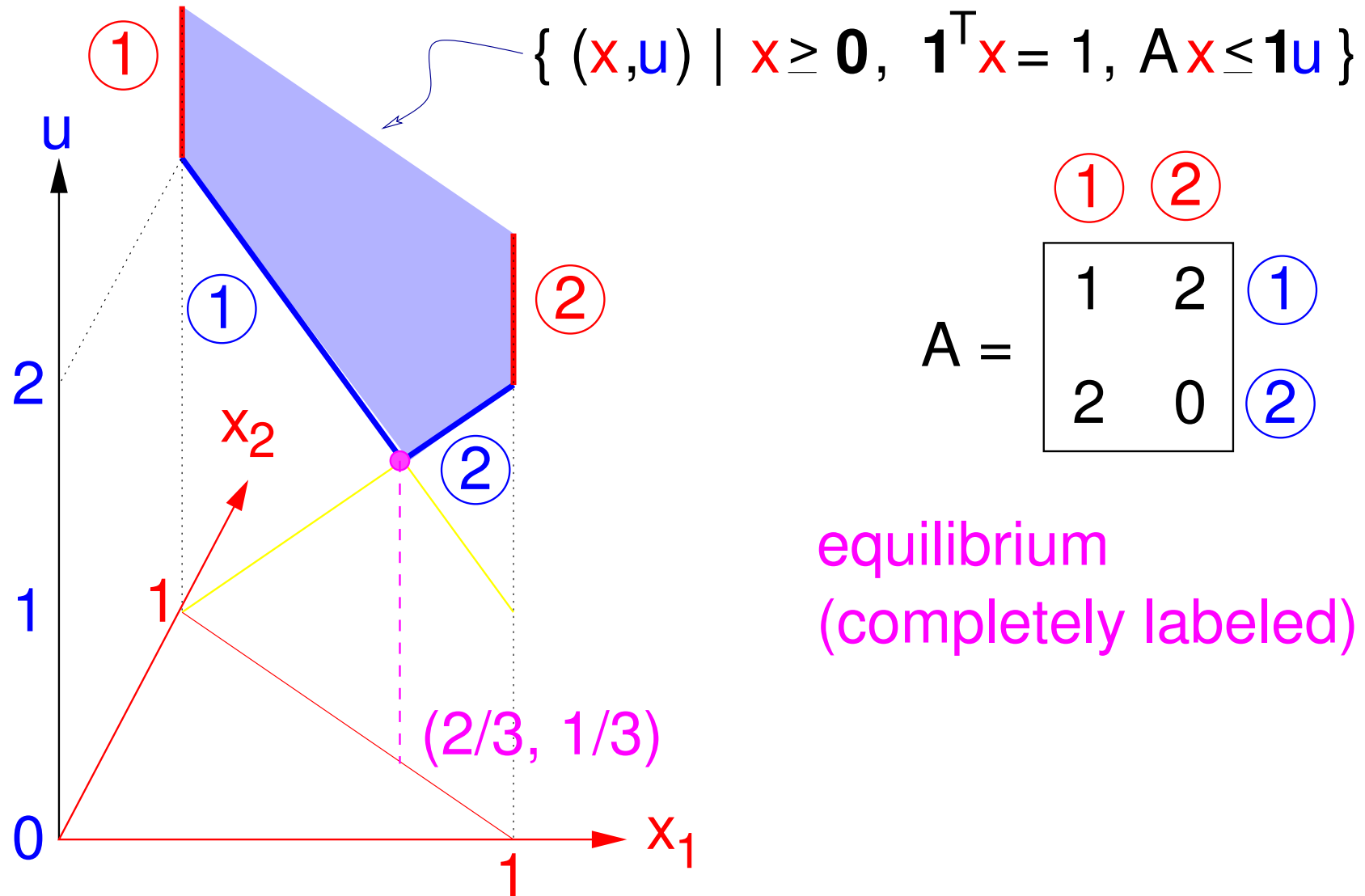
Best response polyhedron



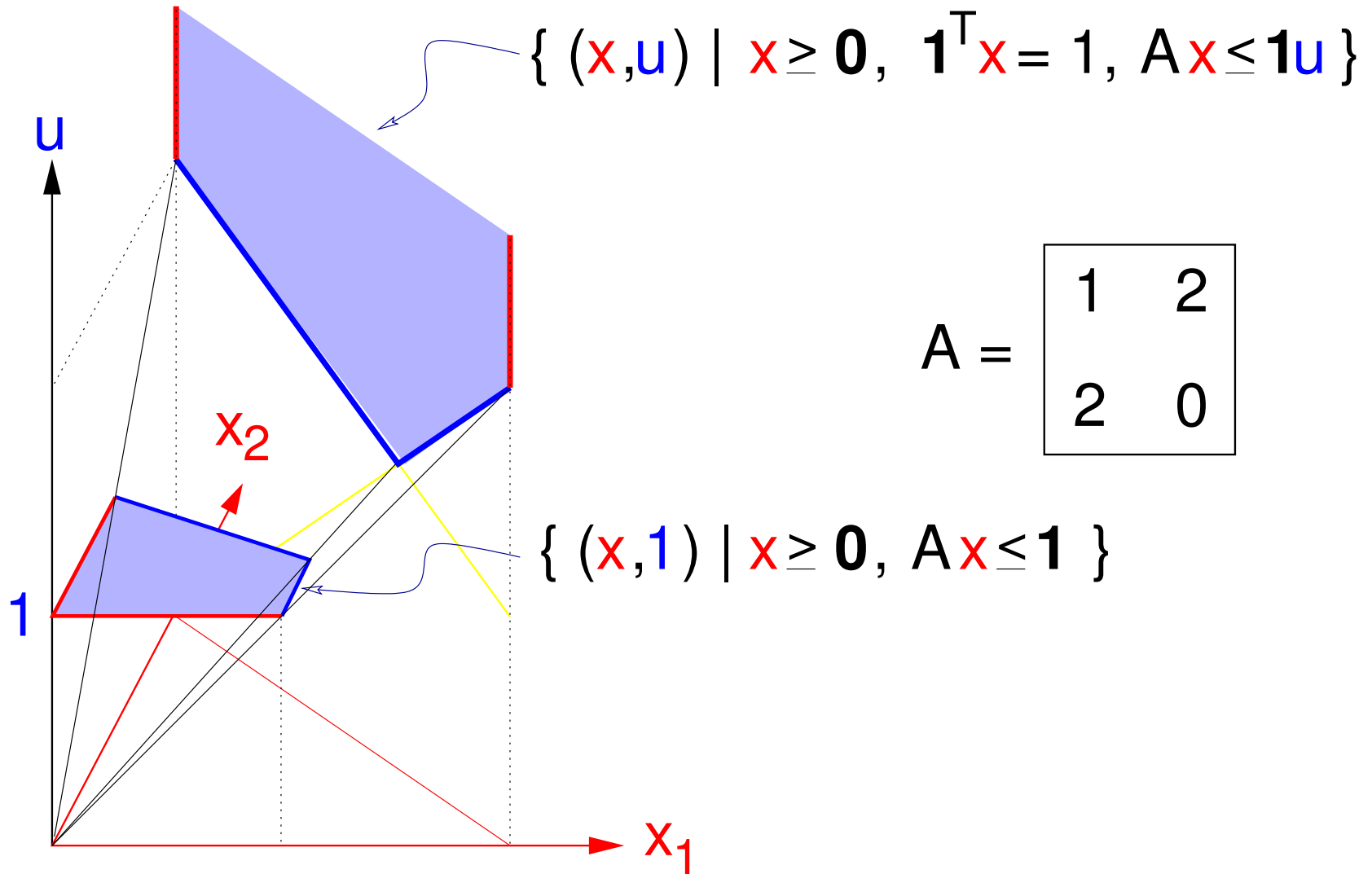
Best response polyhedron



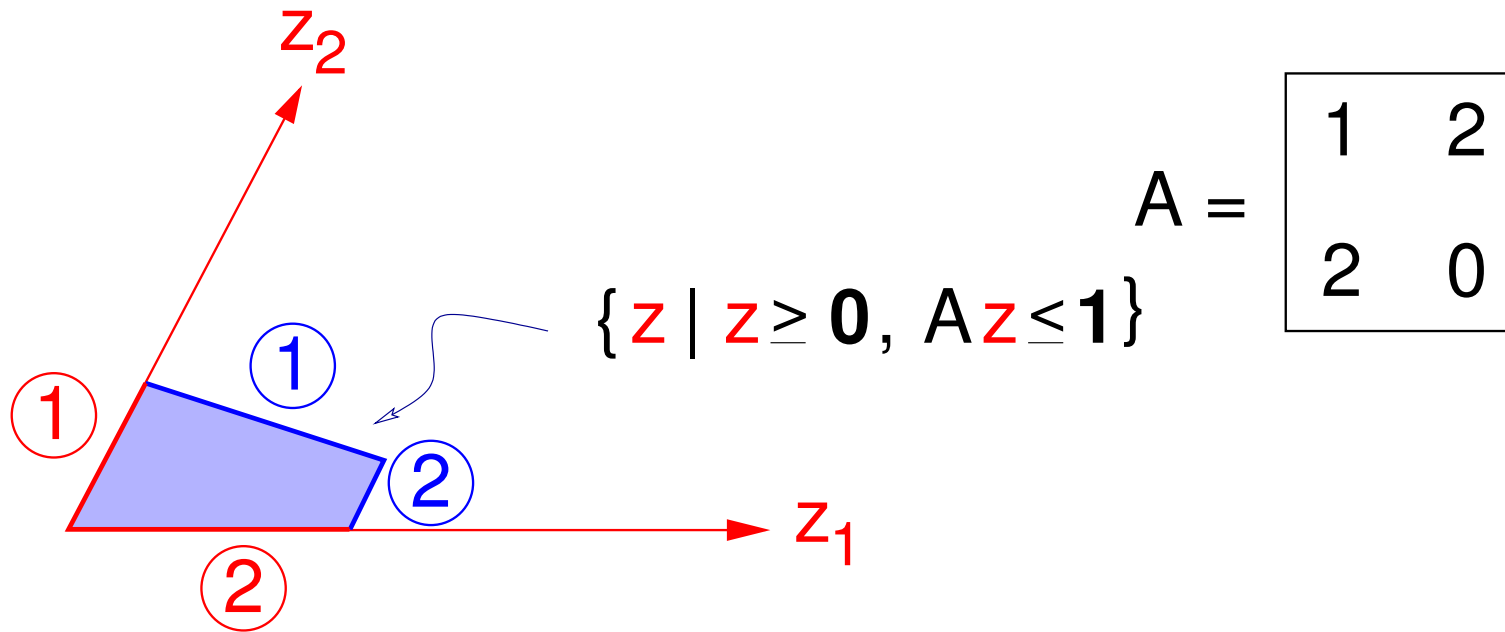
Best response polyhedron



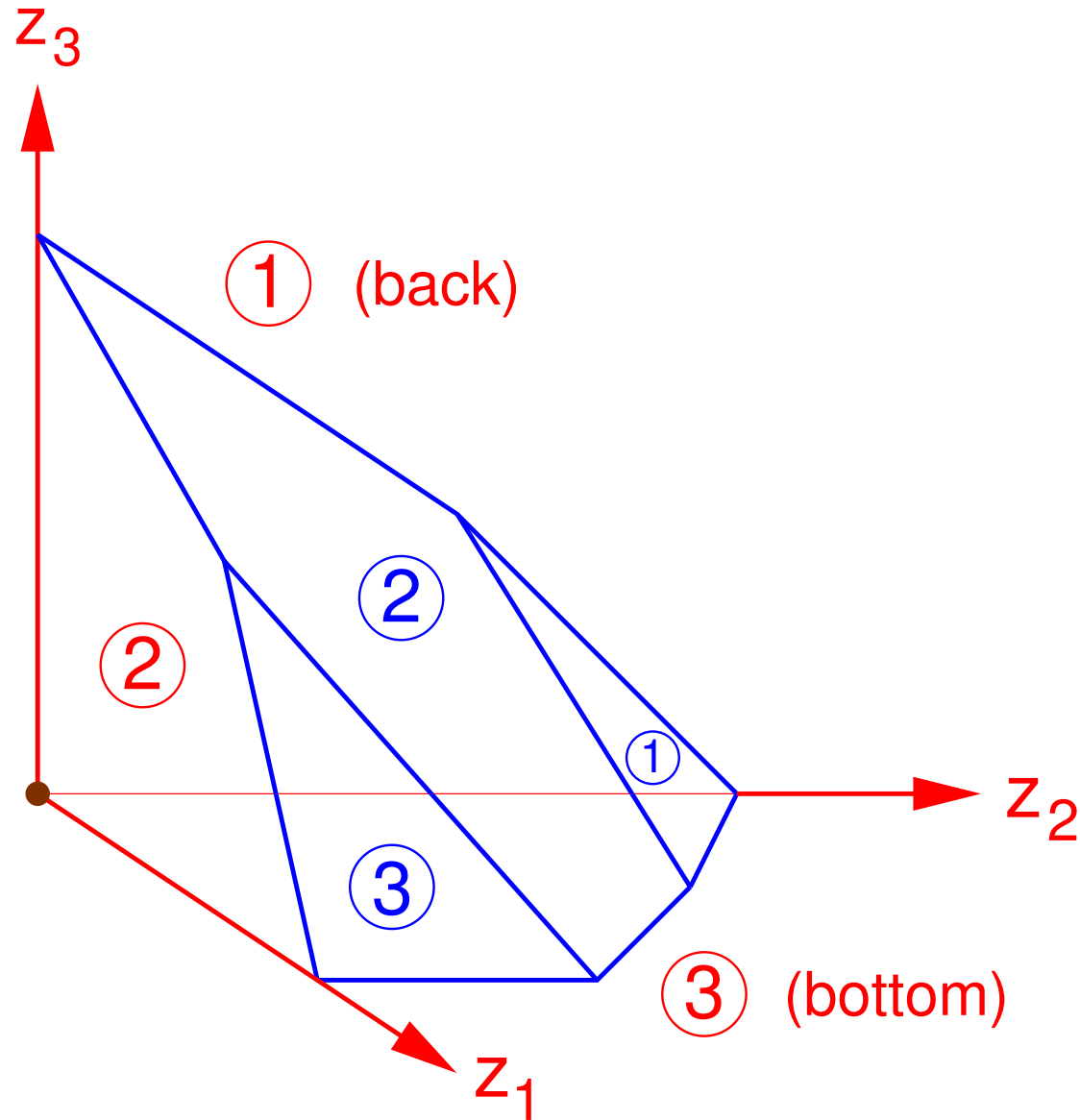
Projective transformation



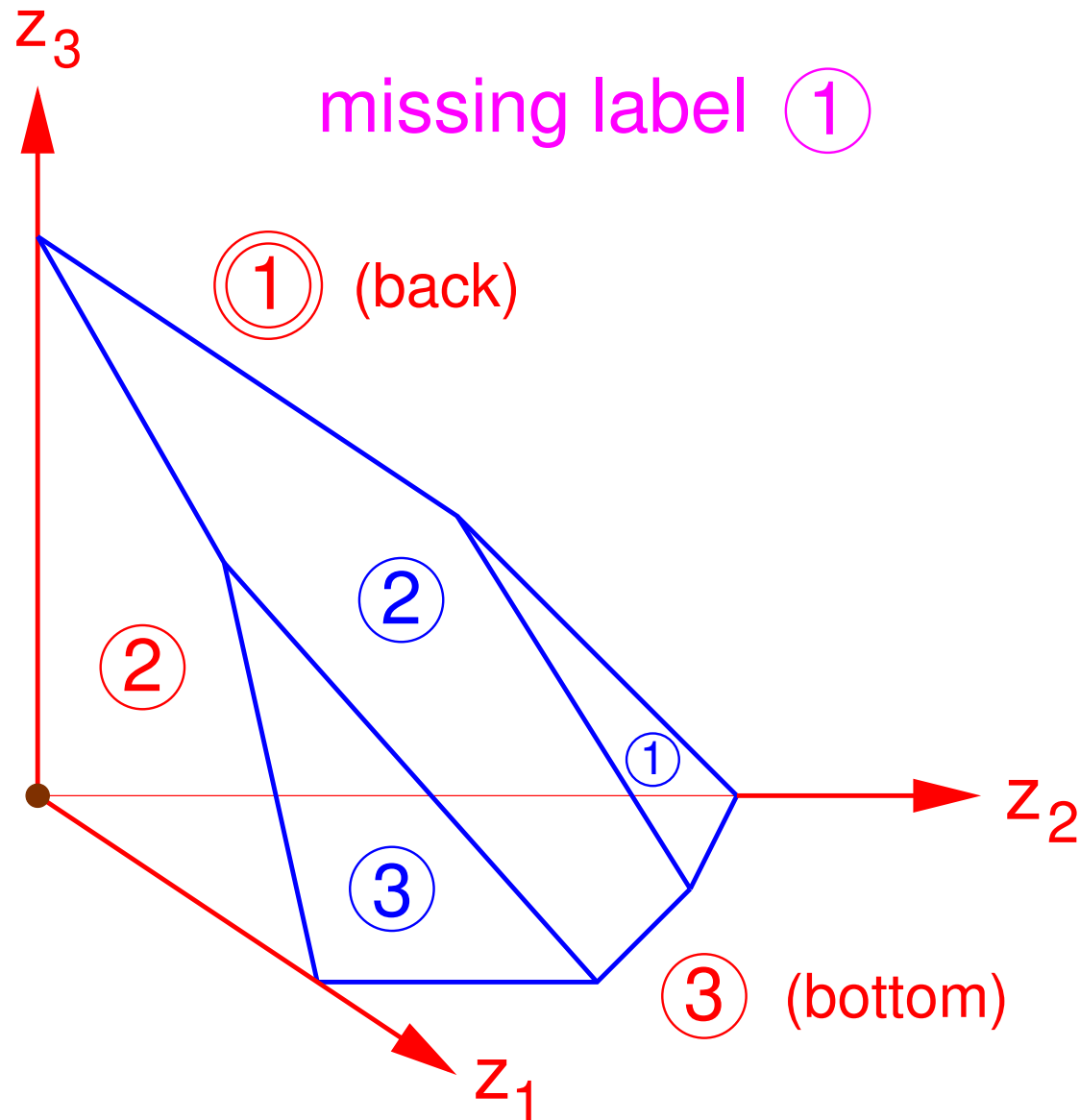
Best response polytope



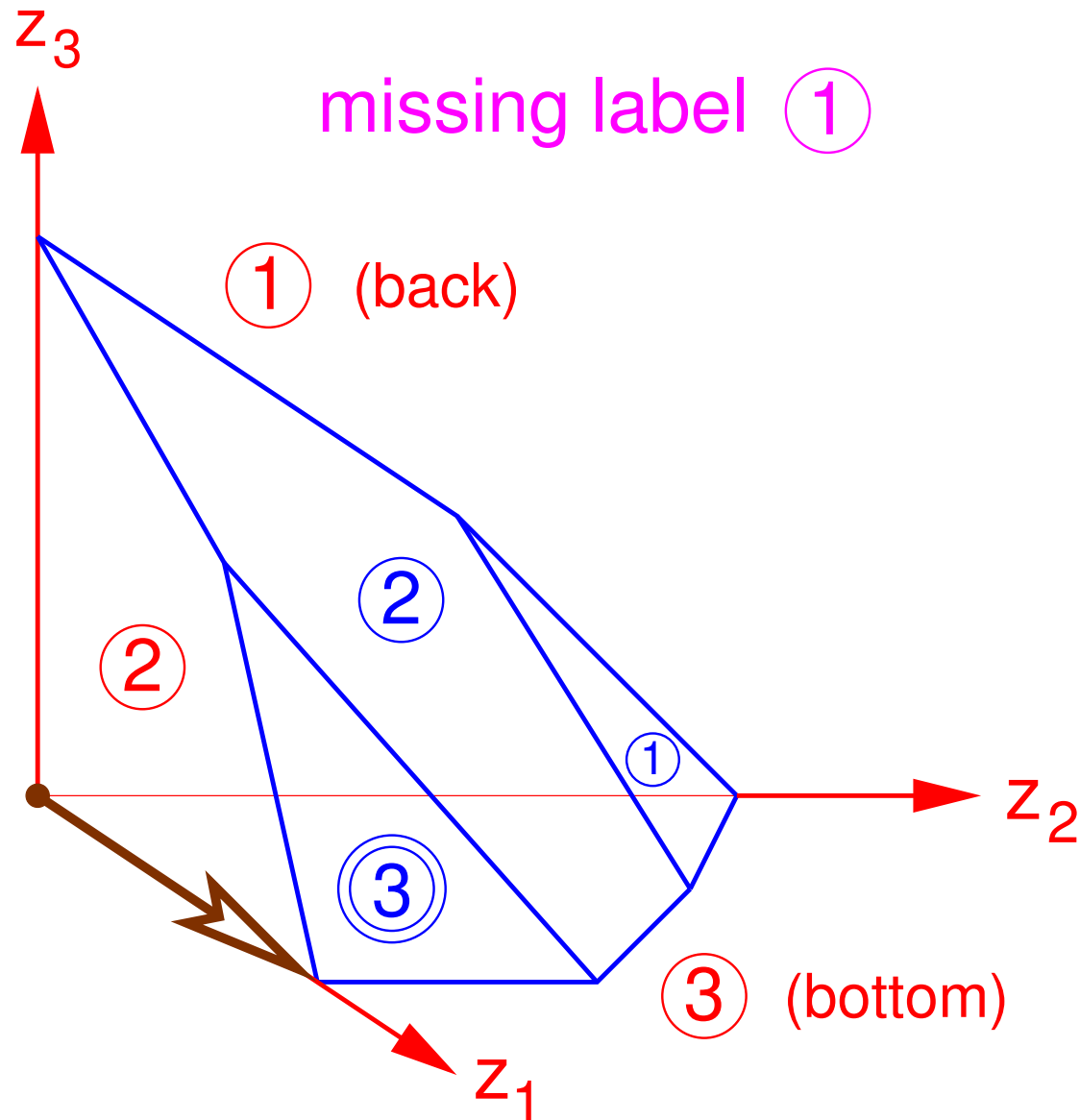
Symmetric Lemke–Howson algorithm



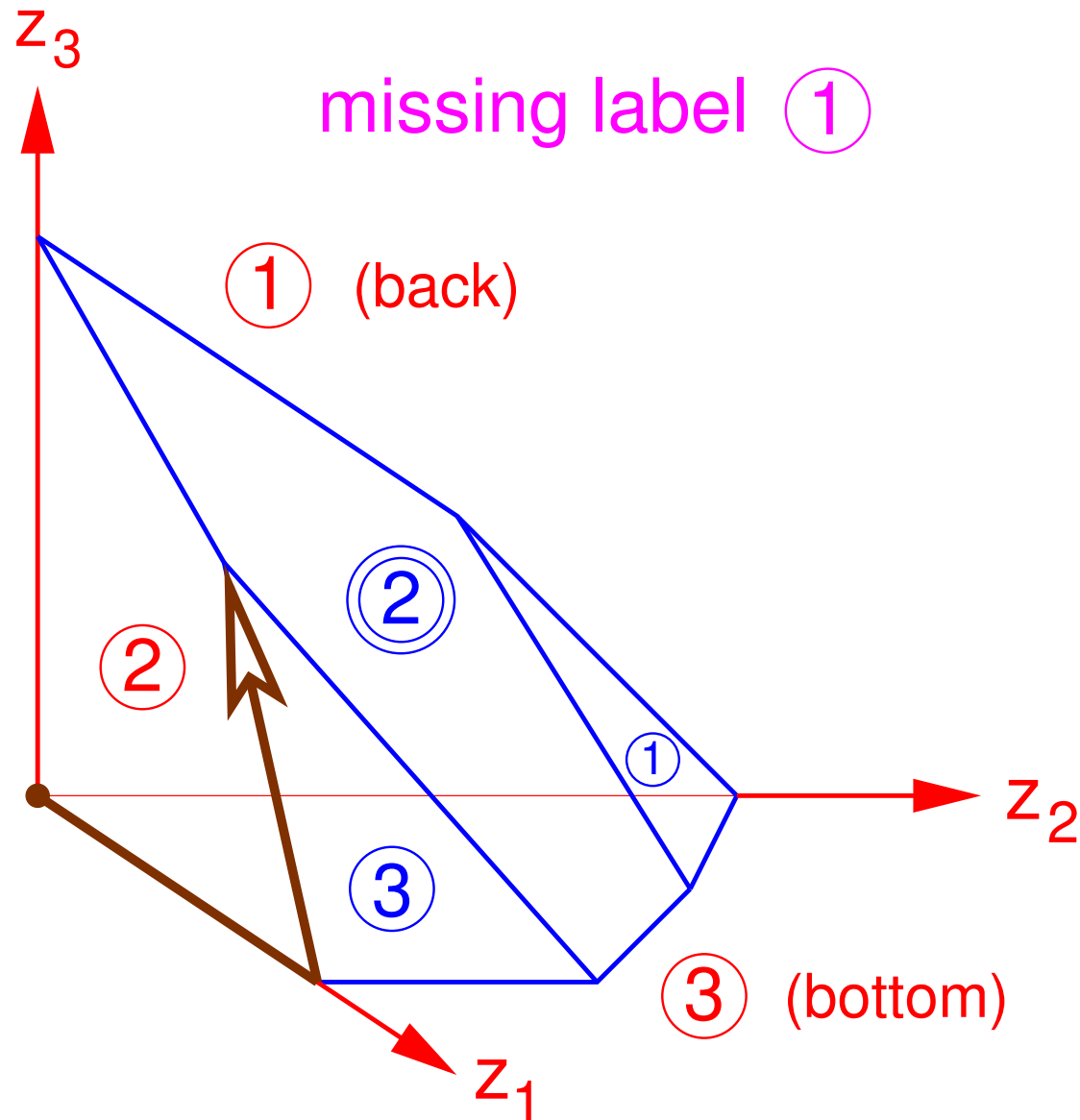
Symmetric Lemke–Howson algorithm



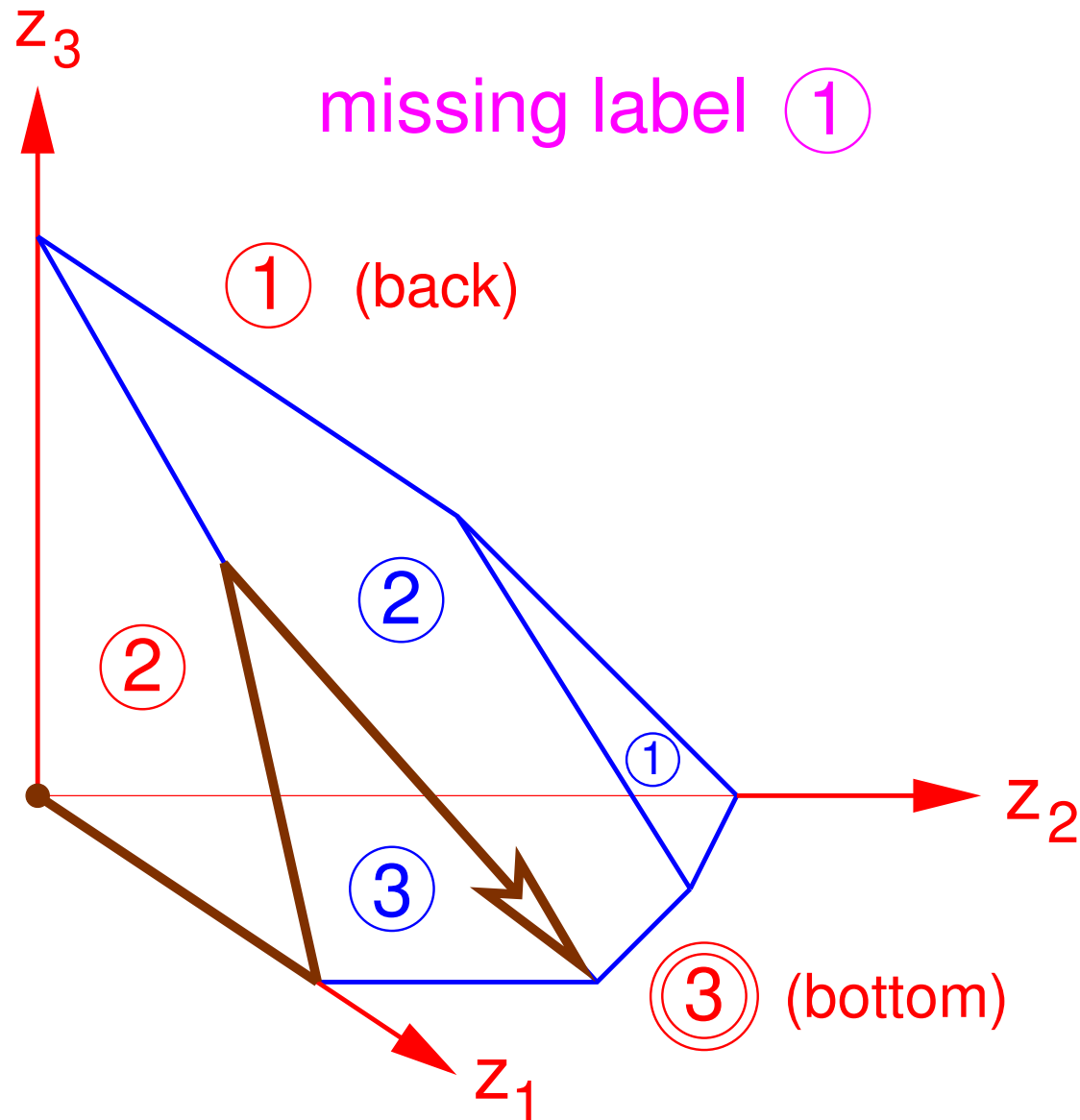
Symmetric Lemke–Howson algorithm



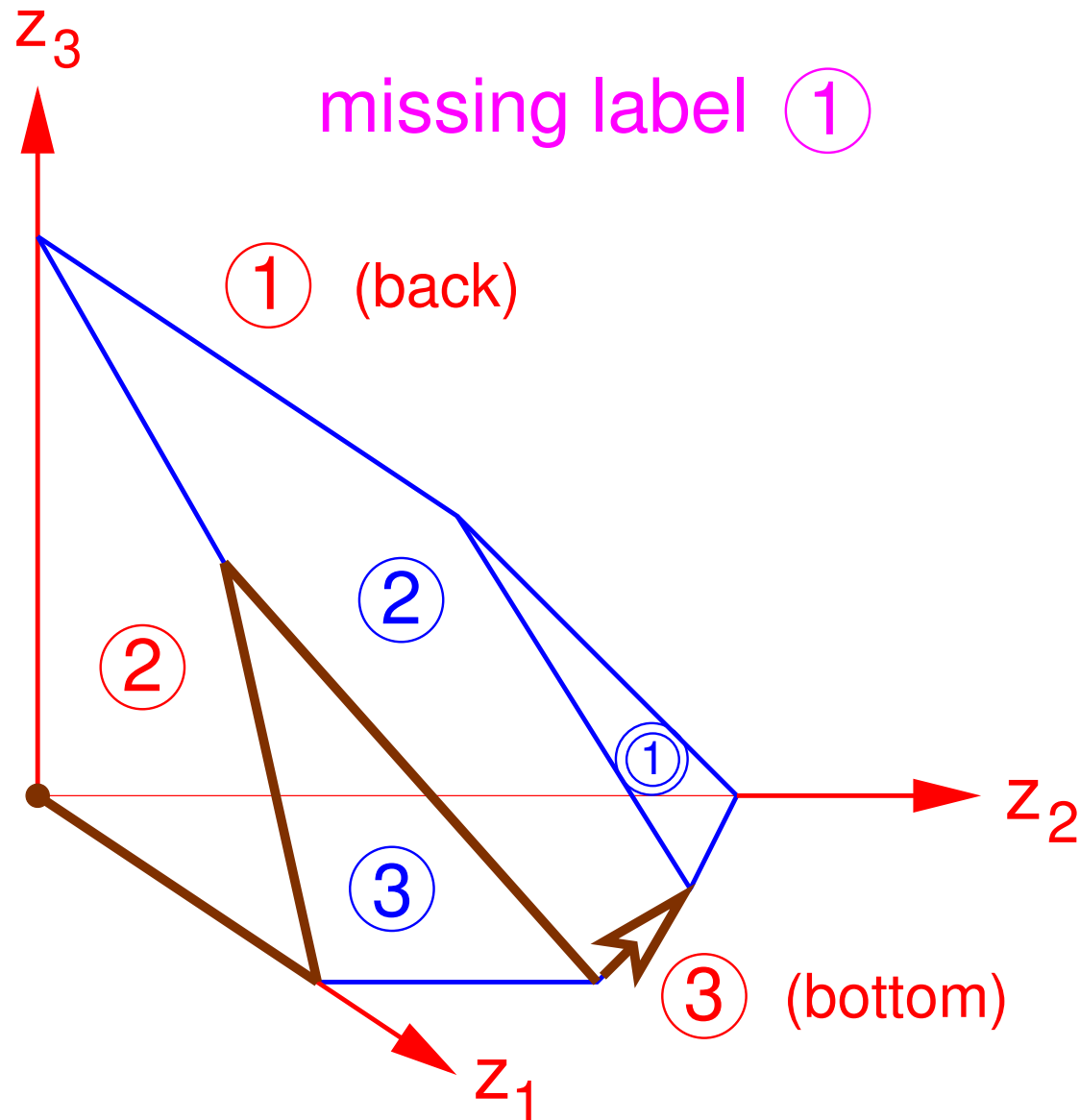
Symmetric Lemke–Howson algorithm



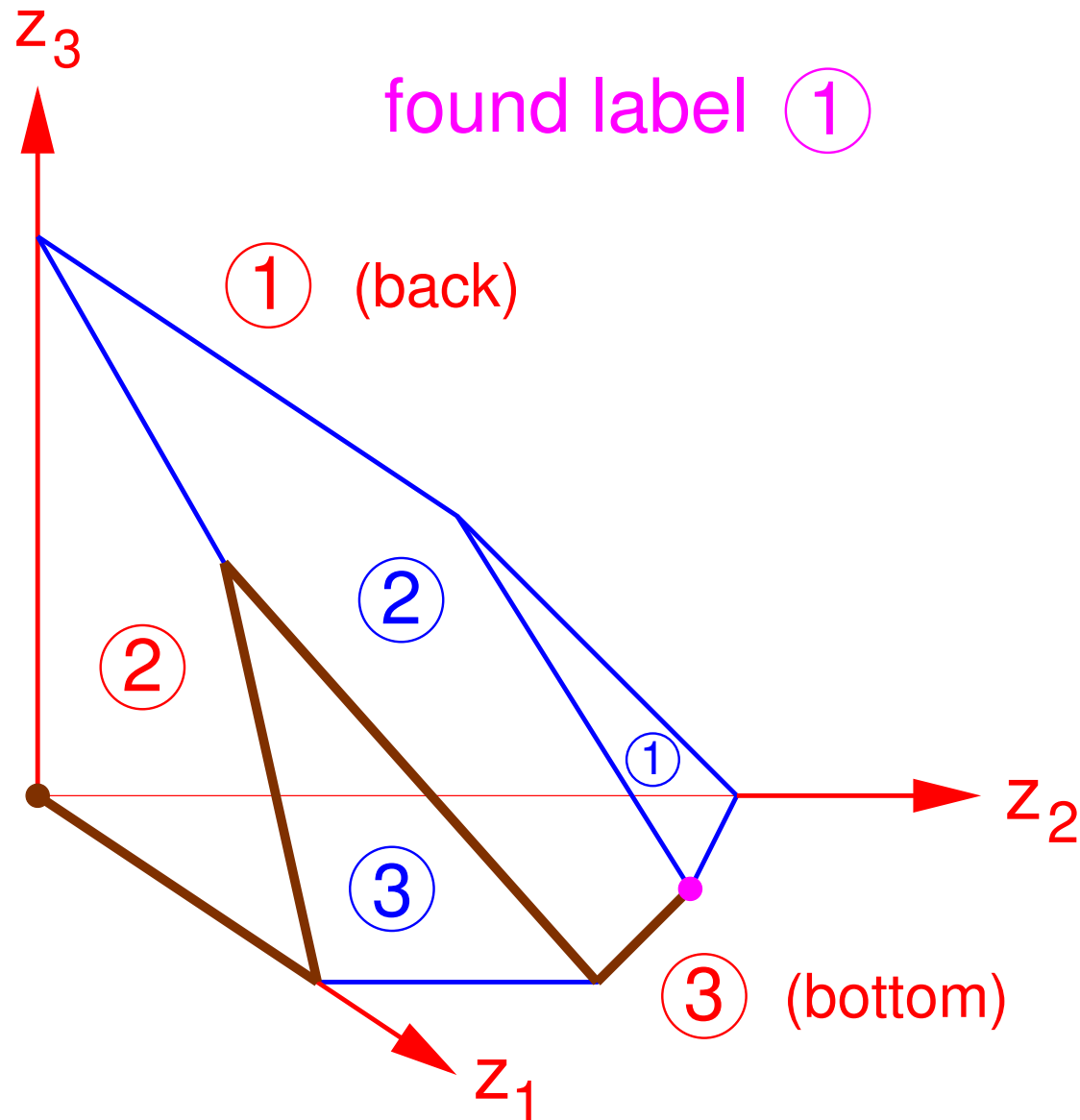
Symmetric Lemke–Howson algorithm



Symmetric Lemke–Howson algorithm



Symmetric Lemke–Howson algorithm



Why Lemke-Howson works

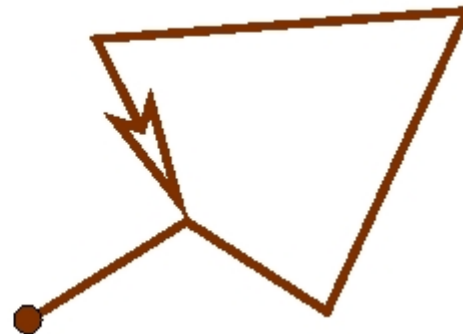
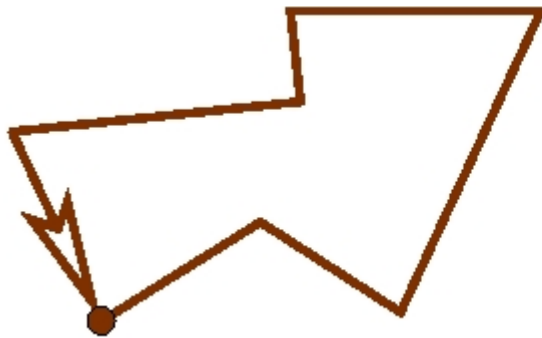
LH finds at least one Nash equilibrium because

- **finitely many** "vertices"

for nondegenerate (generic) games:

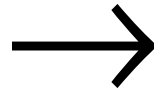
- **unique** starting edge given missing label
- **unique** continuation

⇒ precludes "coming back" like here:



Costs instead of payoffs

1	2
2	0



2	1
1	3

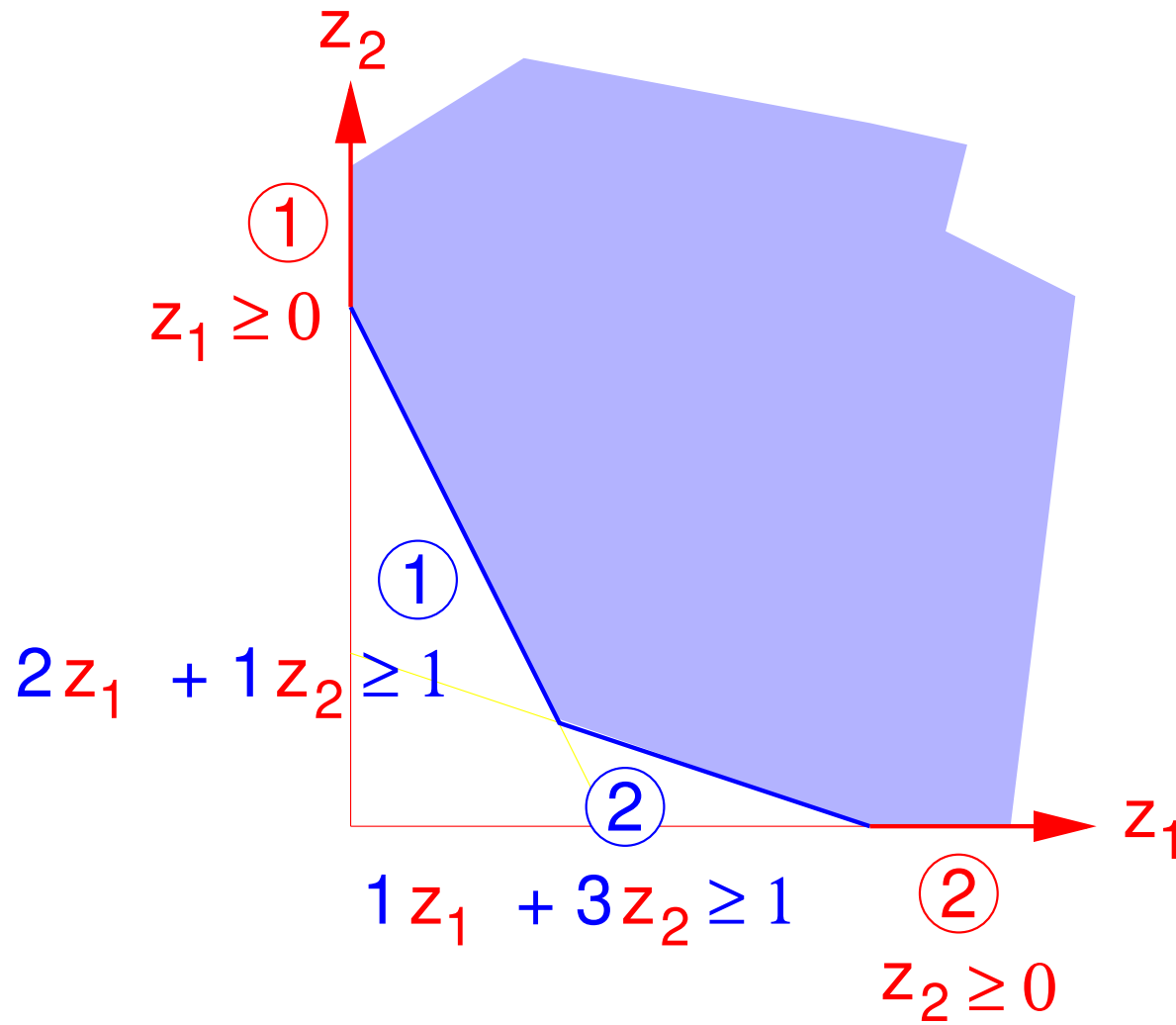
a_{ik}
payoff

$3 - a_{ik}$
cost

with new cost matrix $A > 0$:

equilibrium $z \Leftrightarrow z \geq 0 \perp Az \geq 1$

Polyhedral view



Lemke's algorithm

given LCP

$$z \geq 0 \quad \perp \quad w = q + Mz \quad \geq 0$$

Lemke's algorithm

augmented LCP

$$z \geq 0 \quad \perp \quad w = q + Mz + dz_0 \geq 0$$
$$z_0 \geq 0$$

Lemke's algorithm

augmented LCP

$$z \geq 0 \quad \perp \quad w = q + Mz + dz_0 \geq 0$$
$$z_0 \geq 0$$

where

$d > 0$ covering vector
 z_0 extra variable

$z_0 = 0$ \Leftrightarrow $z \perp w$ solves original LCP

Lemke's algorithm

augmented LCP

$$z \geq 0 \quad \perp \quad w = q + Mz + dz_0 \geq 0$$
$$z_0 \geq 0$$

Initialization:

$$z = 0 \quad \perp \quad w = q \quad + dz_0 \geq 0$$

$z_0 \geq 0$ minimal $\Rightarrow w_i = 0$ for some i

pivot z_0 in, w_i out,

\Rightarrow can increase z_i while maintaining $z \perp w$.

Lemke's algorithm for

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad d = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} w_1 \\ w_2 \end{array} = \begin{array}{c} -1 \\ -1 \end{array} + \begin{array}{c} 2 \\ 1 \end{array} z_1 + \begin{array}{c} 1 \\ 3 \end{array} z_2 + \begin{array}{c} 2 \\ 1 \end{array} z_0$$

$$\begin{array}{c} w_1 \\ z_0 \end{array} = \begin{array}{c} 1 \\ 1 \end{array} + \begin{array}{c} 0 \\ -1 \end{array} z_1 + \begin{array}{c} -5 \\ -3 \end{array} z_2 + \begin{array}{c} -2 \\ -1 \end{array} w_2$$

$$\begin{array}{c}
 w_1 \\
 w_2
 \end{array}
 =
 \begin{array}{c}
 -1 \\
 -1
 \end{array}
 +
 \begin{array}{c}
 2 \\
 1
 \end{array}
 z_1
 +
 \begin{array}{c}
 1 \\
 3
 \end{array}
 z_2
 +
 \begin{array}{c}
 2 \\
 1
 \end{array}
 z_0$$

$$\begin{array}{c}
 w_1 \\
 z_0
 \end{array}
 =
 \begin{array}{c}
 1 \\
 1
 \end{array}
 +
 \begin{array}{c}
 0 \\
 -1
 \end{array}
 z_1
 +
 \begin{array}{c}
 -5 \\
 -3
 \end{array}
 z_2
 +
 \begin{array}{c}
 -2 \\
 -1
 \end{array}
 w_2$$

$$\begin{array}{c}
 z_2 \\
 z_0
 \end{array}
 =
 \begin{array}{c}
 0.2 \\
 0.4
 \end{array}
 +
 \begin{array}{c}
 0 \\
 -1
 \end{array}
 z_1
 +
 \begin{array}{c}
 -0.2 \\
 0.6
 \end{array}
 w_1
 +
 \begin{array}{c}
 -0.4 \\
 0.2
 \end{array}
 w_2$$

$$\begin{array}{c}
 \boxed{w_1} \\
 \\
 \boxed{z_0}
 \end{array}
 =
 \begin{array}{c}
 \boxed{1} \\
 \\
 \boxed{1}
 \end{array}
 +
 \begin{array}{c}
 \boxed{0} \\
 \\
 \boxed{-1}
 \end{array}
 \boxed{z_1}
 +
 \begin{array}{c}
 \boxed{-5} \\
 \\
 \boxed{-3}
 \end{array}
 \boxed{z_2}
 +
 \begin{array}{c}
 \boxed{-2} \\
 \\
 \boxed{-1}
 \end{array}
 \boxed{w_2}$$

$$\begin{array}{c}
 \boxed{z_2} \\
 \\
 \boxed{z_0}
 \end{array}
 =
 \begin{array}{c}
 \boxed{0.2} \\
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 +
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 \boxed{0.2}
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 \boxed{w_2}$$

$$\begin{array}{c}
 \boxed{z_2} \\
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 =
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 \boxed{z_0}
 +
 \begin{array}{c}
 \boxed{-0.2} \\
 \\
 \boxed{0.6}
 \end{array}
 \boxed{w_1}
 +
 \begin{array}{c}
 \boxed{-0.4} \\
 \\
 \boxed{0.2}
 \end{array}
 \boxed{w_2}$$

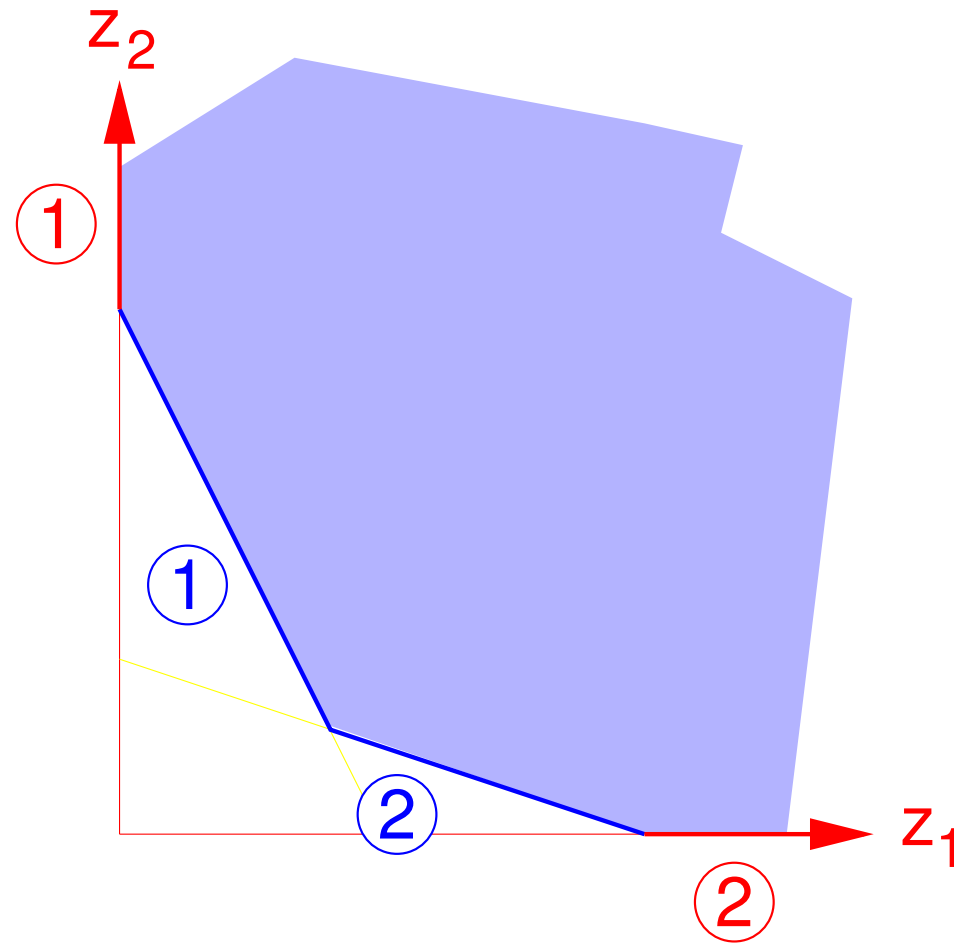
Polyhedral view of Lemke

Polyhedral view of Lemke

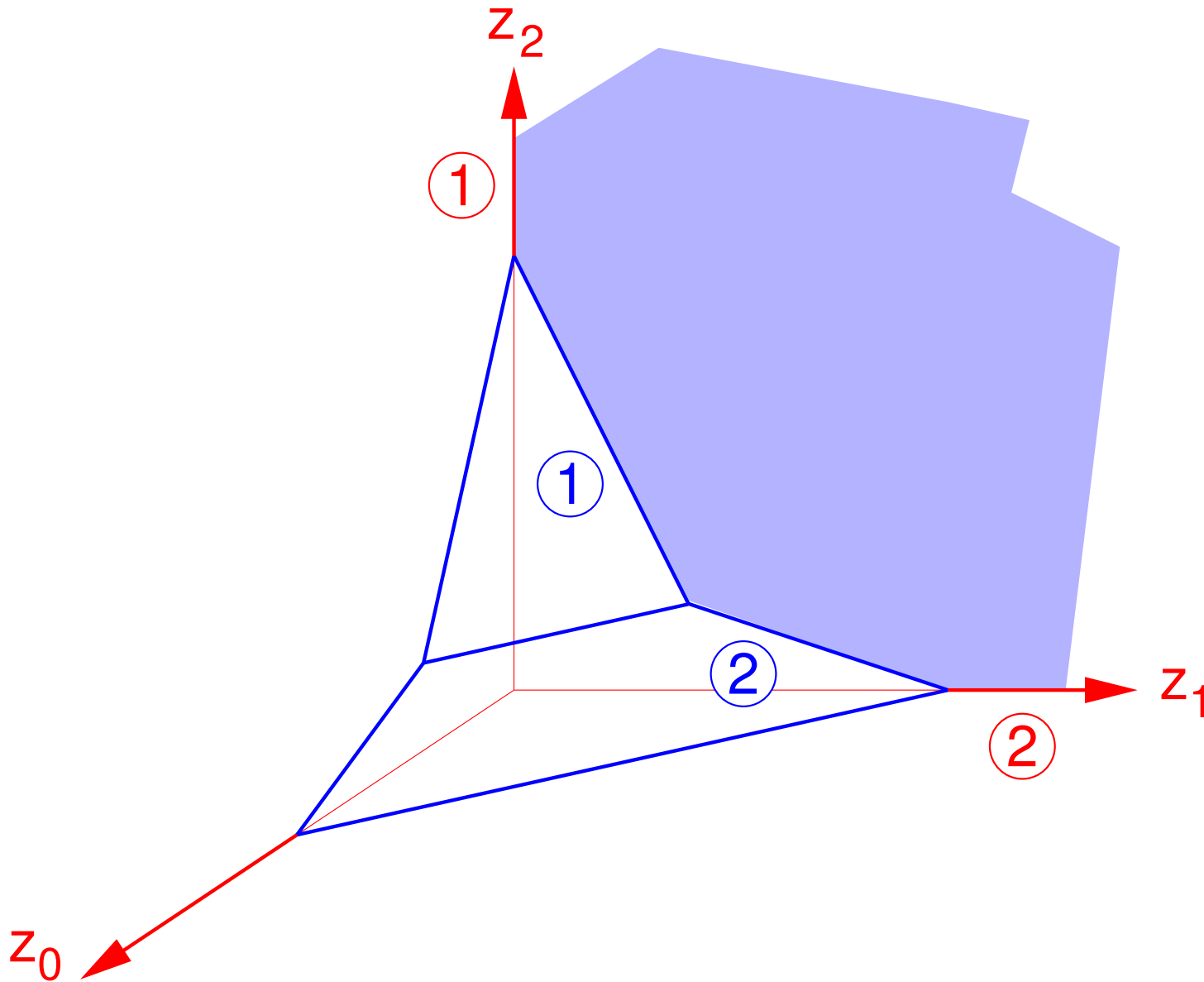


Polyhedral view of Lemke

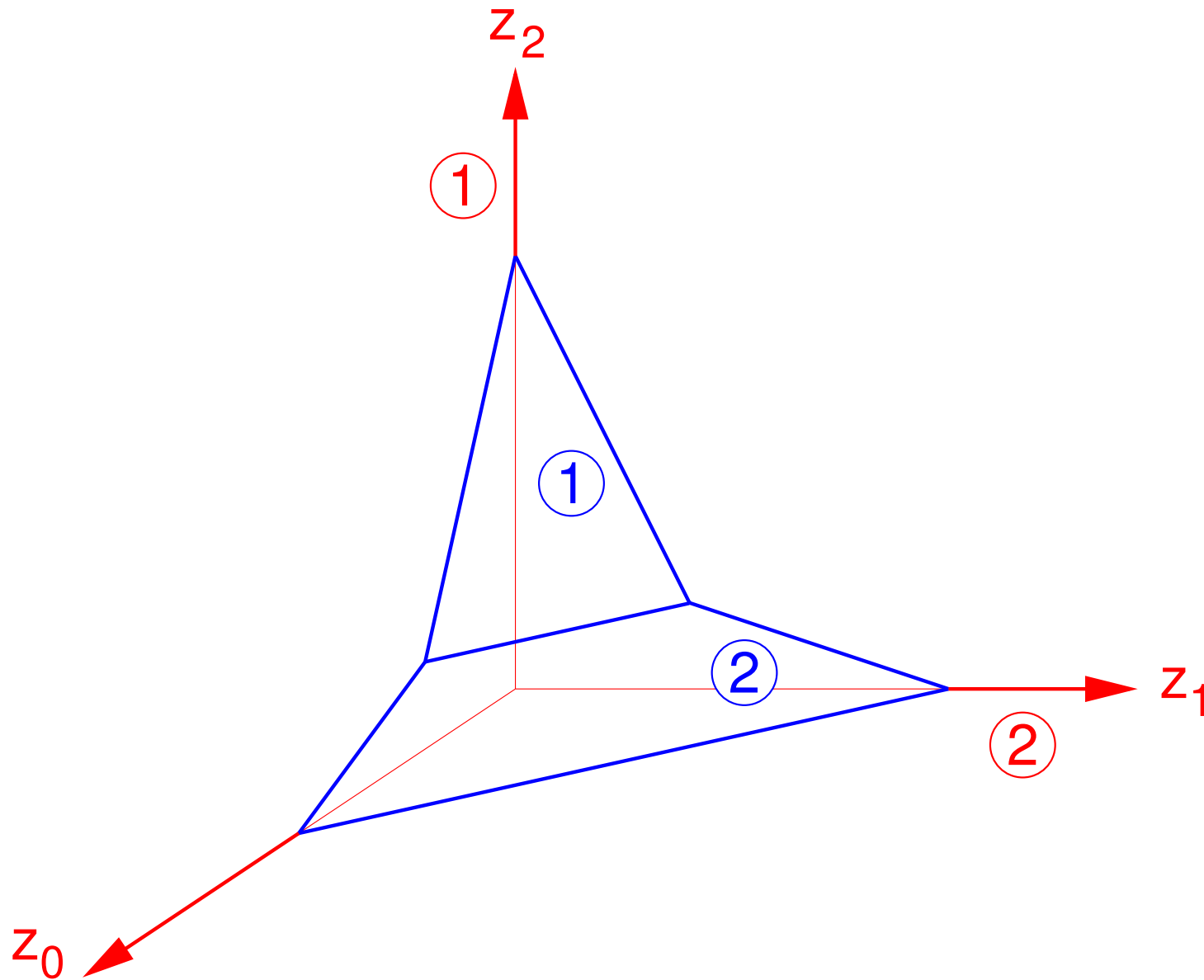
Polyhedral view of Lemke



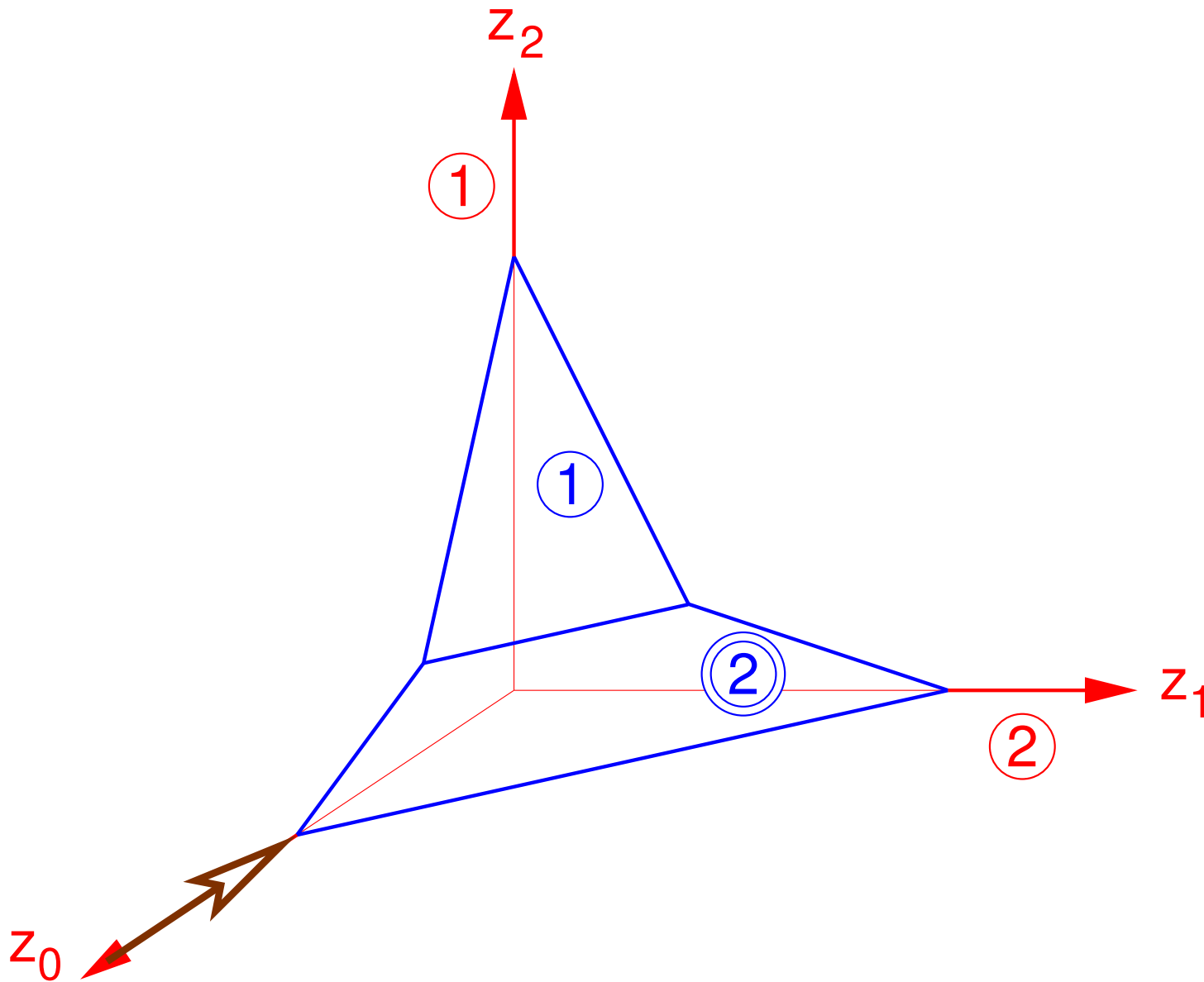
Polyhedral view of Lemke



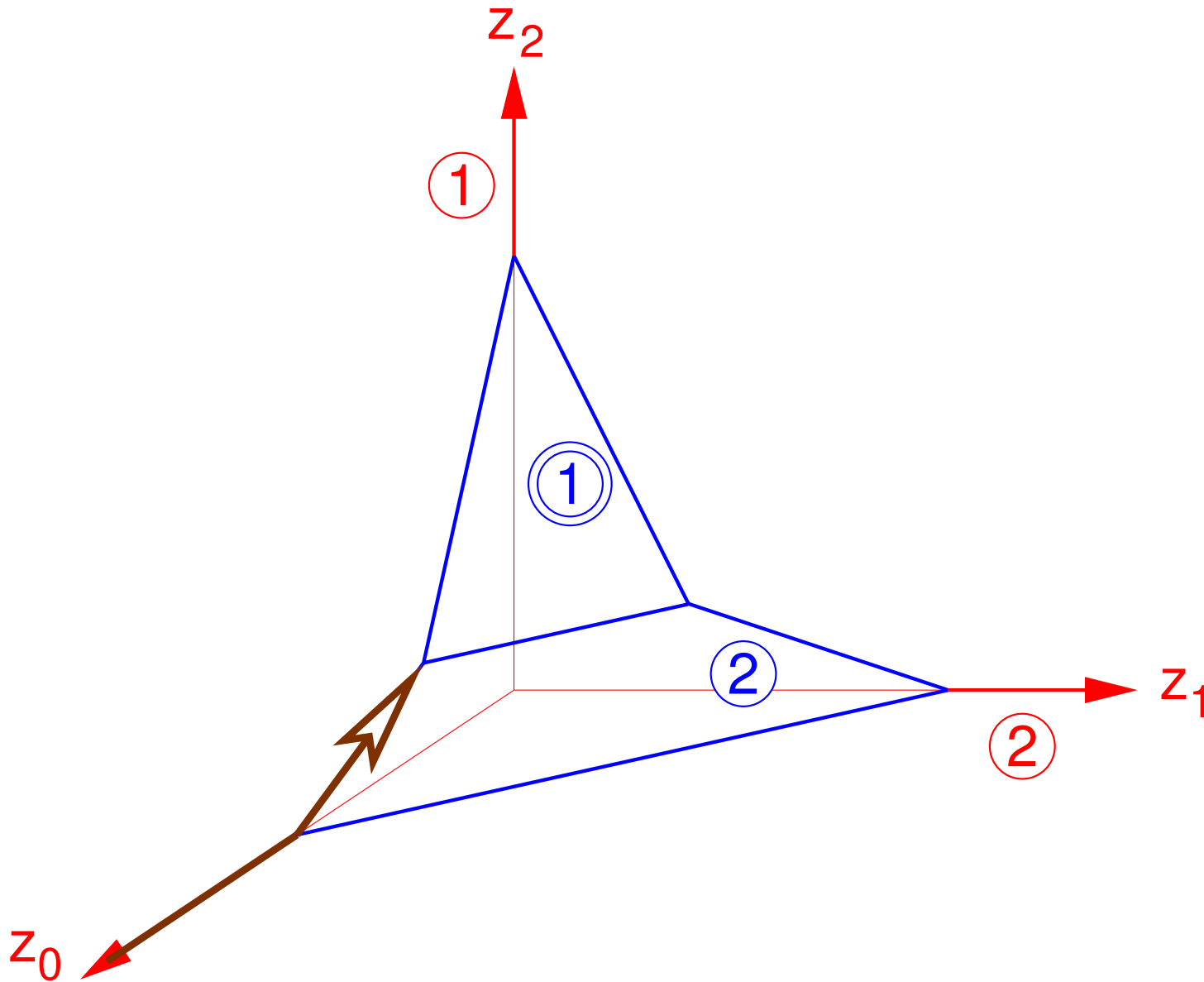
Polyhedral view of Lemke



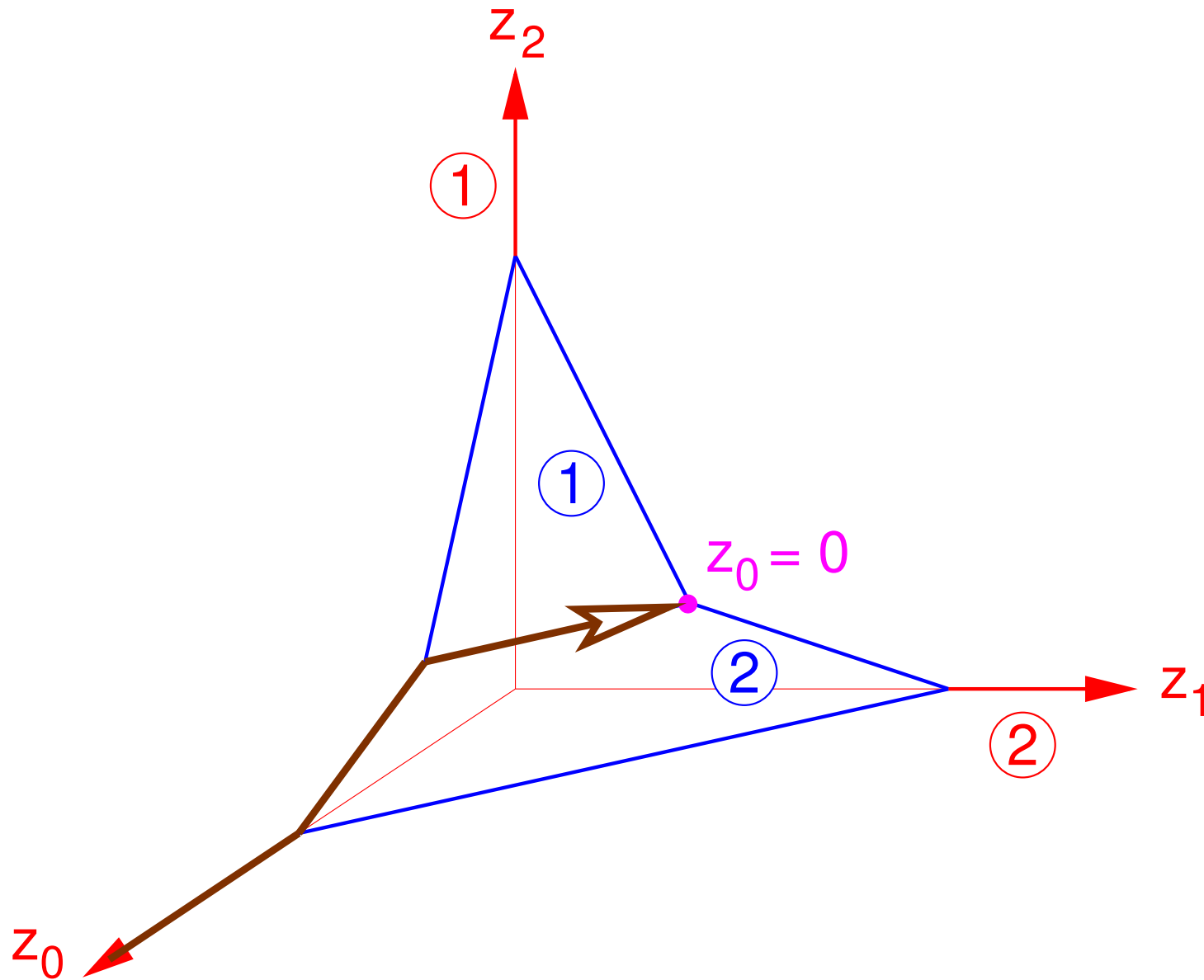
Polyhedral view of Lemke



Polyhedral view of Lemke



Polyhedral view of Lemke



Complementary cones

$$\text{LCP} \quad z \geq 0 \quad \perp \quad w = q + Mz \geq 0$$

$$\Leftrightarrow \quad z \geq 0 \quad \perp \quad w \geq 0, \quad \boxed{-q = Mz - w}$$

\Leftrightarrow $-q$ belongs to a **complementary cone**:

$$\boxed{-q \in \mathbf{C}(\alpha) = \text{cone} \{ M_i, -e_j \mid i \in \alpha, j \notin \alpha \}}$$

for some $\alpha \subseteq \{1, \dots, n\}$, $M = [M_1 \ M_2 \ \dots \ M_n]$

$$\alpha = \{ i \mid z_i > 0 \}$$

Polyhedra versus cones

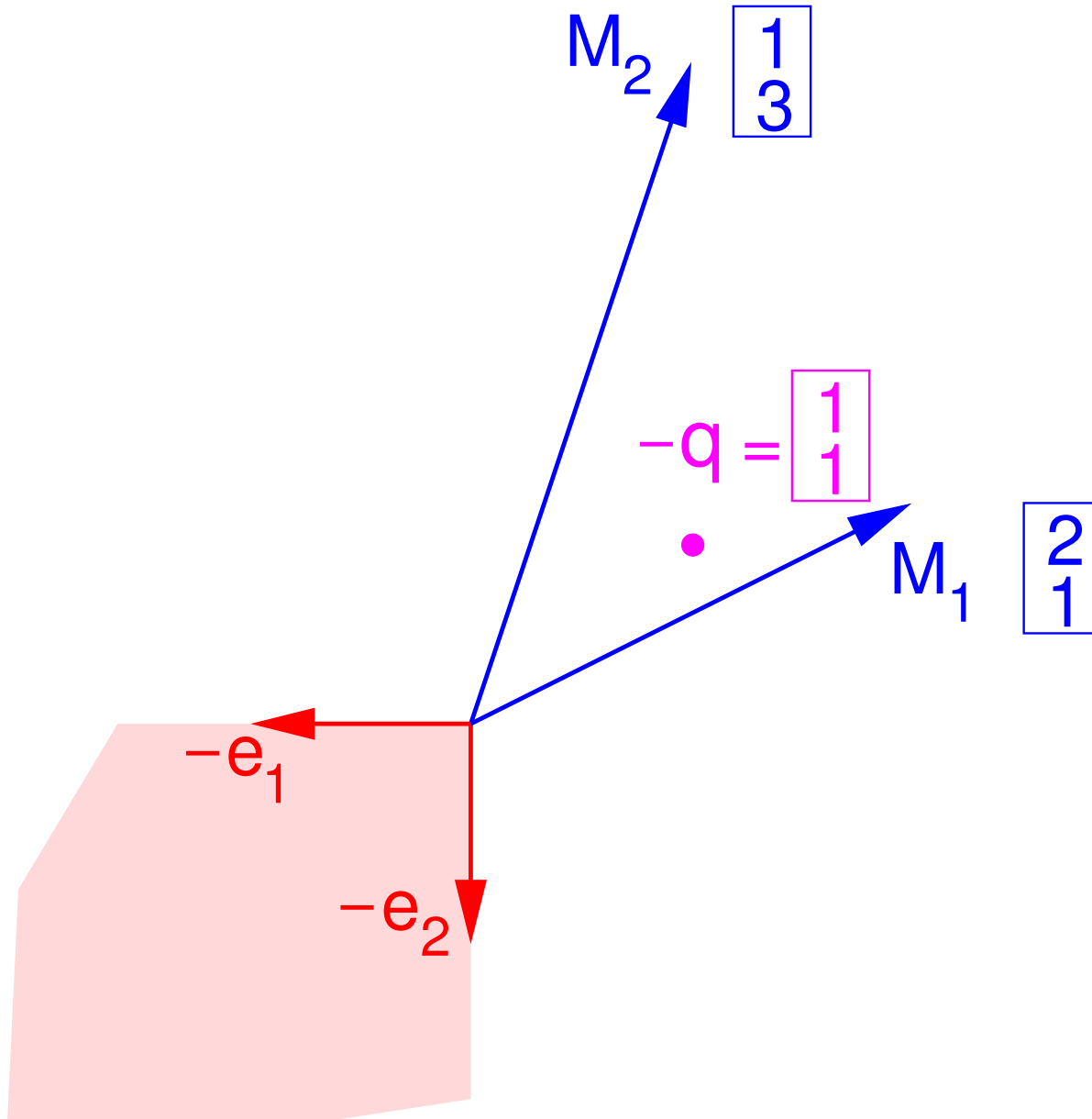
polyhedral view :

- gives feasibility,
want complementary **vertex**

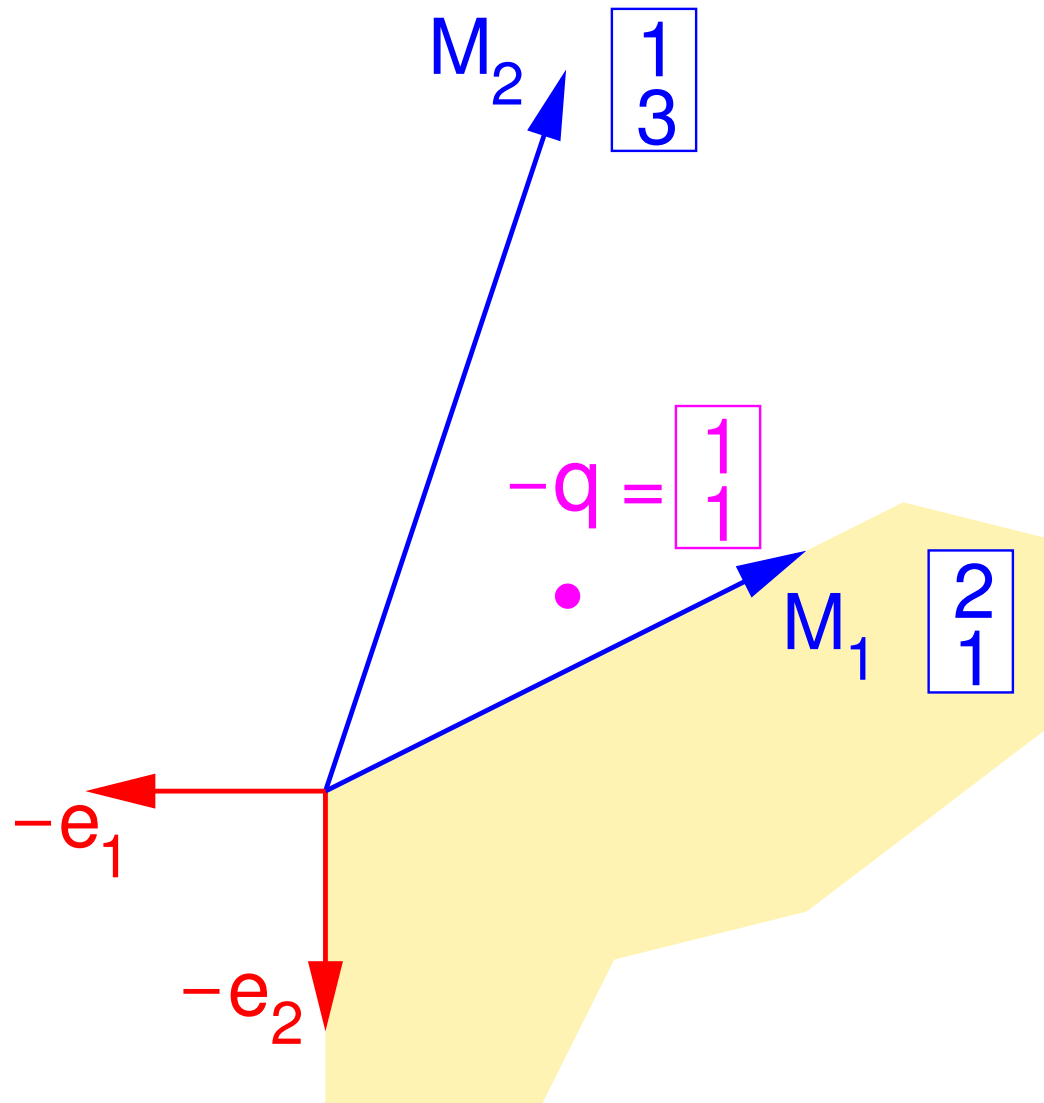
complementary cones :

- gives complementarity and feasibility,
want **α** giving **cone $C(\alpha)$** containing **$-q$**

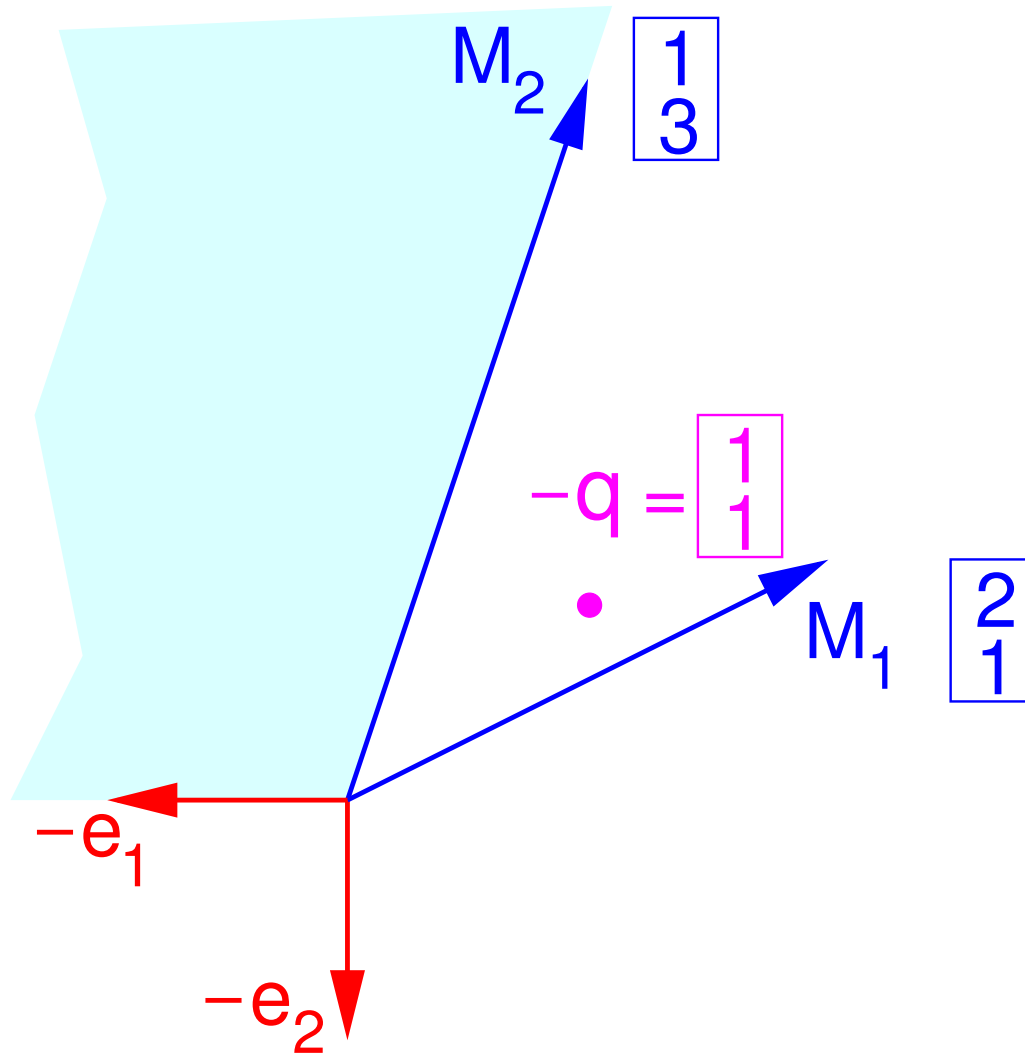
Complementary cone $C(\{\})$



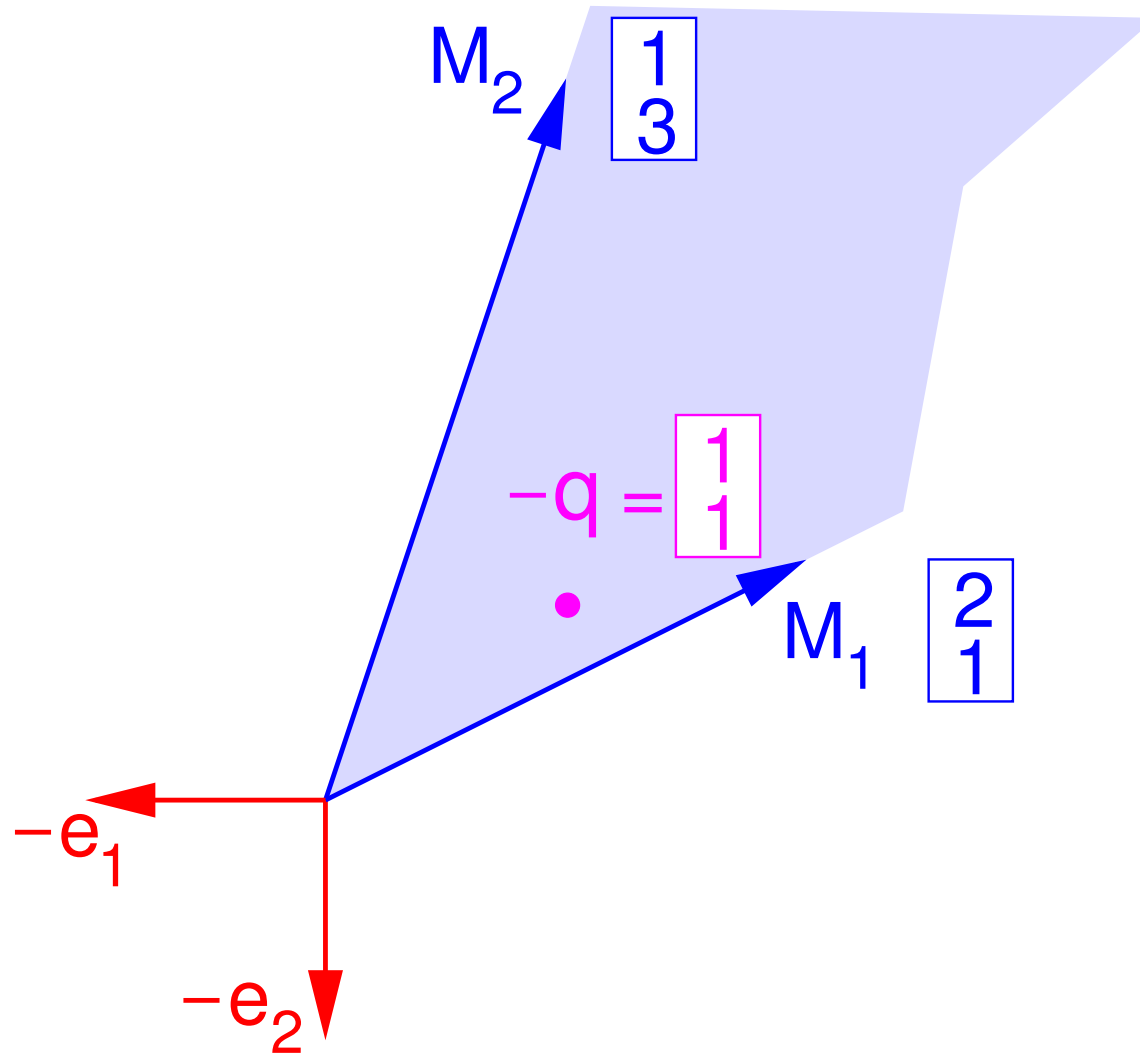
Complementary cone $C(\{1\})$



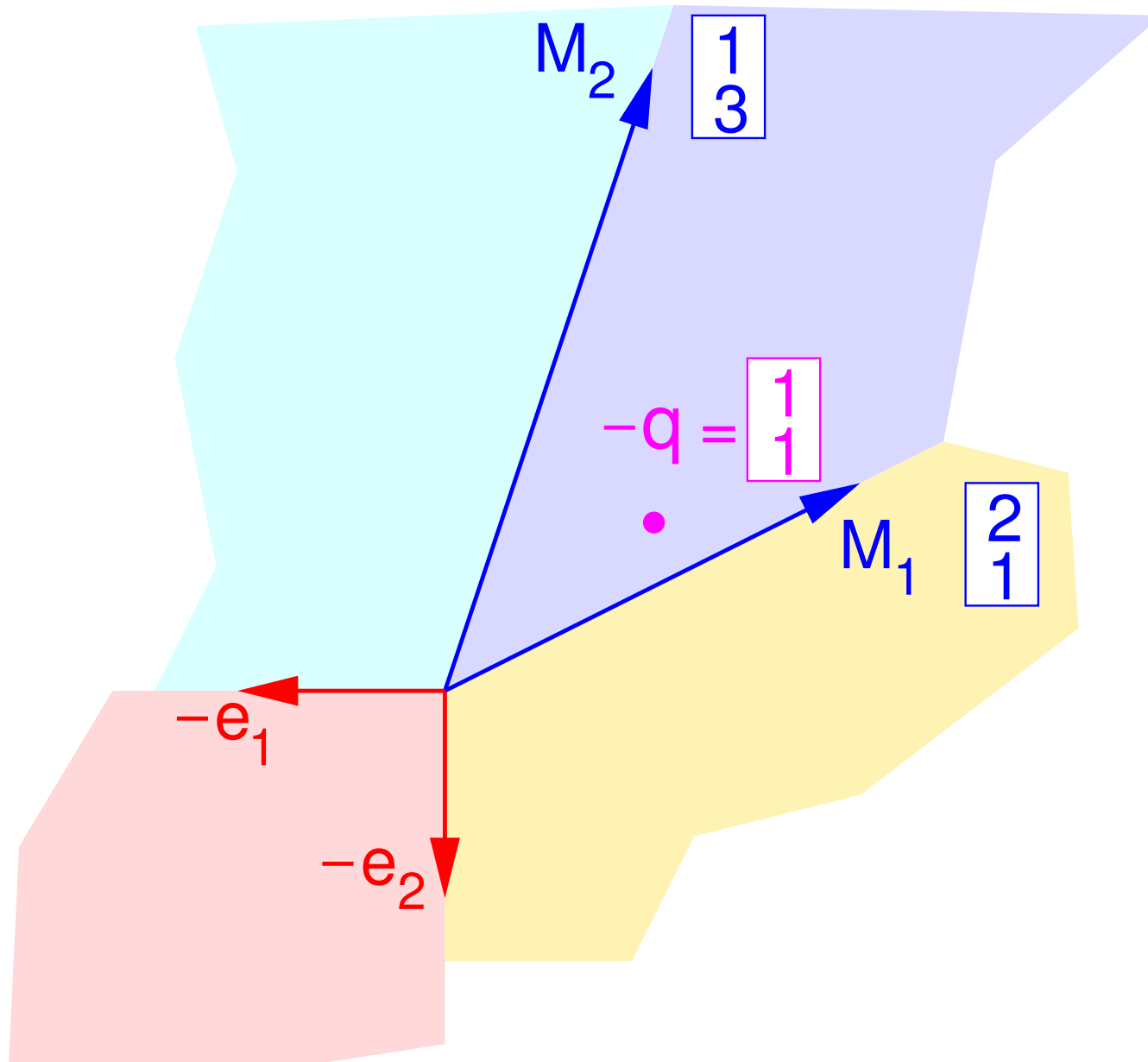
Complementary cone $C(\{2\})$



Complementary cone $C(\{1,2\})$



All complementary cones



LCP map

Let $\alpha \subseteq \{1, \dots, n\}$,

α -orthant = cone $\{ e_i, -e_j \mid i \in \alpha, j \notin \alpha \}$,

$C(\alpha)$ = cone $\{ M_i, -e_j \mid i \in \alpha, j \notin \alpha \}$,

$x_i^+ = \max(x_i, 0)$, $x_i^- = \min(x_i, 0)$

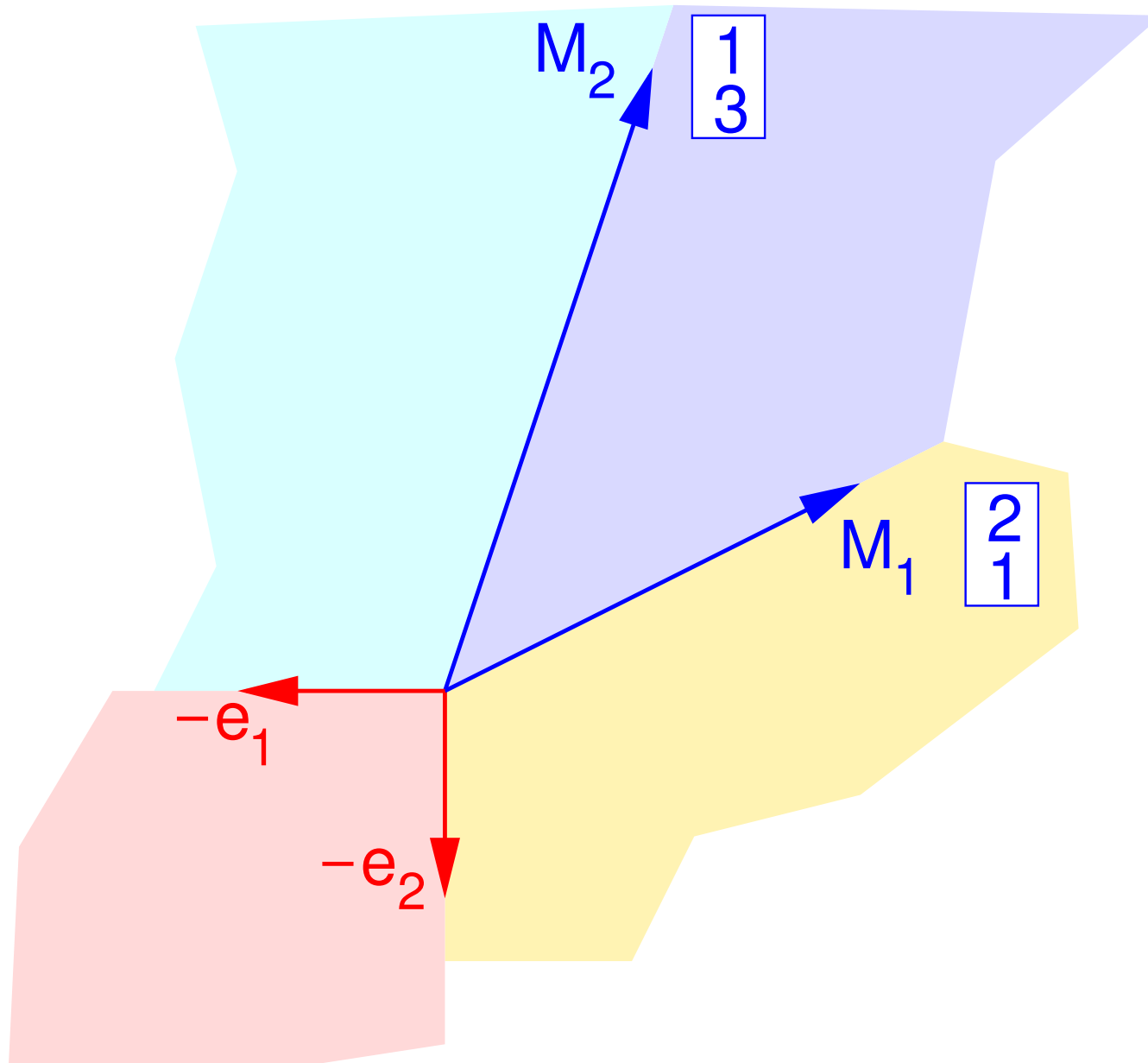
LCP map:

$$F(x) = Mx^+ + x^-$$

so

$$F(\alpha\text{-orthant}) = C(\alpha)$$

Bijjective LCP map F



P-matrix

P-matrix

\Leftrightarrow every **principal minor** is positive:

$$\det (M_{\alpha\alpha}) > 0 \quad \text{for all } \alpha \subseteq \{1, \dots, n\}$$

e.g.

$$\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 3 \\ \hline \end{array}$$

$$\det (M_{1,1}) = 2 > 0$$

$$\det (M_{2,2}) = 3 > 0$$

$$\det (M_{12,12}) = \det (M) = 5 > 0$$

P-matrix

P-matrix

\Leftrightarrow every **principal minor** is positive:

$$\det (M_{\alpha\alpha}) > 0 \quad \text{for all } \alpha \subseteq \{1, \dots, n\}$$

P-matrix

\Leftrightarrow **F** bijective

$$\Leftrightarrow \forall q \in \mathbf{R}^n \quad \exists! z \quad \text{s.t.} \quad z \geq \mathbf{0} \quad \perp \quad Mz \geq -q$$

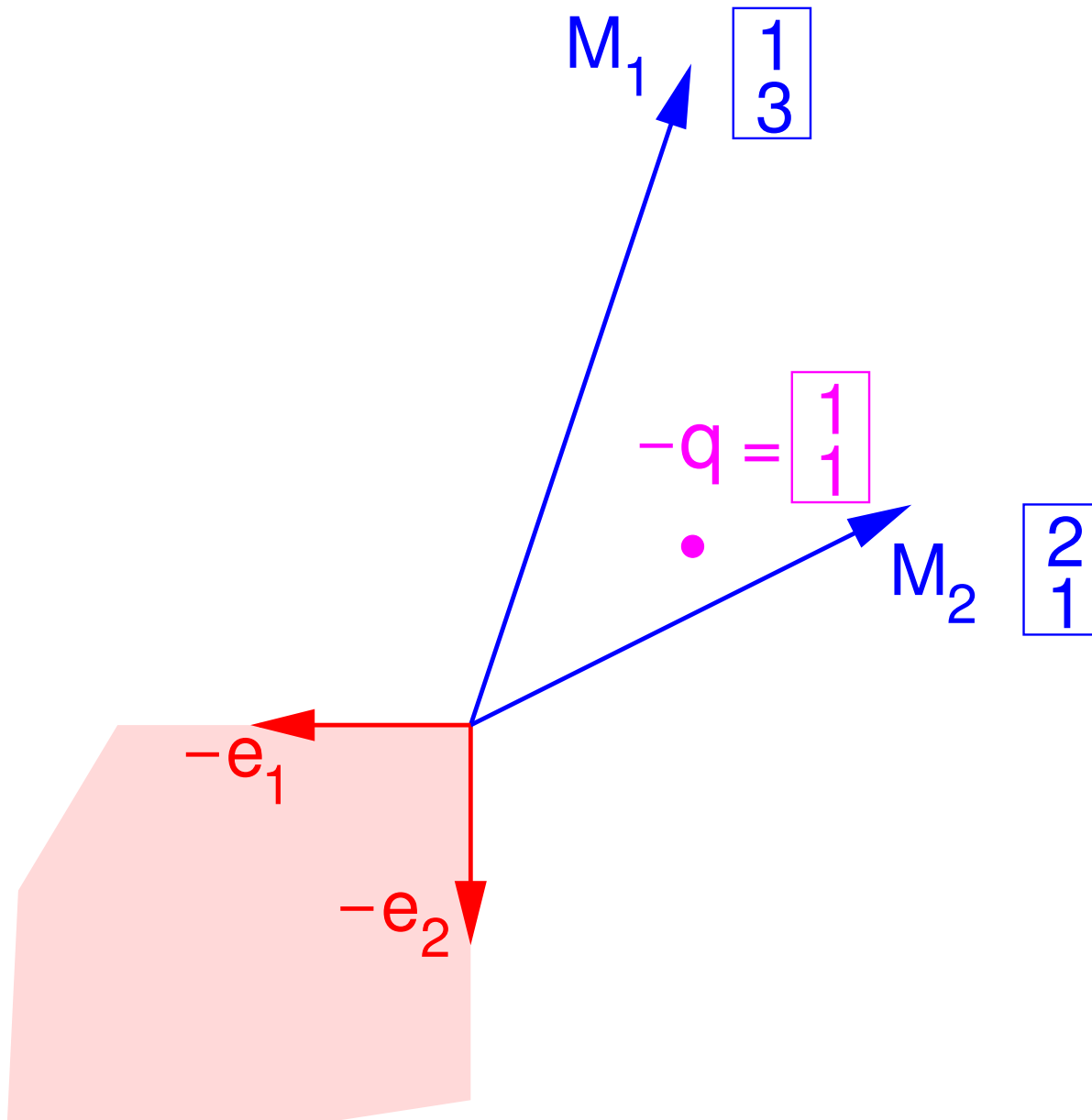
Not a P-matrix

Example:

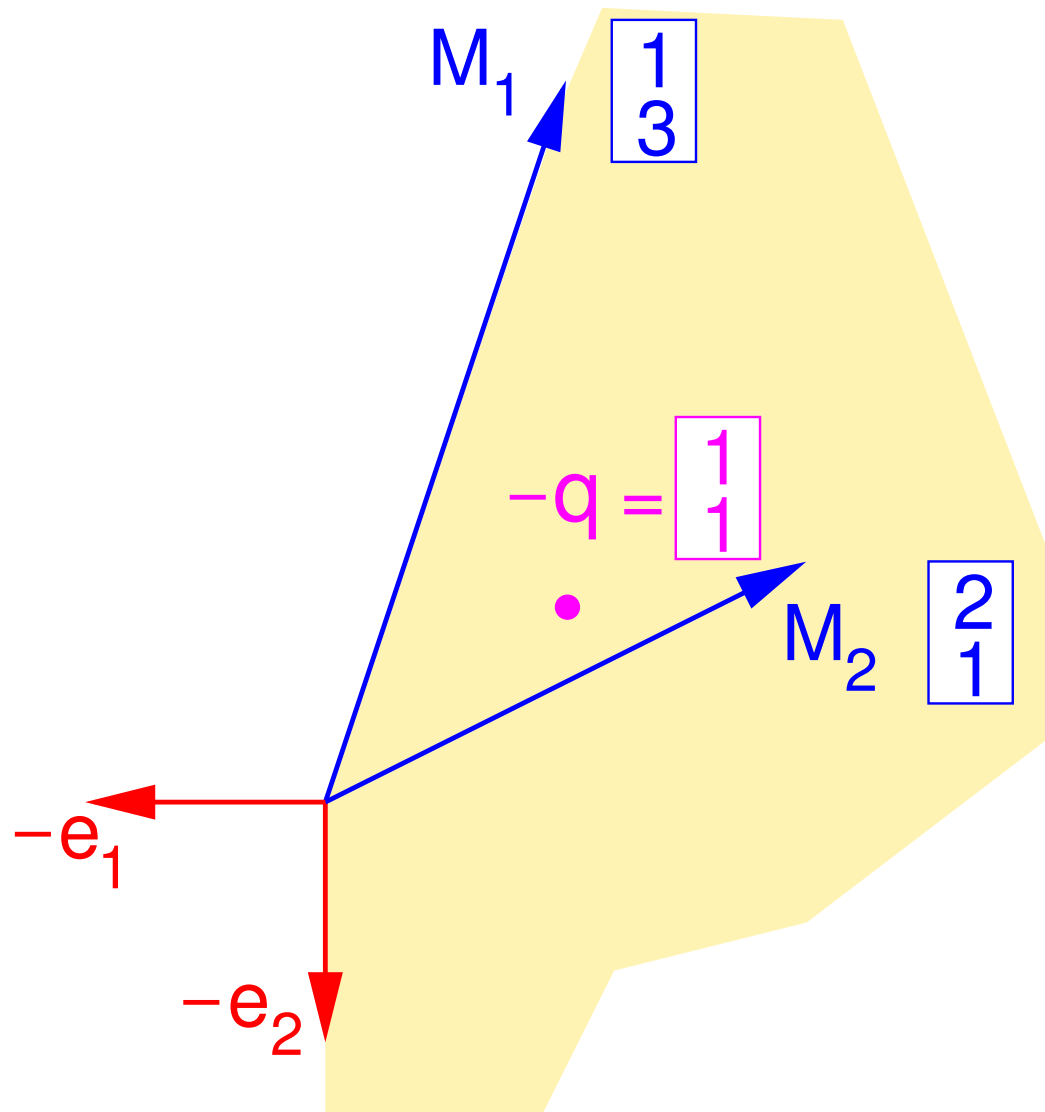
1	2
3	1

$$\det (M_{12,12}) = \det (M) = -5 < 0$$

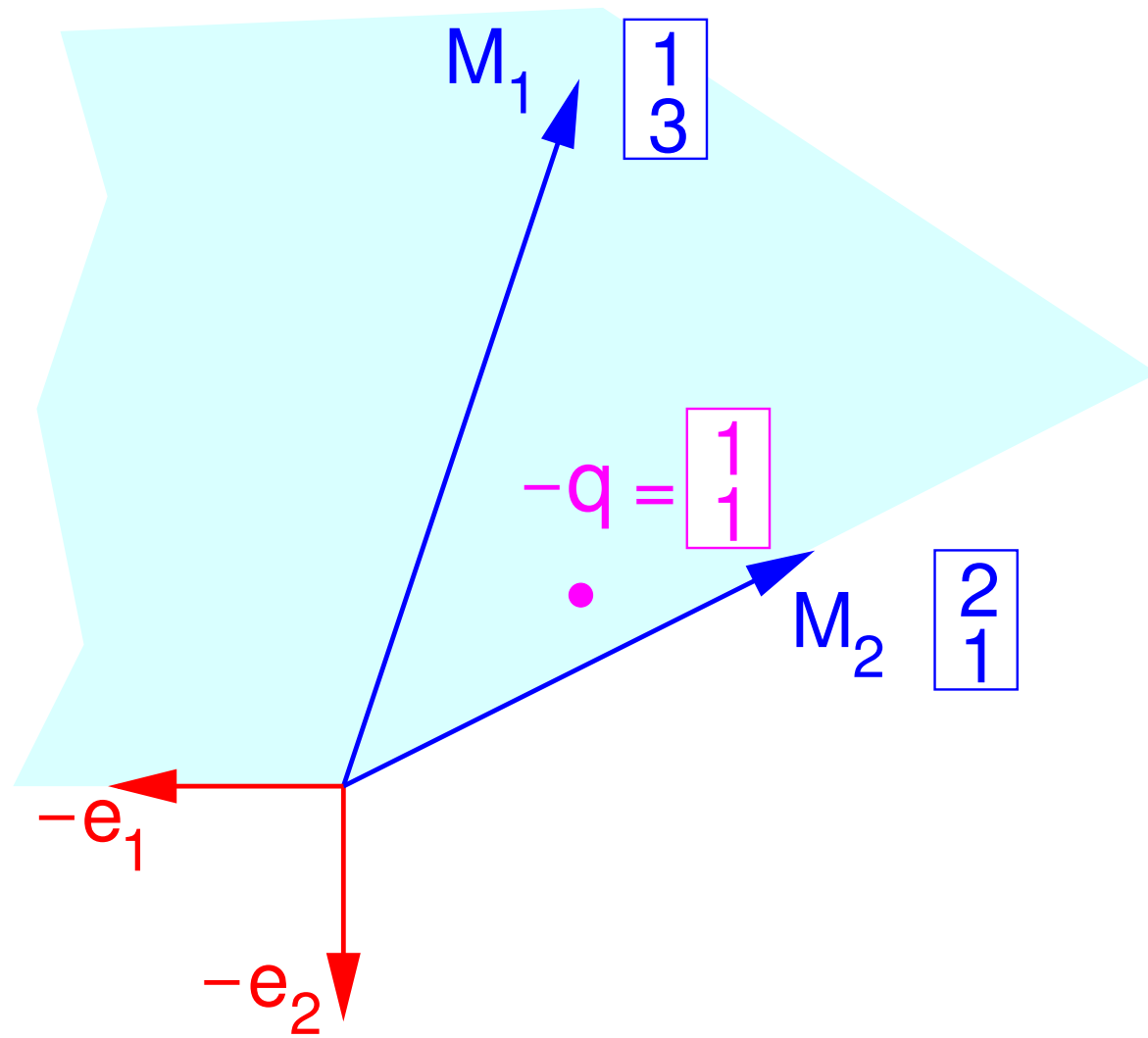
Complementary cone $C(\{\})$



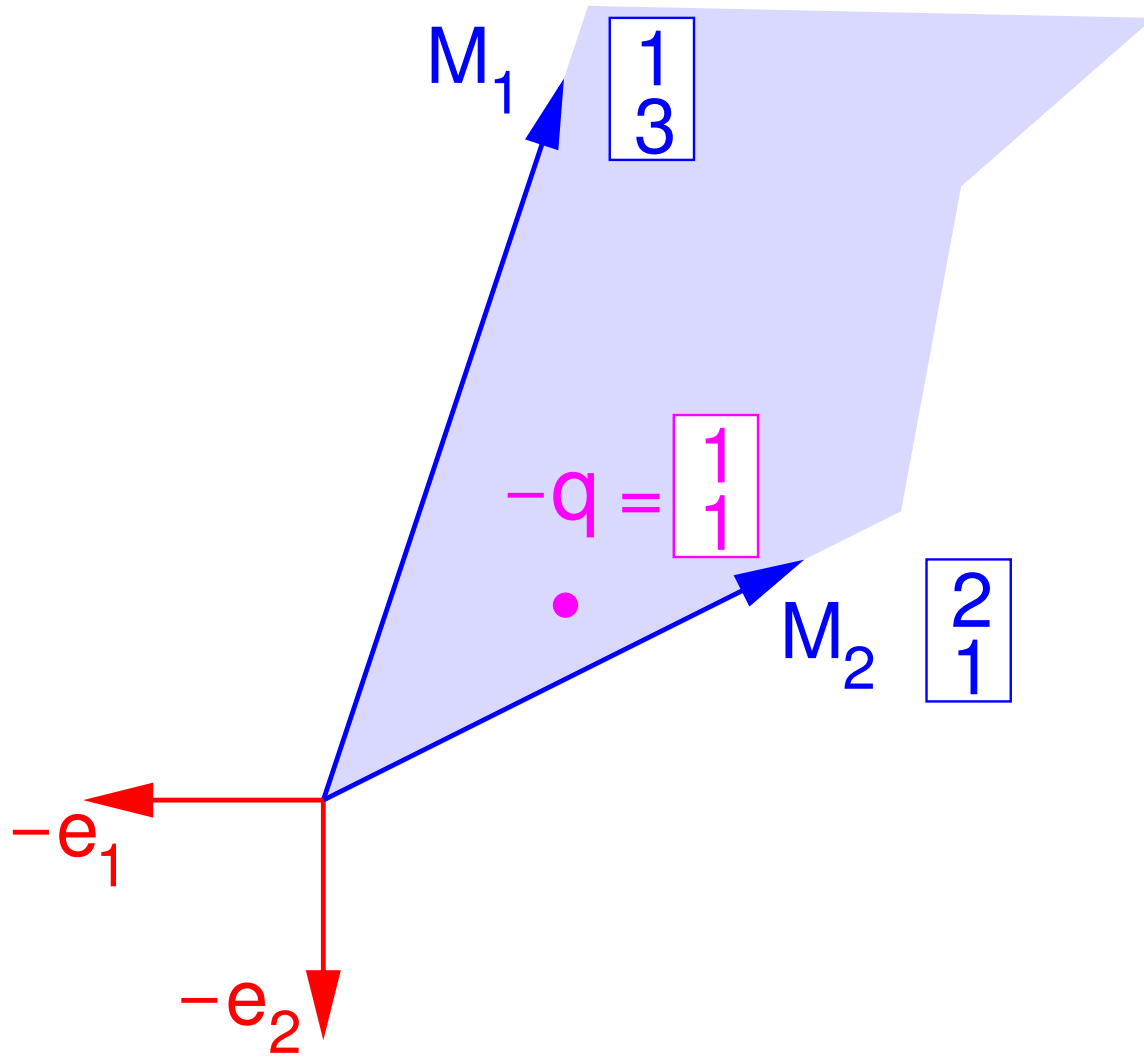
Complementary cone $C(\{1\})$



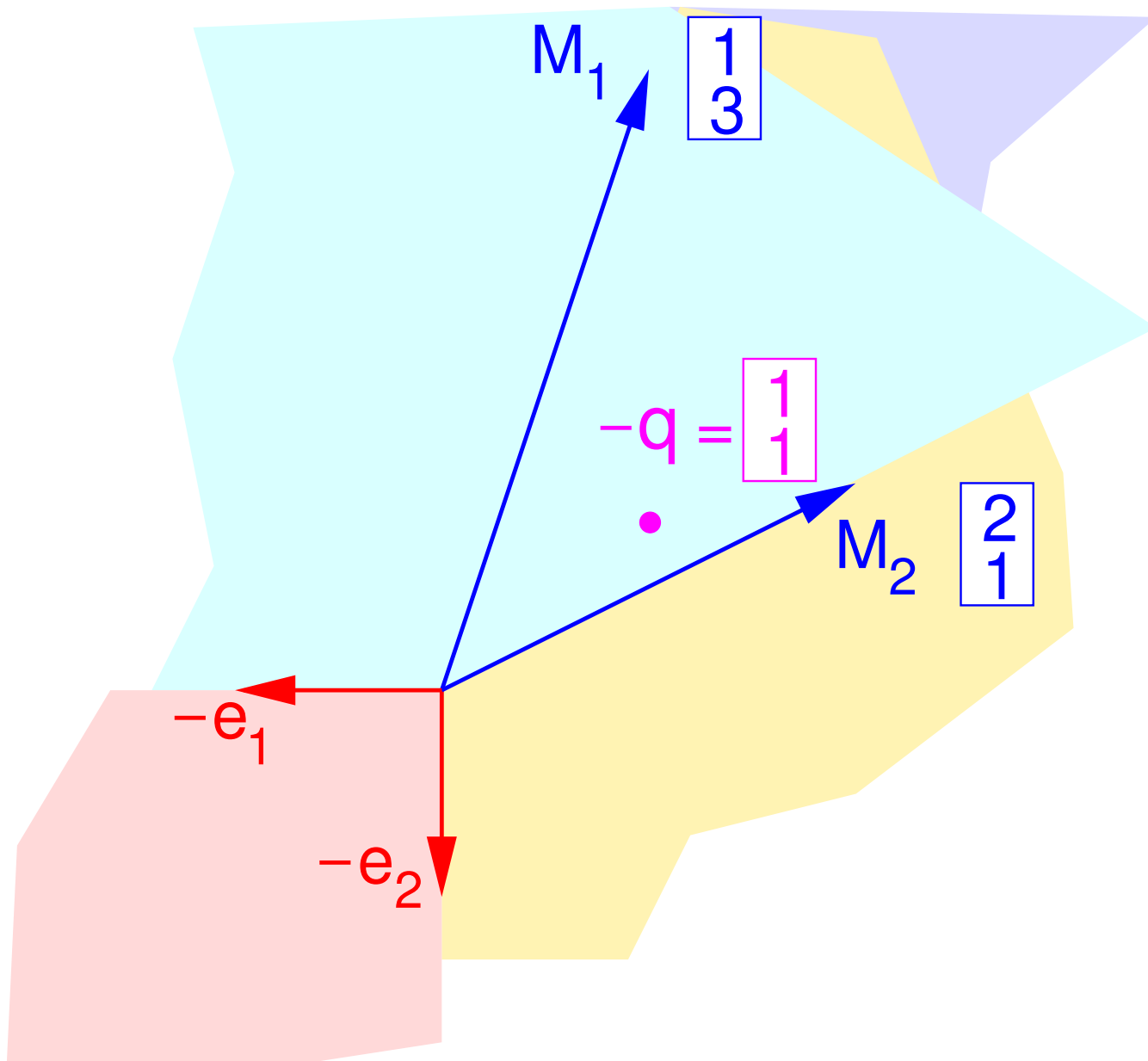
Complementary cone $C(\{2\})$



Complementary cone $C(\{1,2\})$



Non-injective LCP map F



F is surjective for $M > 0$

Given: $p \in \mathbf{R}^n$.

Claim: $\exists x : F(x) = Mx^+ + x^- = p$

Proof (solving $F(\mathbf{x}) = \mathbf{p}$)

Let $\mathbf{p} \in \mathbf{R}^n$, $\alpha = \{i \mid p_i > 0\}$.

Step 1. Consider only rows $i \in \alpha$. Solution \mathbf{x}^+ to

$$\forall i \in \alpha \quad x_i \perp \sum_{j \in \alpha} m_{ij} x_j \geq p_i$$

Proof (solving $F(x) = p$)

Let $p \in \mathbf{R}^n$, $\alpha = \{i \mid p_i > 0\}$.

Step 1. Consider only rows $i \in \alpha$. Solution x^+ to

$$\forall i \in \alpha \quad x_i \perp \sum_{j \in \alpha} (m_{ij} / p_i) x_j \geq 1$$

exists as Nash equilibrium (game matrix m_{ij} / p_i).

Proof (solving $F(x) = p$)

Let $p \in \mathbf{R}^n$, $\alpha = \{i \mid p_i > 0\}$.

Step 1. Consider only rows $i \in \alpha$. Solution x^+ to

$$\forall i \in \alpha \quad x_i \perp \sum_{j \in \alpha} (m_{ij} / p_i) x_j \geq 1$$

exists as Nash equilibrium (game matrix m_{ij} / p_i).

Step 2. $\forall k \notin \alpha$ choose $-x_k^- = w_k \geq 0$ so that

$$\sum_{j \in \alpha} m_{kj} x_j^+ - w_k = p_k (\leq 0).$$

Gives $F(x) = p$.

Lemke via complementary cones

Invert the piecewise linear map $F(x)$ along the line segment $[-d, -q]$:

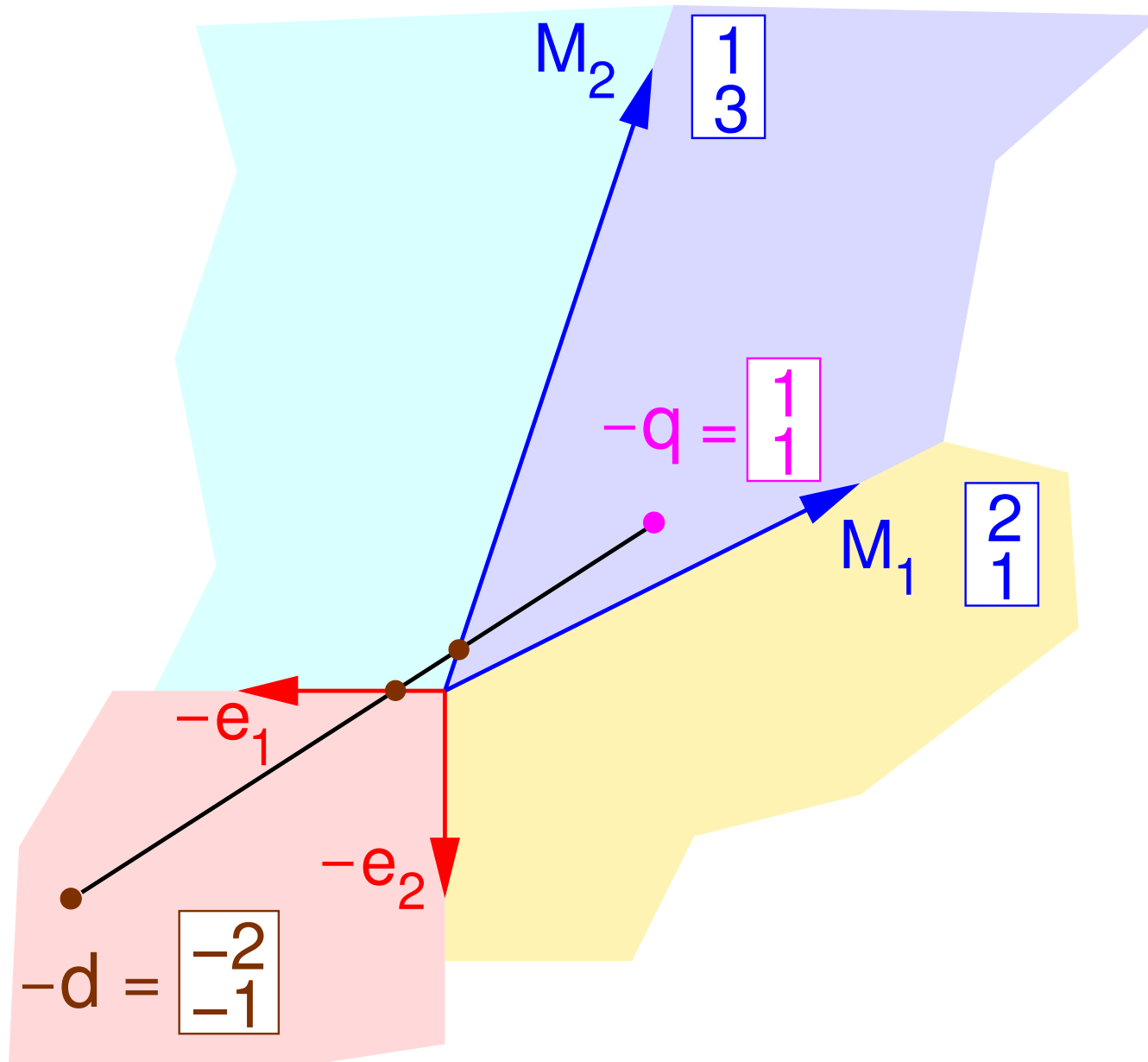
$$F(x) = Mx^+ + x^- = (-d)(1-t) + (-q)t \quad (0 \leq t \leq 1)$$

$t > 0$:

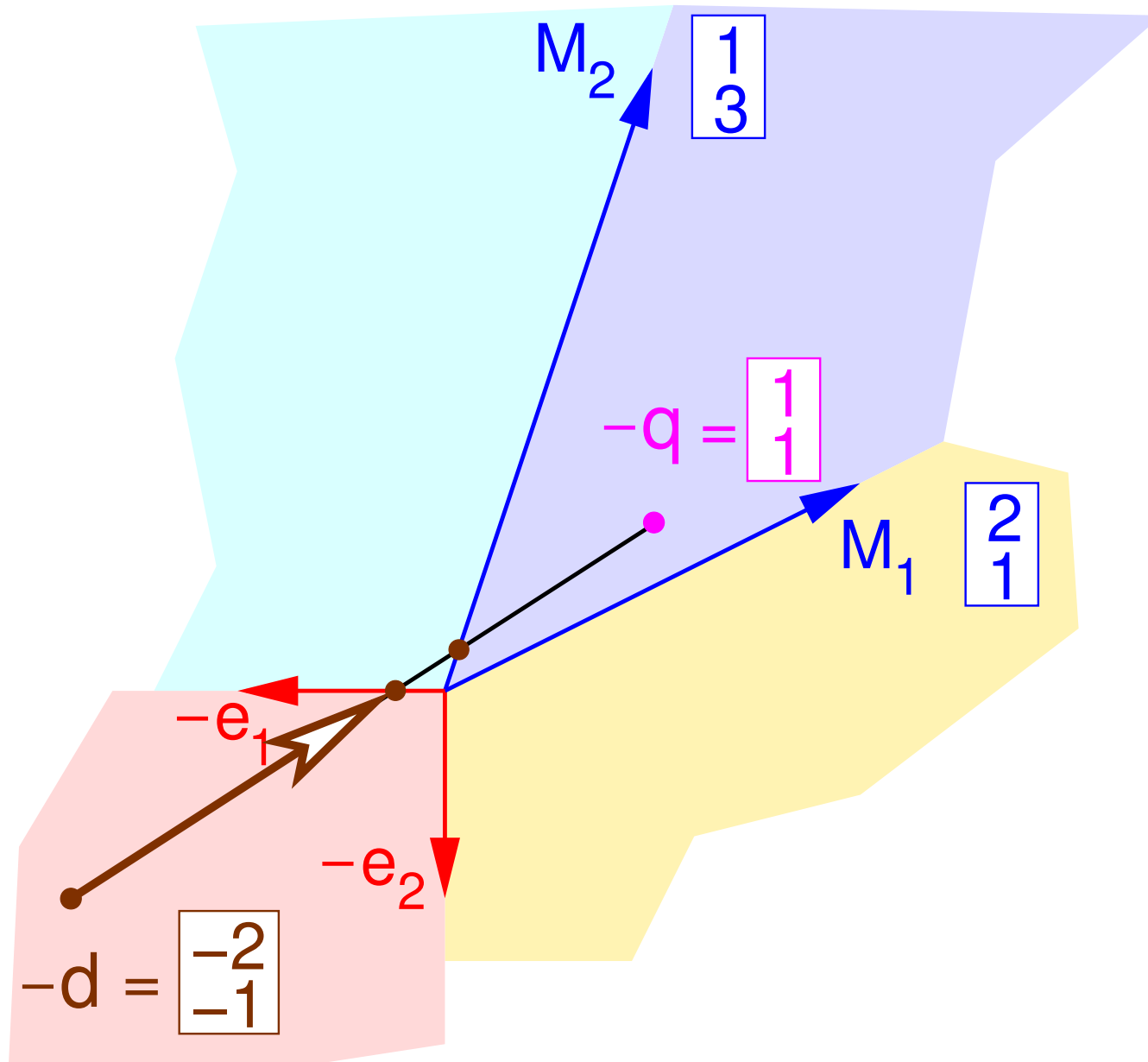
$$\Leftrightarrow Mx^+(1/t) + x^-(1/t) = (-d)(1-t)/t + (-q)$$

$$\Leftrightarrow Mz - w = (-d)z_0 + (-q), \quad z \geq 0 \perp w \geq 0.$$

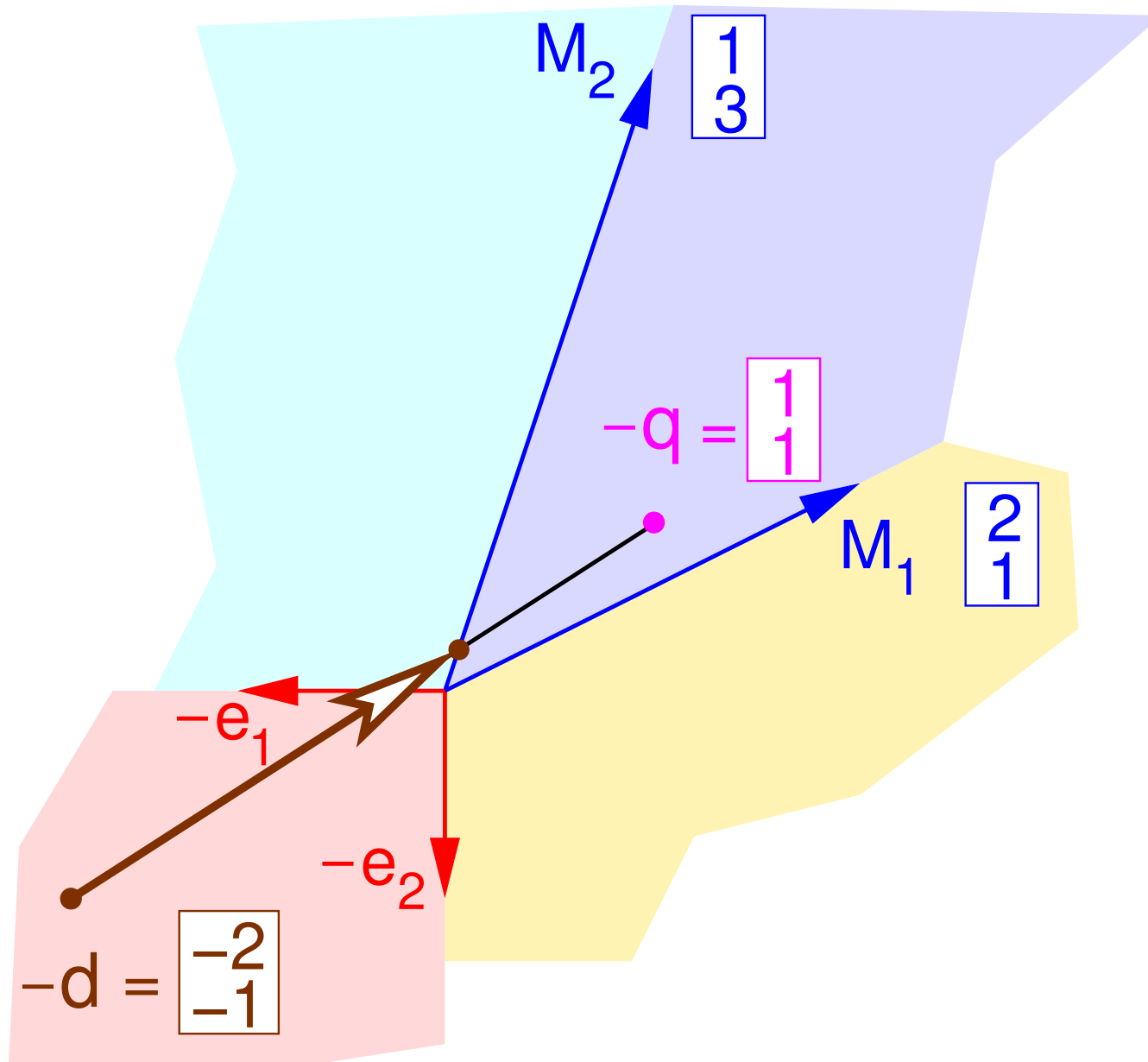
Inverting the LCP map F



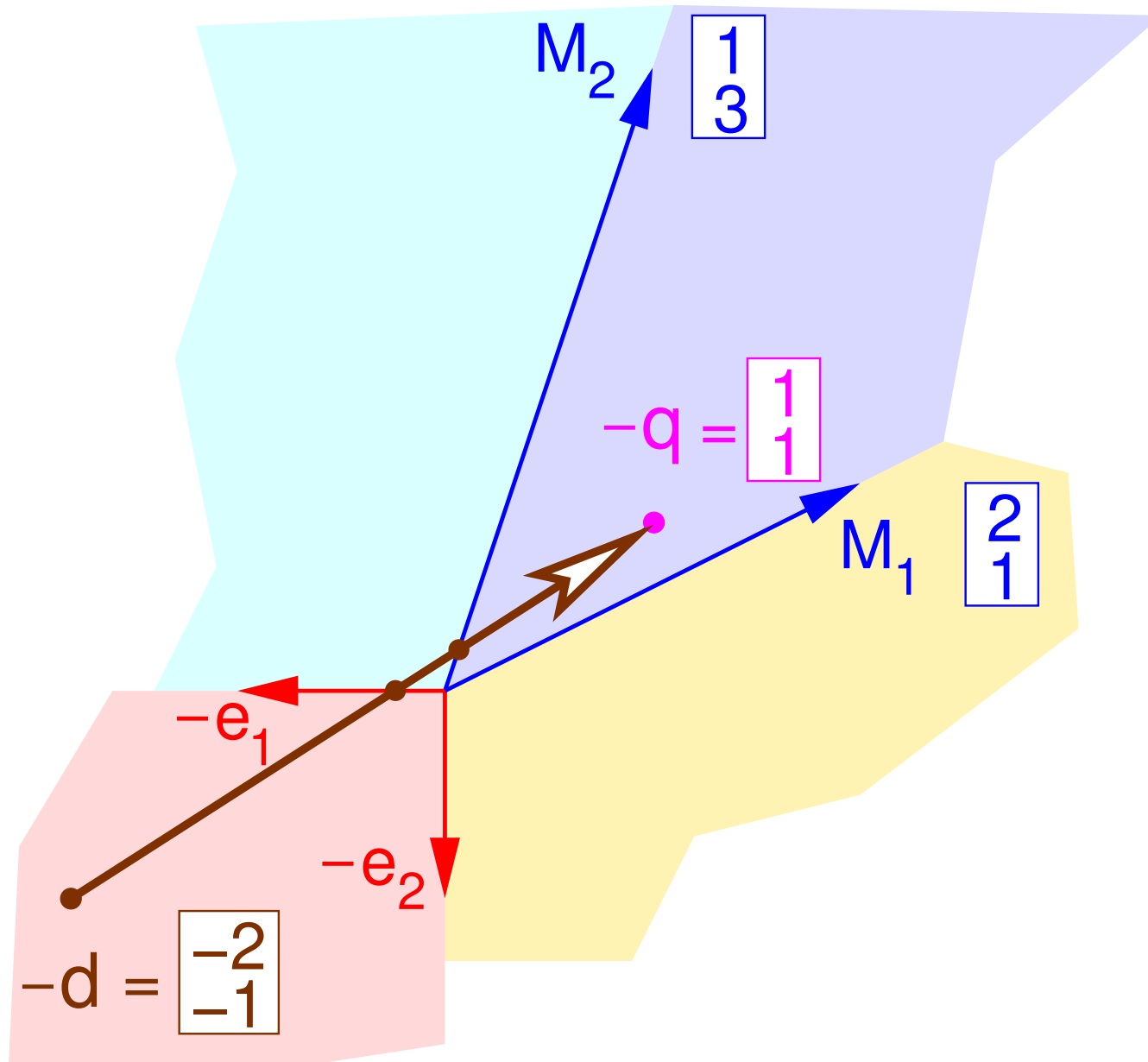
Inverting the LCP map F



Inverting the LCP map F



Inverting the LCP map F



Lemke-Howson: $-d = \text{unit vector}$

Theorem:

Symmetric Lemke-Howson with missing label k
= Lemke started at $-d = e_k$ in cone $\mathbf{C}(\{k\})$

- Proof:**
- initialize by pivoting z_0 in, w_k out
(still infeasible!), w_k stays in negative unit column
 - pivot z_k in (note $M_k > 0$), gives start in cone $\mathbf{C}(\{k\})$

Example with missing label 1

w_1		-1		2		1		-1	
	=		+		z_1 +		z_2 +		z_0
w_2		-1		1		3		0	

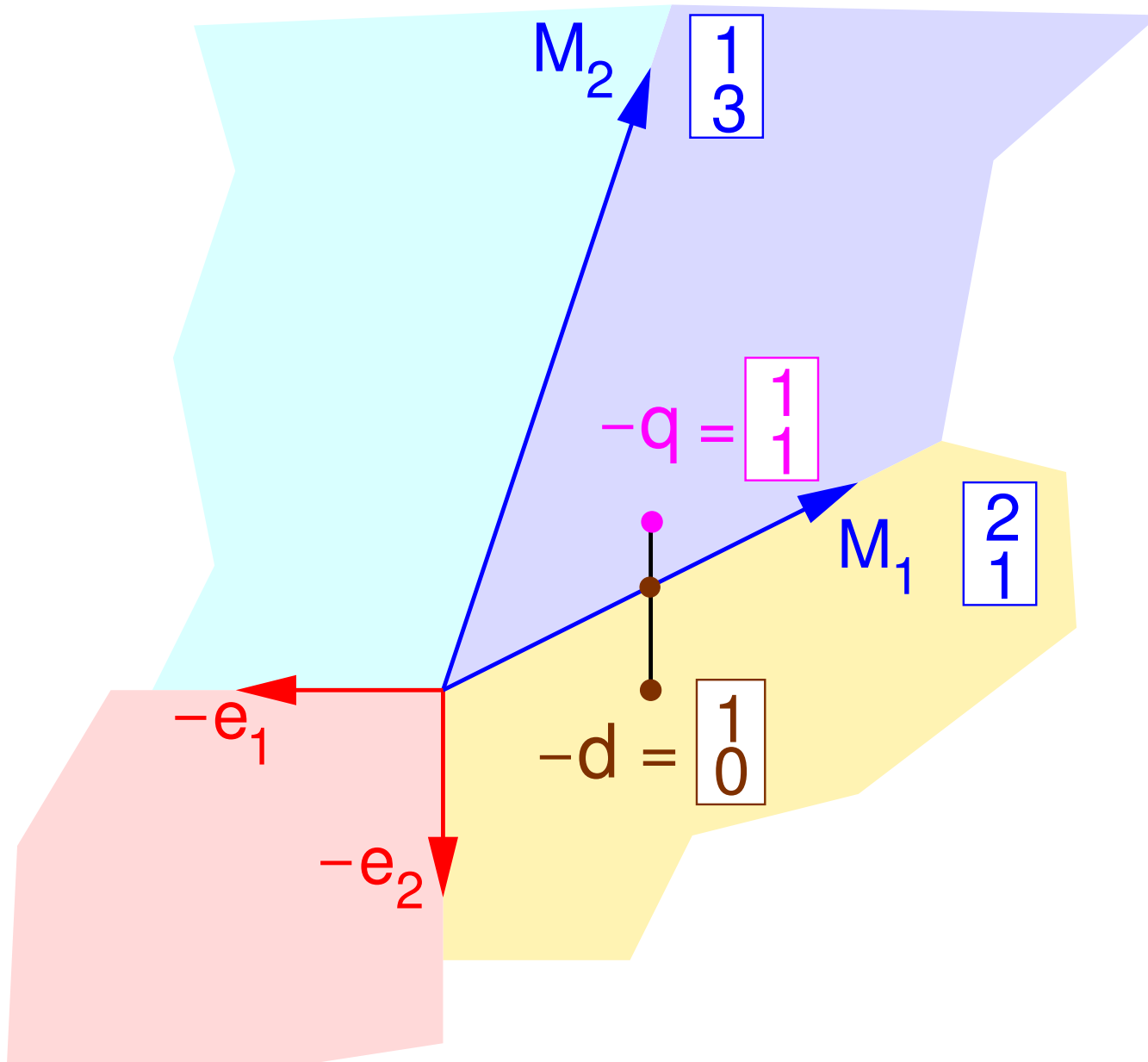
z_0		-1		2		1		-1	
	=		+		z_1 +		z_2 +		w_1
w_2		-1		1		3		0	

$$\begin{array}{c} z_0 \\ w_2 \end{array} = \begin{array}{c} -1 \\ -1 \end{array} + \begin{array}{c} 2 \\ 1 \end{array} z_1 + \begin{array}{c} 1 \\ 3 \end{array} z_2 + \begin{array}{c} -1 \\ 0 \end{array} w_1$$

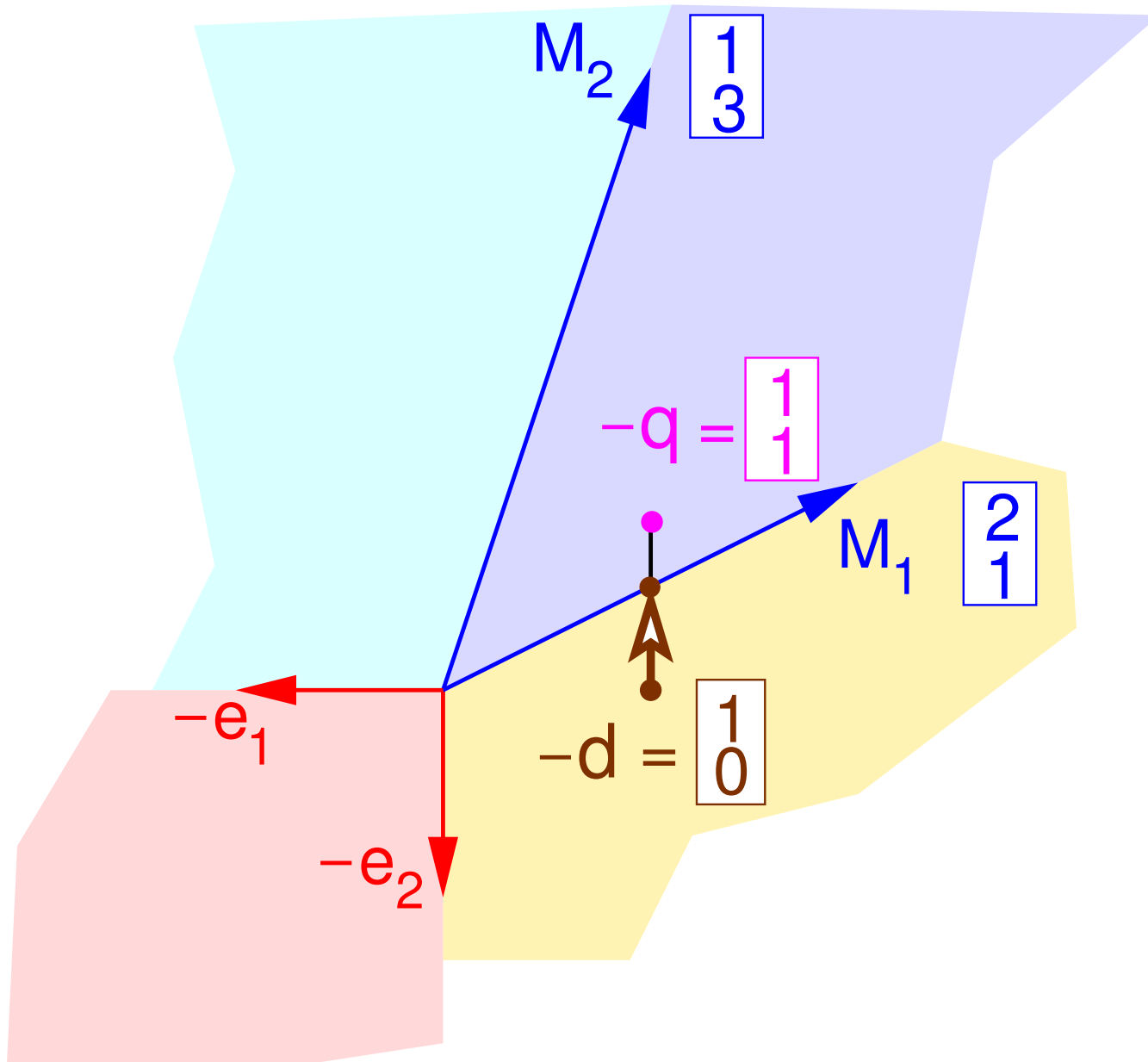
$$\begin{array}{c} z_0 \\ z_1 \end{array} = \begin{array}{c} 1 \\ 1 \end{array} + \begin{array}{c} 2 \\ 1 \end{array} w_2 + \begin{array}{c} -5 \\ -3 \end{array} z_2 + \begin{array}{c} -1 \\ 0 \end{array} w_1$$

$$\begin{array}{c} z_2 \\ z_1 \end{array} = \begin{array}{c} 0.2 \\ 0.4 \end{array} + \begin{array}{c} 0.4 \\ -0.2 \end{array} w_2 + \begin{array}{c} -0.2 \\ 0.6 \end{array} z_0 + \begin{array}{c} -0.2 \\ 0 \end{array} w_1$$

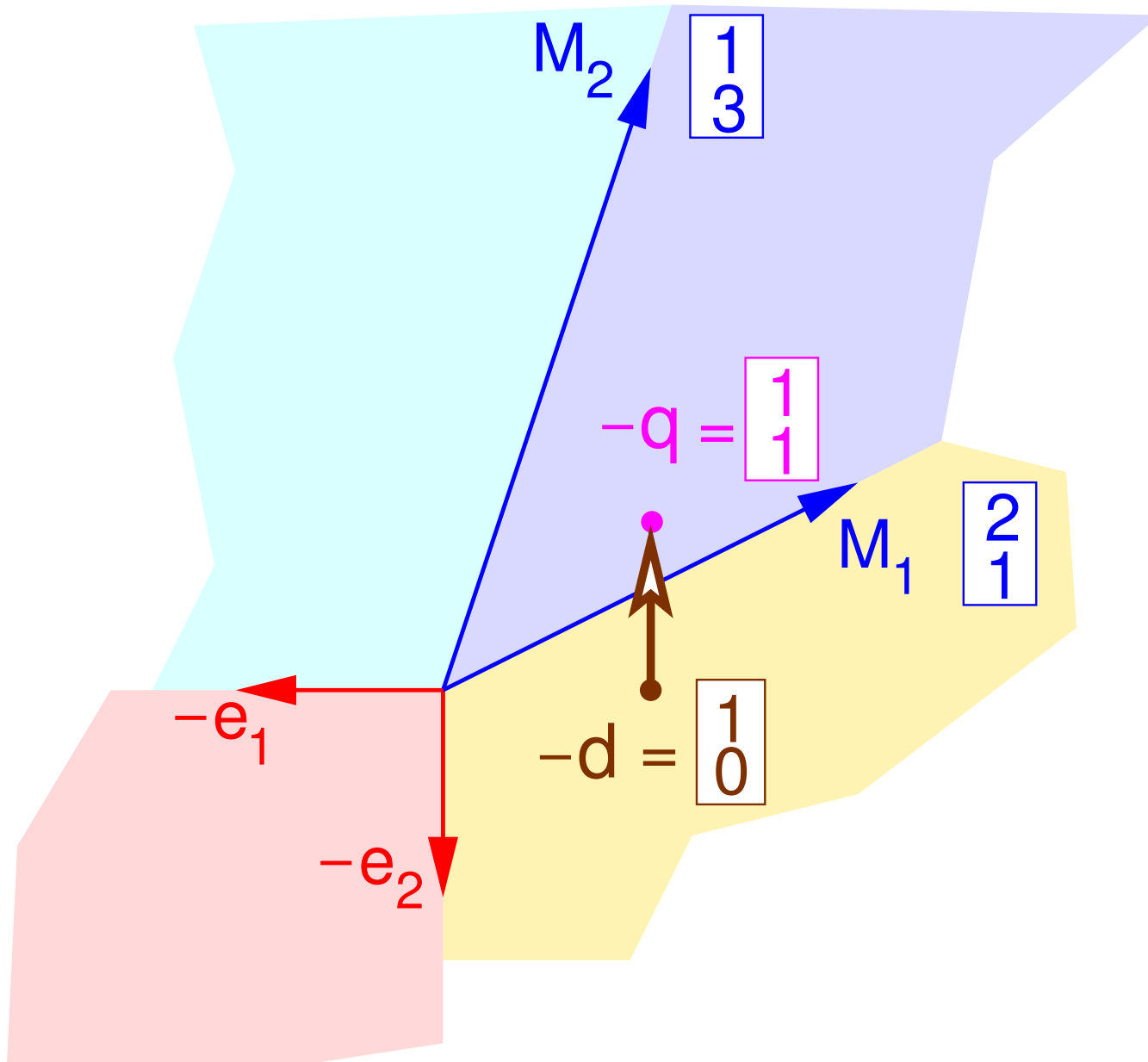
Start at unit vector



Start at unit vector



Start at unit vector



Complexity

A result of **Morris** implies that the symmetric LH can be **best-case exponential** (i.e., for **any** missing label).

Savani & von Stengel showed that for bimatrix games LH can be **best-case exponential** (i.e., for **any** missing label).

Murty and **Goldfarb** (independently):

Lemke's algorithm derived from an **LP** can be **exponential** for the specific covering vectors $(0, \dots, 0, 1, \dots, 1)^T$ resp. $(1, \dots, 1, 0, \dots, 0)^T$.

Megiddo: Lemke for **random M** (not $> \mathbf{0}$) has **expected**

- **exponential** running time when $\mathbf{d} = (1, 1, \dots, 1)^T$
- **quadratic** running time when $\mathbf{d} = (\varepsilon, \varepsilon^2, \dots, \varepsilon^n)^T$.